OPTIMAL CONTROL OF A DYNAMIC SEIRDS MODEL FOR THE SPREAD OF INFECTIOUS DISEASES

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Abstract. This article is devoted to the problem of optimal control of a reaction-diffusion system for an SEIRDS-type epidemiological model, where the dynamics evolve in a spatially heterogeneous environment. The control variables are the transmission rates β_e, β_1, and β_2, corresponding respectively to the contagion resulting from contact with asymptomatic and symptomatic individuals. The aim is to optimize the number of exposed and infected individuals at a final time T within the framework of the controlled evolution of the system. More precisely, the aim is to determine the optimal rates $\bar{\beta}_e$, $\bar{\beta}_1$, and $\bar{\beta}_2$ so that the numbers of exposed E and infected I_1 and I_2 do not exceed, at the final time T, the pre-established thresholds e, i_1, and i_2. In this article, we demonstrate the existence of these optimal controls in a suitable functional framework, and we derive the necessary first-order optimality conditions based on the adjoint variables.

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1. Introduction

In recent decades, various mathematical models have been developed to analyze the evolution of infectious diseases and curb their spread. These tools, whether statistical or mathematical in nature, provide valuable information, enabling decision-makers to implement effective policies [9]. Time series and compartmental models are frequently used to predict and simulate the dynamics of infectious diseases, thereby offering key instruments for epidemic management [11]. In fact, these models have helped to better understand these phenomena and have guided decision-makers towards the most appropriate decisions regarding the effectiveness of the measures implemented [5]. The major impact

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of infectious diseases on the development of human society highlights the importance of adopting strong prevention and control policies to protect public health. The recent COVID-19 pandemic has illustrated the essential role of global surveillance systems and rapid response mechanisms, capable of mitigating the negative impacts on both the economy and human health. Indeed, this pandemic led to a significant slowdown of the global economy, disrupting numerous economic sectors [6].

For several decades, the scientific literature on epidemic mathematical models has expanded with numerous contributions, often based on compartmental models [14]. These models divide the population into different compartments based on qualitative characteristics, such as "susceptible," "infected," and "recovered." These models have naturally allowed for the introduction of diffusion terms. For a recent overview of mathematical models of viral pandemics, we refer to [2]. It should be noted that epidemic models including spatial diffusion have been studied for a long time [12]. Very recently, a new epidemic diffusion model with nonlinear transmission rates and diffusion coefficients has been introduced and tested [13], while in [1], the authors proved well-posedness results for an initial boundary value problem associated with a variant of the compartmental model for COVID-19 studied in [13]. However, most models are based on ordinary differential equations (ODEs), but here we explore a compartmental model using partial differential equations (PDEs) to better represent spatial variations. By exploring these models, we hope to contribute to a deeper understanding and more effective management of infectious diseases.

The following diagram shows the contagion dynamics between the compartments of our model.

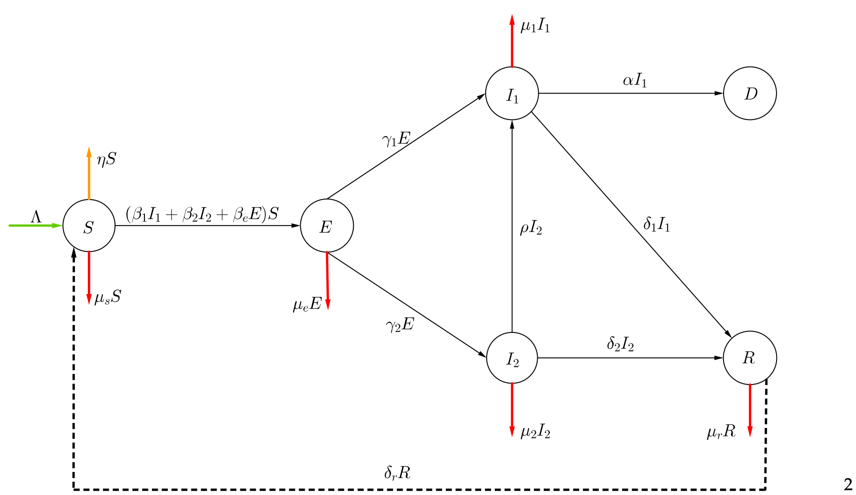

FIGURE 1. Flow chart describing the dynamics of contagion between the compartmental sub-groups considered in our model.

From the diagram, the state equations representing the spatio-temporal variations of the compartments in our model are established as follows:

$$\begin{cases} \partial_t S - \operatorname{div}(k_s \nabla S) = \Lambda + \delta_r R - (\beta_1 I_1 + \beta_2 I_2 + \beta_e E)\, S - (\eta + \mu_s)\, S \quad \text{in} \quad Q \\ \partial_t E - \operatorname{div}(k_e \nabla E) = (\beta_1 I_1 + \beta_2 I_2 + \beta_e E)\, S - \gamma_1 E - \gamma_2 E - \mu_e E \quad \text{in} \quad Q \\ \partial_t I_2 - \operatorname{div}(k_2 \nabla I_2) = \gamma_2 E - \rho I_2 - \delta_2 I_2 - \mu_2 I_2 \quad \text{in} \quad Q \\ \partial_t I_1 - \operatorname{div}(k_1 \nabla I_1) = \gamma_1 E + \rho I_2 - \alpha I_1 - \delta_1 I_1 - \mu_1 I_1 \quad \text{in} \quad Q \\ \partial_t R - \operatorname{div}(k_r \nabla R) = \delta_1 I_1 + \delta_2 I_2 - \delta_r R - \mu_r R \quad \text{in} \quad Q \\ (S, E, I_1, I_2, R)(0) = (S_0, E_0, I_{1_0}, I_{2_0}, R_0) \qquad \text{in} \quad \Omega \\ \dfrac{\partial S}{\partial \nu} = \dfrac{\partial E}{\partial \nu} = \dfrac{\partial I_1}{\partial \nu} = \dfrac{\partial I_2}{\partial \nu} = \dfrac{\partial R}{\partial \nu} = 0, \quad \text{in} \quad \Sigma_T := \partial\Omega \times (0, T). \end{cases} \tag{1.1}$$

where

♦ $T > 0$, Ω is an open bounded spatial domain in $\mathbb{R}^d$, $d \geq 2$, and $\partial\Omega$ denotes its boundary, assumed to be regular.

♦ We denote by ν the outward normal vector to Ω.

♦ $S(x,t)$, $E(x,t)$, $I_1(x,t)$, $I_2(x,t)$, $R(x,t)$, and $D(x,t)$ represent the respective densities at time $t \in [0,T)$ and location $x \in \Omega$ of susceptible individuals (those who can contract the disease), exposed individuals (those who carry the disease but do not yet show symptoms, although they can transmit the disease), detected infectious individuals (those who show symptoms, tested positive, and can transmit the disease), undetected infectious individuals (those who are sick but unaware of their status and can still transmit the disease), recovered individuals (after an infectious period but not necessarily immune), and finally individuals who have died from the disease.

As can be observed, the equation for D depends only on I_1 and does not influence the other equations. Consequently, D can be considered as an independent compartment since knowing I_1 allows one to determine D.

$$D(t,x) = D(0) + \alpha \int_Q I_1$$

where $D(0)$ is the initial condition for the death compartment (generally $D(0) = 0$ if we assume no deaths at the beginning of the epidemic). Additionally, we define the parameters involved in our model, which we assume to be all positive:

Table 1: Parameter Descriptions

Parameters	Description
β_1	Contribution of known infectious individuals to the infection force
β_2	Contribution of unknown infectious individuals to the infection force
β_e	Contribution of exposed individuals to the infection force
η	Vaccination or immunity gain rate
δ_r	Immunity loss rate depending on time
γ_1	Progression rate of exposed individuals to the detected infectious compartment
γ_2	Progression rate of exposed individuals to the undetected infectious compartment
ρ	Progression rate of undetected infectious individuals to the detected infectious compartment
δ_1	Recovery rate of known infectious individuals
δ_2	Recovery rate of unknown infectious individuals
Λ	Natural birth rate
α	Disease-induced mortality rate for known infectious individuals
μ_k	Natural mortality rate in compartment $k = s, e, 1, 2, r$

The exposed population and the infectious population refer to asymptomatic and symptomatic individuals, respectively. As observed in the COVID-19 epidemic, exposed individuals can also spread the disease. This means that the transmission or contagion parameters β_1, β_2, and β_e play a crucial role in the spread of an epidemic. It would therefore be interesting and highly beneficial to control these different parameters in order to slow the progression of an epidemic through social prevention policies or measures. With this in mind, we consider the transmission rates resulting from contact with asymptomatic or symptomatic individuals β_1, β_2, and β_e as control variables for our optimal control problem, which we introduce in Section 3.

The article is structured into three distinct sections, each contributing specifically to our research. Section 1, as mentioned earlier, serves as an introduction to our study. In Section 2, we list our assumptions and notations, and we state our well-posedness results for the state problem (1.1). Finally, Section 3, which addresses the main objective of this article, is dedicated to the study of the optimal control problem, for which we prove the existence of an optimal control and derive a first-order optimality condition.

2. Assumptions and Well-posedness Results

In this section, we make specific assumptions and present our well-resolved results. First, we assume that the set $\Omega \subset \mathbb{R}^d, 1 \leq d \leq 3$, is bounded, connected and regular. Then, if X is a Banach space, $\|\cdot\|_X$ is its norm. For simplicity, we use the same symbol for the norm in X and in all powers of X. We also introduce

$$H := L^2(\Omega) \quad \text{and} \quad V := H^1(\Omega).$$

We have the dense and continuous embeddings $V \subset H \cong H^* \subset V^*$, such that

$$\langle u, v\rangle = \int_\Omega uv$$

for all $u \in H$ and $v \in V$, where $\langle \cdot, \cdot \rangle$ is the dual pairing between V^* and V.

We assume that

♦ $k_s, k_e, k_1, k_2, k_r : Q \to \mathbb{R}$ are positive functions in $L^\infty(Q)$ satisfying

$$k_* \leq k_s(x,t), k_e(x,t), k_1(x,t), k_2(x,t), k_r(x,t) \leq k^* \quad \text{a.e. } (x,t) \in Q \tag{2.1}$$

with k_* and k^* being strictly positive constants.

♦ $\beta_1, \beta_2, \beta_e : Q \to \mathbb{R}$ are positive functions in $L^\infty(Q)$ satisfying

$$0 \leq \beta_1(x,t), \beta_2(x,t), \beta_e(x,t) \leq M, \quad \text{a.e. } (x,t) \in Q \tag{2.2}$$

where M is a positive constant.

$$\Lambda,\ \gamma_1,\ \gamma_2,\ \delta_1,\ \delta_2,\ \eta,\ \rho,\ \alpha,\ \mu_s,\ \mu_e,\ \mu_1,\ \mu_2, \text{ and } \mu_r \text{ are all positive constants.} \tag{2.3}$$

♦ $\delta_r \in L^\infty(0,T)$ and satisfies

$$0 \le \delta_r(t) \le \delta^* \quad \text{a.e. } t \in (0,T) \tag{2.4}$$

with δ^* being a positive constant.

♦ For the initial data, we assume that

$$S_0, E_0, I_{1_0}, I_{2_0}, R_0 \in L^\infty(\Omega) \quad \text{and} \quad D_0 = 0 \tag{2.5}$$

are positive functions.

Additionally, the quantities appearing in the control problem, more precisely in (3.2) and (3.1), must satisfy the following assumptions:

$$(\mathbf{H}_1)\ \theta_i, \varpi_i \ge 0, i = e, 1, 2, \text{ but not all equal to } 0 \text{ simultaneously}, \tag{2.6}$$

$$(\mathbf{H}_2)\ \beta_i^{\min}, \beta_i^{\max} \in L^\infty(Q), i = e, 1, 2, \text{ are non-negative.} \tag{2.7}$$

We also use continuous embedding in dimensions three

$$V \hookrightarrow L^p(\Omega) \quad \text{for} \quad p \in [1, 2^\star] := [1, 6] \tag{2.8}$$

where this embedding is compact if $p < 6$. In particular, there exists a positive constant C_Ω that depends only on the Ω domain, such that

$$\|v\|_{L^6(\Omega)} \le C_\Omega \|v\|_V \quad \text{for all } v \in V, \tag{2.9}$$

$$\|v\|_{L^p(\Omega)} \le \delta \|\nabla v\|_H + C_{\Omega,\delta,p}\|v\|_H \quad \text{for all } v \in V, p \in [1,6) \text{ and } \delta > 0, \tag{2.10}$$

where $C_{\Omega,\delta,p}$ is a constant depending on p and δ. We now recall Young's inequality, which we will use very frequently for estimations.

$$ab \le \delta a^2 + \frac{1}{4\delta} b^2 \quad \text{for all } a, b \in \mathbb{R} \text{ and } \delta > 0.$$

Now, we define the notion of solution for our state problem (1.1) under the assumptions (2.1)-(2.5).

Definition 2.1. *Suppose (2.1)-(2.5). Given $S_0, E_0, I_{1_0}, I_{2_0}, R_0 \in L^\infty(\Omega)$, a weak solution of the system (1.1) is a quintuple of positive functions (S, E, I_1, I_2, R) satisfying the regularity properties*

$$S, E, I_1, I_2, R \in H^1(0,T;V^*) \cap L^2(0,T;V) \hookrightarrow C^0([0,T];H), \tag{2.11}$$

$$S, E, I_1, I_2, R \ge 0 \quad \textit{a.e. in } Q, \tag{2.12}$$

$$S, E, I_1, I_2, R \in L^\infty(Q), \tag{2.13}$$

and satisfying the variational equations

$$\langle \partial_t S, v\rangle + \int_\Omega k_s \nabla S \cdot \nabla v = \int_\Omega \left[\Lambda + \delta_r R - (\beta_1 I_1 + \beta_2 I_2 + \beta_e E)\, S - (\eta + \mu_s)\, S\right] v \tag{2.14}$$

$$\langle \partial_t E, v\rangle + \int_\Omega k_e \nabla E \cdot \nabla v = \int_\Omega \left[(\beta_1 I_1 + \beta_2 I_2 + \beta_e E)\, S - \gamma_1 E - \gamma_2 E - \mu_e E\right] v \tag{2.15}$$

$$\langle \partial_t I_2, v\rangle + \int_\Omega k_2 \nabla I_2 \cdot \nabla v = \int_\Omega \left(\gamma_2 E - \rho I_2 - \delta_2 I_2 - \mu_2 I_2\right) v \tag{2.16}$$

$$\langle \partial_t I_1, v\rangle + \int_\Omega k_1 \nabla I_1 \cdot \nabla v = \int_\Omega \left(\gamma_1 E + \rho I_2 - \delta_1 I_1 - \alpha I_1 - \mu_1 I_1\right) v \tag{2.17}$$

$$\langle \partial_t R, v\rangle + \int_\Omega k_r \nabla R \cdot \nabla v = \int_\Omega \left(\, \delta_1 I_1 + \delta_2 I_2 - \delta_r R - \mu_r R\right) v \tag{2.18}$$

a.e. in $(0, T)$ *for all* $v \in V$*, as well as the initial condition*

$$(S, E, I_1, I_2, R, D)\,(0) = (S_0, E_0, I_{1_0}, I_{2_0}, R_0, 0)\,. \tag{2.19}$$

The results regarding the well-posedness of the problem (1.1) are given by the following theorems, whose proofs can be found in [7].

Theorem 2.1. *(See [7], theorem 1). Under the assumptions (2.1)-(2.4) on the structure of the system and (2.5) on the initial data, there exists a unique solution* (S, E, I_1, I_2, R) *satisfying the regularity conditions (2.11)-(2.13), which solves the variational problem (2.14)-(2.19) and also satisfies the stability estimate*

$$\|(S, E, I_1, I_2, R)\|_W \leq K_1 \quad \text{where } W = C^0([0,T]; H) \cap L^2(0,T; V) \cap L^\infty(Q) \tag{2.20}$$

with a positive constant $K_1 > 0$ *that depends only on* Ω*,* T*, the constants* k_**,* k^**,* M*,* δ^**,* γ_1*,* γ_2*,* δ*, and* ρ*, as well as the initial data.*

The second result is an estimate of the continuous dependence of the solution to the problem (1.1) with respect to the different contact rates β_1, β_2, and β_e.

Theorem 2.2. *(See [7], theorem 2). Suppose (2.1)-(2.4) on the structure of the system and (2.5) on the initial data. Let* $\beta_1^{(j)}, \beta_2^{(j)}, \beta_e^{(j)}, j = 1, 2$*, be positive functions in* $L^\infty(Q)$ *with norms bounded by* $\beta^\star$*, and let* $\left(S^{(j)}, E^{(j)}, I_1^{(j)}, I_2^{(j)}, R^{(j)}\right)$ *be the corresponding solutions. Then the inequality*

$$\begin{aligned} &\left\|\left(S^{(1)}, E^{(1)}, I_1^{(1)}, I_2^{(1)}, R^{(1)}\right) - \left(S^{(2)}, E^{(2)}, I_1^{(2)}, I_2^{(2)}, R^{(2)}\right)\right\|_{C^0([0,T];H)\cap L^2(0,T;V)} \\ &\qquad \leq K_2 \left\|\left(\beta_1^{(1)}, \beta_2^{(1)}, \beta_e^{(1)}\right) - \left(\beta_1^{(2)}, \beta_2^{(2)}, \beta_e^{(2)}\right)\right\|_{L^2(0,T;H)} \end{aligned} \tag{2.21}$$

holds with a positive constant K_2 *that depends only on the structure of the system,* Ω*,* T*, the initial data, and the constant* M*.*

3. The Optimal Control Problem

In this section, we consider the following optimal control problem:

Minimize the cost functional $\mathcal{J}\left((S,E,I_1,I_2,R)\;;\;\cdot\;\right):\mathcal{U}_{ad}\to\mathbb{R}$

$$\begin{aligned}\mathcal{J}\left((S,E,I_1,I_2,R);\beta_1,\beta_2,\beta_e\right):=&\frac{\theta_e}{2}\int_\Omega|E-e|^2+\frac{\theta_1}{2}\int_\Omega|I_1-i_1|^2+\frac{\theta_2}{2}\int_\Omega|I_2-i_2|^2\\&+\frac{1}{2}\int_Q\left(\varpi_1|\beta_1|^2+\varpi_2|\beta_2|^2+\varpi_e|\beta_e|^2\right)\end{aligned}\tag{3.1}$$

Subject to the control constraint

$$\begin{aligned}(\beta_1,\beta_2,\beta_e)\in\mathcal{U}_{ad}:=\Big\{(\beta_1,\beta_2,\beta_e)\in\left(L^\infty(Q)^+\right)^3:\beta_1^{\min}\le\beta_1\le\beta_1^{\max},\beta_2^{\min}\le\beta_2\le\beta_2^{\max}\\\text{and }\beta_e^{\min}\le\beta_e\le\beta_e^{\max}\ \text{ a.e. in }Q\}\end{aligned}\tag{3.2}$$

and the state system (1.1)

$$\begin{cases}\partial_t S-\operatorname{div}(k_s\nabla S)=\Lambda+\delta_r R-(\beta_1 I_1+\beta_2 I_2+\beta_e E)\,S-(\eta+\mu_s)\,S & \text{in}\quad Q\\ \partial_t E-\operatorname{div}(k_e\nabla E)=(\beta_1 I_1+\beta_2 I_2+\beta_e E)\,S-\gamma_1 E-\gamma_2 E-\mu_e E & \text{in}\quad Q\\ \partial_t I_2-\operatorname{div}(k_2\nabla I_2)=\gamma_2 E-\rho I_2-\delta_2 I_2-\mu_2 I_2 & \text{in}\quad Q\\ \partial_t I_1-\operatorname{div}(k_1\nabla I_1)=\gamma_1 E+\rho I_2-\alpha I_1-\delta_1 I_1-\mu_1 I_1 & \text{in}\quad Q\\ \partial_t R-\operatorname{div}(k_r\nabla R)=\delta_1 I_1+\delta_2 I_2-\delta_r R-\mu_r R & \text{in}\quad Q\\ (S,E,I_1,I_2,R)(0)=(S_0,E_0,I_{1_0},I_{2_0},R_0) & \text{in}\quad\Omega\\ \dfrac{\partial S}{\partial\nu}=\dfrac{\partial E}{\partial\nu}=\dfrac{\partial I_1}{\partial\nu}=\dfrac{\partial I_2}{\partial\nu}=\dfrac{\partial R}{\partial\nu}=0, & \text{in}\quad\Sigma_T:=\partial\Omega\times(0,T).\end{cases}$$

where E, I_1, and I_2 (in the expression of $\mathcal{J}$) are components of the weak solution (S,E,I_1,I_2,R) of (1.1) corresponding to $(\beta_e,\beta_1,\beta_2)$. For reasons of simplicity, we will very often write $\mathcal{J}\left((S,E,I_1,I_2,R);\beta_1,\beta_2,\beta_e\right)$ by $\mathcal{J}\left(\beta_1,\beta_2,\beta_e\right)$.

Aim here is to keep the number of exposed or infected individuals below certain threshold values at the final time T, specifically the values e, i_1, and i_2 for the compartments E, I_1, and I_2, respectively. For similar work, we refer the reader to [4], where a similar approach was used to keep the prostate index in a prostate tumor growth model below a certain threshold value. We can also cite [3]. Optimal control is of great interest in epidemics, and at this point, we can refer the reader to very few recent articles on this subject [10], where control or identification problems for various coefficients in ODE models (without diffusion) have been studied, and [10] for a control problem in a reaction-diffusion model.

3.1. Existence of Optimal Control.

In this subsection, we prove the existence of an optimal control $(\bar\beta_e,\bar\beta_1,\bar\beta_2)$ and derive the first-order optimality conditions.

Theorem 3.1. *Suppose the assumptions (2.1)-(2.5) and (2.6)-(2.7) hold. Then the optimal control problem (3.1)-(3.2)-(1.1) admits at least one solution* $(\bar{\beta}_e, \bar{\beta}_1, \bar{\beta}_2) \in \mathcal{U}_{ad}$*, such that if* $(\bar{S}, \bar{E}, \bar{I}_1, \bar{I}_2, \bar{R})$ *is the solution of the state system (1.1) associated with* $(\bar{\beta}_e, \bar{\beta}_1, \bar{\beta}_2)$*, we have*

$$\mathcal{J}(\bar{\beta}_e, \bar{\beta}_1, \bar{\beta}_2) = \min_{(\beta_e,\beta_1,\beta_2)\in\mathcal{U}_{ad}} \mathcal{J}(\beta_e, \beta_1, \beta_2). \tag{3.3}$$

Proof. Let $\left(\beta_1^{(n)}, \beta_2^{(n)}, \beta_e^{(n)}\right)_{n\in\mathbb{N}} \subset \mathcal{U}_{\text{ad}}^{\mathbb{N}}$ be a minimizing sequence for $\mathcal{J}$ such that

$$\inf_{(\beta_e,\beta_1,\beta_2)\in\mathcal{U}_{\text{ad}}} \{\mathcal{J}\} \leq \mathcal{J}\left(\beta_1^{(n)}, \beta_2^{(n)}, \beta_e^{(n)}\right) \leq \inf_{(\beta_e,\beta_1,\beta_2)\in\mathcal{U}_{\text{ad}}} \{\mathcal{J}\} + \frac{1}{n} \tag{3.4}$$

and $\{(S_n, E_n, I_{1_n}, I_{2_n}, R_n)\}$, the sequence of corresponding states for $\left(\beta_1^{(n)}, \beta_2^{(n)}, \beta_e^{(n)}\right)$, with the regularities given by Definition 2.1.

Since $\left(\beta_1^{(n)}, \beta_2^{(n)}, \beta_e^{(n)}\right)_{n\in\mathbb{N}} \subset \mathcal{U}_{\text{ad}}^{\mathbb{N}}$, we have that $\left(\beta_i^{(n)}\right)_{i=e,1,2}$ is uniformly bounded in $L^\infty(Q_T)$. Therefore, by the Banach-Alaoglu theorem, we can deduce that there exists $(\bar{\beta}_e, \bar{\beta}_1, \bar{\beta}_2) \in (L^\infty(Q_T))^3$ such that, after extracting a subsequence,

$$\beta_e^{(n)} \overset{*}{\rightharpoonup} \bar{\beta}_e, \quad \beta_1^{(n)} \overset{*}{\rightharpoonup} \bar{\beta}_1, \quad \beta_2^{(n)} \overset{*}{\rightharpoonup} \bar{\beta}_2 \text{ weakly* in } L^\infty(Q_T). \tag{3.5}$$

Moreover, since $\mathcal{U}_{\text{ad}}$ is convex and closed in the space $L^2(Q)$, it is also sequentially closed for the weak topology, and thus $(\bar{\beta}_e, \bar{\beta}_1, \bar{\beta}_2) \in \mathcal{U}_{\text{ad}}$.

Now, consider the solutions $\{(S_n, E_n, I_{1_n}, I_{2_n}, R_n)\} \in H^1(0,T;V^*) \cap L^2(0,T;V)$ corresponding to $\left(\beta_1^{(n)}, \beta_2^{(n)}, \beta_e^{(n)}\right)$ for each $n \in \mathbb{N}$. By the uniform estimate (2.20), these solutions are uniformly bounded with respect to n in $H^1(0,T;V^*) \cap L^2(0,T;V)$. Therefore, again by Banach-Alaoglu, we can say that, after extracting a subsequence,

$$S_n \rightharpoonup \bar{S}, \quad E_n \rightharpoonup \bar{E}, \quad I_{1_n} \rightharpoonup \bar{I}_1, \quad I_{2_n} \rightharpoonup \bar{I}_2, \quad \text{and} \quad R_n \rightharpoonup \bar{R} \tag{3.6}$$

weakly in $H^1(0,T;V^*) \cap L^2(0,T;V)$, hence weakly in $C^0([0,T];H)$, strongly in $L^p(Q)$ for $1 \leq p < +\infty$, and almost everywhere in Q. Thus, by the same arguments used to prove the existence of a solution (see subsection 3.1 in the proof of Theorem 2.1 in [7]), we see that $(\bar{S}, \bar{E}, \bar{I}_1, \bar{I}_2, \bar{R})$ is the solution of (2.14)-(2.18) corresponding to $(\bar{\beta}_e, \bar{\beta}_1, \bar{\beta}_2)$.

Thus, taking the $\liminf$ in $\mathcal{J}\left(\beta_1^{(n)}, \beta_2^{(n)}, \beta_e^{(n)}\right)$, it follows that

$$\begin{aligned}
\liminf_{n\to+\infty} \mathcal{J}\left(\beta_1^{(n)}, \beta_2^{(n)}, \beta_e^{(n)}\right) &= \liminf_{n\to+\infty} \left[\frac{\theta_e}{2}\int_\Omega |E_n - e|^2 + \frac{\theta_1}{2}\int_\Omega |I_{1_n} - i_1|^2 + \frac{\theta_2}{2}\int_\Omega |I_{2_n} - i_2|^2 \right. \\
&\qquad \left. + \frac{1}{2}\int_Q \left(\varpi_1|\beta_1^{(n)}|^2 + \varpi_2|\beta_2^{(n)}|^2 + \varpi_e|\beta_e^{(n)}|^2\right)\right] \\
&= \lim_{n\to+\infty} \left[\frac{\theta_e}{2}\int_\Omega |E_n - e|^2 + \frac{\theta_1}{2}\int_\Omega |I_{1_n} - i_1|^2 + \frac{\theta_2}{2}\int_\Omega |I_{2_n} - i_2|^2\right] \\
&\qquad + \liminf_{n\to+\infty} \frac{1}{2}\int_Q \left(\varpi_1|\beta_1^{(n)}|^2 + \varpi_2|\beta_2^{(n)}|^2 + \varpi_e|\beta_e^{(n)}|^2\right)\Bigg]
\end{aligned}$$

$$= \left[\frac{\theta_e}{2}\int_\Omega |\bar{E}-e|^2 + \frac{\theta_1}{2}\int_\Omega |\bar{I}_1 - i_1|^2 + \frac{\theta_2}{2}\int_\Omega |\bar{I}_2 - i_2|^2\right] + \liminf_{n\to+\infty} \frac{1}{2}\int_Q \left(\varpi_1 |\beta_1^{(n)}|^2 + \varpi_2 |\beta_2^{(n)}|^2 + \varpi_e |\beta_e^{(n)}|^2\right)\Bigg] \tag{3.7}$$

Exploiting the lower semicontinuity of the norm in $L^2(Q)$, we obtain

$$\liminf_{n\to+\infty} \mathcal{J}\left(\beta_1^{(n)}, \beta_2^{(n)}, \beta_e^{(n)}\right) \geq \left[\frac{\theta_e}{2}\int_\Omega |\bar{E}-e|^2 + \frac{\theta_1}{2}\int_\Omega |\bar{I}_1 - i_1|^2 + \frac{\theta_2}{2}\int_\Omega |\bar{I}_2 - i_2|^2\right] + \frac{1}{2}\int_Q \left(\varpi_1 |\bar{\beta}_1|^2 + \varpi_2 |\bar{\beta}_2|^2 + \varpi_e |\bar{\beta}_e|^2\right)$$

that is,

$$\mathcal{J}(\bar{\beta}_e, \bar{\beta}_1, \bar{\beta}_2) \leq \liminf_{n\to+\infty} \mathcal{J}\left(\beta_1^{(n)}, \beta_2^{(n)}, \beta_e^{(n)}\right). \tag{3.8}$$

Therefore,

$$\mathcal{J}(\bar{\beta}_e, \bar{\beta}_1, \bar{\beta}_2) = \min_{(\beta_e,\beta_1,\beta_2)\in\mathcal{U}_{\text{ad}}} \mathcal{J}(\beta_e, \beta_1, \beta_2).$$

This completes the proof of Theorem 3.1. □

3.2. **The State-to-Control Mapping.**

Linearized System. In this section, we introduce the state-to-control mapping $\mathcal{S}$ and prove its Fréchet differentiability. We define

$\mathbb{Y} := \left(H^1(0,T;V^*) \cap L^2(0,T;V)\right)^5, \mathcal{S} : \mathcal{U}_{ad} \to \mathbb{Y}$ by setting

$\mathcal{S}(\beta_e, \beta_1, \beta_2)$ as the solution (S, E, I_1, I_2, R) of (1.1) corresponding to $(\beta_e, \beta_1, \beta_2)$.

To achieve the result of Fréchet differentiability of $\mathcal{S}$, we introduce the linearized system of (1.1). We fix an optimal control $(\bar{\beta}_e, \bar{\beta}_1, \bar{\beta}_2)$ and the corresponding state $(\bar{S}, \bar{E}, \bar{I}_1, \bar{I}_2, \bar{R})$. Given a variation $\left(\tilde{\beta}_e, \tilde{\beta}_1, \tilde{\beta}_2\right) \in (L^\infty(Q))^3$, for any $(\beta_e, \beta_1, \beta_2) := \left(\bar{\beta}_e + \tilde{\beta}_e, \bar{\beta}_1 + \tilde{\beta}_1, \bar{\beta}_2 + \tilde{\beta}_2\right) \in \mathcal{U}_{ad}$ such that $\mathcal{S}(\beta_e, \beta_1, \beta_2) = (S, E, I_1, I_2, R)$, the linearized problem consists of finding $(\xi_s, \xi_e, \xi_1, \xi_2, \xi_r) := \left(S - \bar{S}, E - \bar{E}, I_1 - \bar{I}_1, I_2 - \bar{I}_2, R - \bar{R}\right) \in \mathbb{Y}$ satisfying the following linear system

$$\begin{cases} \partial_t \xi_s - \operatorname{div}(k_s \nabla \xi_s) + A_1 \xi_s + B_1 \xi_e + C_1 \xi_1 + D_1 \xi_2 - \delta_r \xi_r = -\left(K_1 \tilde{\beta}_1 + L_1 \tilde{\beta}_2 + M_1 \tilde{\beta}_e\right) \\ \partial_t \xi_e - \operatorname{div}(k_e \nabla \xi_e) - (A_2 \xi_s + B_1 \xi_e + C_1 \xi_1 + D_1 \xi_2) + (\gamma_1 + \gamma_2 + \mu_e)\xi_e = K_1 \tilde{\beta}_1 + L_1 \tilde{\beta}_2 + M_1 \tilde{\beta}_e \\ \partial_t \xi_2 - \operatorname{div}(k_2 \nabla \xi_2) = \gamma_2 \xi_e - (\rho + \delta_2 + \mu_2)\xi_2 \\ \partial_t \xi_1 - \operatorname{div}(k_1 \nabla \xi_1) = \gamma_1 \xi_e + \rho \xi_2 - (\alpha + \delta_1 + \mu_1)\xi_1 \\ \partial_t \xi_r - \operatorname{div}(k_r \nabla \xi_r) = \delta_1 \xi_1 + \delta_2 \xi_2 - (\delta_r + \mu_r)\xi_r \end{cases} \tag{3.9}$$

in the spacetime cylinder $Q := \Omega \times (0,T)$, with

$$(\xi_s, \xi_e, \xi_1, \xi_2, \xi_r)(0) = (0,0,0,0,0) \quad \text{in} \quad \Omega, \tag{3.10}$$

and

$$\frac{\partial \xi_s}{\partial \nu} = \frac{\partial \xi_e}{\partial \nu} = \frac{\partial \xi_1}{\partial \nu} = \frac{\partial \xi_2}{\partial \nu} = \frac{\partial \xi_r}{\partial \nu} = 0, \quad \text{in} \quad \Sigma_T := \partial\Omega \times (0,T). \tag{3.11}$$

where

$$\begin{aligned} A_1 &= \bar{\beta}_1 \bar{I}_1 + \bar{\beta}_2 \bar{I}_2 + \bar{\beta}_e \bar{E}, \quad B_1 = \bar{S}\bar{\beta}_e, \quad C_1 = \bar{S}\bar{\beta}_1, \\ D_1 &= \bar{S}\bar{\beta}_2, \quad K_1 = \bar{S}\bar{I}_1, \quad L_1 = \bar{S}\bar{I}_2, \quad M_1 = \bar{S}\bar{E}. \end{aligned} \tag{3.12}$$

In variational form, the linearized system associated with $\left(\tilde{\beta}_e, \tilde{\beta}_1, \tilde{\beta}_2\right)$ is written as

$$\begin{aligned} \langle \partial_t \xi_s, v\rangle + \int_\Omega \left[(A_1 + \eta + \mu_s)\,\xi_s + B_1\xi_e + C_1\xi_1 + D_1\xi_2 - \delta_r\xi_r\right] v + \int_\Omega k_s \nabla\xi_s \cdot \nabla v \\ = -\int_\Omega \left(K_1\tilde{\beta}_1 + L_1\tilde{\beta}_2 + M_1\tilde{\beta}_e\right) v \end{aligned} \tag{3.13}$$

$$\begin{aligned} \langle \partial_t \xi_e, v\rangle - \int_\Omega \left[A_1\xi_s + \left(B_1 - (\gamma_1 + \gamma_2 + \mu_e)\right)\xi_e + C_1\xi_1 + D_1\xi_2\right] v + \int_\Omega k_e \nabla\xi_e \cdot \nabla v \\ = \int_\Omega \left(K_1\tilde{\beta}_1 + L_1\tilde{\beta}_2 + M_1\tilde{\beta}_e\right) v \end{aligned} \tag{3.14}$$

$$\langle \partial_t \xi_2, v\rangle - \int_\Omega \left[\gamma_2\xi_e - (\rho + \delta_2 + \mu_2)\xi_2\right] v + \int_\Omega k_2 \nabla\xi_2 \cdot \nabla v = 0 \tag{3.15}$$

$$\langle \partial_t \xi_1, v\rangle - \int_\Omega \left[\gamma_1\xi_e + \rho\xi_2 - (\alpha + \delta_1 + \mu_1)\xi_1\right] v + \int_\Omega k_1 \nabla\xi_1 \cdot \nabla v = 0 \tag{3.16}$$

$$\langle \partial_t \xi_r, v\rangle - \int_\Omega \left[\delta_1\xi_1 + \delta_2\xi_2 - (\delta_r + \mu_r)\xi_r\right] v + \int_\Omega k_r \nabla\xi_r \cdot \nabla v = 0 \tag{3.17}$$

a.e. in $(0,T)$ for all $v \in V$. The result we introduce now proves that the linearized system (3.13)-(3.17) is well-posed, and the solution satisfies an estimate given by the following theorem.

Theorem 3.2. *Suppose the assumptions (2.1)-(2.5) and (2.6)-(2.7) hold. For all $\left(\bar{\beta}_e, \bar{\beta}_1, \bar{\beta}_2\right) \in \mathcal{U}_{ad}$ and $(h_e, h_1, h_2) \in \mathcal{U}_{ad}$, the corresponding linearized system (3.13)-(3.17) has a unique solution $(\xi_s, \xi_e, \xi_1, \xi_2, \xi_r) \in \mathbb{Y}$. Furthermore, the following estimate*

$$\|(\xi_s, \xi_e, \xi_1, \xi_2, \xi_r)\|_{\mathbb{Y}} \leq C\, \|h\|_{\mathcal{U}_{ad}} \tag{3.18}$$

is satisfied, with a positive constant C that depends only on the structure of the original system, on Ω, T, the initial data, as well as on an upper bound for the norm $\left\|\left(\bar{\beta}_e, \bar{\beta}_1, \bar{\beta}_2\right)\right\|_{L^\infty(Q)}$.

Proof. We apply a Faedo-Galerkin approximation. For this, a specific basis is needed. We introduce the following spectral problem: find $w \in H^1(\Omega)$ and $\lambda \in \mathbb{R}$ such that

$$\begin{cases} \langle \nabla w, \nabla\phi\rangle_{H^1(\Omega), H^1(\Omega)^\star} = \lambda \langle w, \phi\rangle_{H,H}, & \forall \phi \in H^1(\Omega) \\ \nabla w \cdot \eta = 0, & \text{on } \partial\Omega \end{cases} \tag{3.19}$$

The problem (3.19) has a sequence of non-decreasing eigenvalues, $\{\lambda_k\}_{k=1}^{\infty}$, and a corresponding sequence of eigenfunctions, $\{e_k\}_{k=1}^{\infty}$, which form an orthogonal basis in $H^1(\Omega)$ and an orthonormal basis in H. Moreover, we also have $(\Delta e_k, \Delta e_m)_H = 0$ whenever $k \neq m$.

Our goal is to prove the local existence of at least one finite-dimensional Galerkin approximate solution of dimension n for the linearized system (3.13)-(3.11) in the form of sequences $\{\xi_s^n, \xi_e^n, \xi_1^n, \xi_2^n, \xi_r^n\}_{n>0}$ defined for $t \geq 0$ and $x \in \bar{\Omega}$ by

$$\xi_i^n(x,t) := \sum_{k=1}^{n} c_{i,k}^n(t) e_k(x), \quad \text{with} \quad c_{i,k}^n \in H^1(0,T) \quad \text{for } i = s, e, 1, 2, r. \tag{3.20}$$

where $\{e_k(x)\}_{k=1}^{n}$ is an orthonormal basis of eigenfunctions in $V_n = \text{span}\{e_1, \ldots, e_n\}$.

Evidently, the initial conditions associated with these unknown functions are given for $i = s, e, 1, 2, r$ by

$$\xi_i^n(x,0) = \xi_{i,0}^n(x) := \sum_{k=1}^{n} c_{i,k}^n(0) e_k(x); \quad c_{i,k}^n(0) = \langle \xi_{i,0}; e_k \rangle_{H,H}. \tag{3.21}$$

By definition and thanks to (3.11), it is easy to see that

$$(\xi_s^n, \xi_e^n, \xi_1^n, \xi_2^n, \xi_r^n)(0) = (0,0,0,0,0) \quad \text{for all } n \in \mathbb{N}^*, \tag{3.22}$$

Therefore, it is clear that

$$\xi_{i,0}^n \to \xi_{i,0} \in H \text{ as } n \to +\infty \text{ for } i = s, e, 1, 2, r. \tag{3.23}$$

Moreover, due to our choice of bases, we observe that ξ_i^n satisfies the boundary conditions (3.11) for $i = s, e, 1, 2, r$.

By inserting (3.20) into (3.13)-(3.17) and considering $v = e_m$ as a test function, we obtain the following approximate variational problem:

$$\text{Determine } \xi_i^n \in H^1(0,T;V),\ i = s, e, 1, 2, r \text{ such that, for all } m = 1, \cdots, n,$$

$$\begin{aligned} \langle \partial_t \xi_s^n, e_m \rangle_{V,V^\star} = &-\int_\Omega \left[(A_1 + \eta + \mu_s)\, \xi_s^n + B_1 \xi_e^n + C_1 \xi_1^n + D_1 \xi_2^n - \delta_r \xi_r^n \right] e_m \\ &- \int_\Omega k_s \nabla \xi_s^n \cdot \nabla e_m - \int_\Omega \left(K_1 \tilde{\beta}_1 + L_1 \tilde{\beta}_2 + M_1 \tilde{\beta}_e \right) e_m \end{aligned} \tag{3.24}$$

$$\begin{aligned} \langle \partial_t \xi_e^n, e_m \rangle_{V,V^\star} = &\int_\Omega \left[A_1 \xi_s^n + (B_1 - (\gamma_1 + \gamma_2 + \mu_e))\, \xi_e^n + C_1 \xi_1^n + D_1 \xi_2^n \right] e_m \\ &- \int_\Omega k_e \nabla \xi_e^n \cdot \nabla e_m + \int_\Omega \left(K_1 \tilde{\beta}_1 + L_1 \tilde{\beta}_2 + M_1 \tilde{\beta}_e \right) e_m \end{aligned} \tag{3.25}$$

$$\langle \partial_t \xi_2^n, e_m \rangle_{V,V^\star} = \int_\Omega \left[\gamma_2 \xi_e^n - (\rho + \delta_2 + \mu_2)\, \xi_2^n \right] e_m - \int_\Omega k_2 \nabla \xi_2^n \cdot \nabla e_m \tag{3.26}$$

$$\langle \partial_t \xi_1^n, e_m \rangle_{V,V^\star} = \int_\Omega \left[\gamma_1 \xi_e^n + \rho \xi_2^n - (\alpha + \delta_1 + \mu_1) \xi_1^n \right] e_m - \int_\Omega k_1 \nabla \xi_1^n \cdot \nabla e_m \tag{3.27}$$

$$\langle \partial_t \xi_r^n, e_m \rangle_{V,V^\star} = \int_\Omega \left[\delta_1 \xi_1^n + \delta_2 \xi_2^n - (\delta_r + \mu_r) \xi_r^n \right] e_m - \int_\Omega k_r \nabla \xi_r^n \cdot \nabla e_m. \tag{3.28}$$

Observing that $\langle \xi_i^n, e_m\rangle = c_{i,m}^n(t)$ and $\langle \partial_t \xi_i^n, e_m\rangle_{V,V^\star} = {c_{i,m}^n}'(t)$ for $i = s, e, 1, 2, r$, the previous problem reduces to an ODE problem for the unknown functions $c_{i,m}^n$ for $m = 1, \cdots, n$:

$$\begin{cases} {c_{s,m}^n}'(t) + (A_1 + \eta + \mu_s)\, c_{s,m}^n(t) + B_1 c_{e,m}^n(t) + C_1 c_{1,m}^n(t) + D_1 c_{2,m}^n(t) - \delta_r c_{r,m}^n(t) \\ \qquad\qquad\qquad\qquad\qquad\qquad + \sum\limits_{k=1}^{n} b(e_k, e_m) c_{s,k}^n(t) = -g^m \\ {c_{e,m}^n}'(t) - A_1 c_{s,m}^n(t) - (B_1 - (\gamma_1 + \gamma_2 + \mu_e))\, c_{e,m}^n(t) - C_1 c_{1,m}^n(t) - D_1 c_{2,m}^n(t) \\ \qquad\qquad\qquad\qquad\qquad\qquad + \sum\limits_{k=1}^{n} b(e_k, e_m) c_{e,k}^n(t) = g^m \\ {c_{2,m}^n}'(t) - \gamma_2 c_{e,m}^n(t) + (\rho + \delta_2 + \mu_2) c_{2,m}^n(t) + \sum\limits_{k=1}^{n} b(e_k, e_m) c_{2,k}^n(t) = 0 \\ {c_{1,m}^n}'(t) - \gamma_1 c_{e,m}^n(t) + (\alpha + \delta_1 + \mu_1) c_{1,m}^n(t) - \rho c_{2,m}^n(t) + \sum\limits_{k=1}^{n} b(e_k, e_m) c_{1,k}^n(t) = 0 \\ {c_{r,m}^n}'(t) - \delta_1 c_{1,m}^n(t) - \delta_2 c_{2,m}^n(t) + (\delta_r + \mu_r) c_{r,m}^n(t) + \sum\limits_{k=1}^{n} b(e_k, e_m) c_{r,k}^n(t) = 0 \\ (c_{s,m}^n, c_{e,m}^n, c_{1,m}^n, c_{2,m}^n, c_{r,m}^n)(0) = (0,0,0,0,0) \end{cases} \tag{3.29}$$

where $g^m = \displaystyle\int_\Omega \left(K_1\tilde{\beta}_1 + L_1\tilde{\beta}_2 + M_1\tilde{\beta}_e\right) e_m$ et $b(e_k, e_m) = \displaystyle\int_\Omega \nabla e_k \cdot \nabla e_m$. Due to (2.20), the linear system of ODEs (3.29) has bounded measurable coefficients, and by Carathéodory's theorem, the system (3.29) has a unique solution

$$\left(\left\{c_{s,m}^n(t)\right\}_{m=1}^n, \left\{c_{e,m}^n(t)\right\}_{m=1}^n, \left\{c_{1,m}^n(t)\right\}_{m=1}^n, \left\{c_{2,m}^n(t)\right\}_{m=1}^n, \left\{c_{r,m}^n(t)\right\}_{m=1}^n\right) \in (H^1(0,T))^{5n}.$$

Consequently, $\xi_i^n \in H^1(0,T;V)\quad i = s, e, 1, 2, r$ in (3.20) is well-defined and bounded due to the estimate (2.21). Therefore, we have the following convergence:

$$\xi_i^n \rightharpoonup \xi_i, \quad \text{weakly in } H^1(0,T;V,V^*) \text{ for } i = s, e, 1, 2, r \tag{3.30}$$

as $n \to +\infty$, and also

$$\|(\xi_s, \xi_e, \xi_1, \xi_2, \xi_r)\|_{\mathbb{Y}} \le C\,\|h\|_{\mathcal{U}_{ad}} \tag{3.31}$$

is satisfied. □

We are now in a position to establish the Fréchet differentiability of the control-to-state map. We have the following result:

Theorem 3.3. *Assume the hypotheses (2.1)-(2.5) and (2.6)-(2.7). The control-to-state map $\mathcal{S}$ is Fréchet differentiable at every point in $\mathcal{U}_{ad}$.*

Moreover, for any $\bar{\beta} := (\bar{\beta}_e, \bar{\beta}_1, \bar{\beta}_2) \in \mathcal{U}_{ad}$, the Fréchet derivative $D\mathcal{S}(\bar{\beta}) \in \mathcal{L}(\mathcal{U}_{ad}, \mathbb{Y})$ is defined as follows: for any $h := (h_e, h_1, h_2) \in \mathcal{U}_{ad}$, we have

$$D\mathcal{S}(\bar{\beta})h = \left(\xi_s^h, \xi_e^h, \xi_1^h, \xi_2^h, \xi_r^h\right),$$

where $(\xi_s^h, \xi_e^h, \xi_1^h, \xi_2^h, \xi_r^h)$ *is the unique solution of the linearized system (3.13)-(3.17) corresponding to* $\bar{\beta}$ *and the perturbation* h.

Proof. Fix an arbitrary $\bar{\beta} \in \mathcal{U}_{ad}$, such that $(\bar{S}, \bar{E}, \bar{I}_1, \bar{I}_2, \bar{R}) = \mathcal{S}(\bar{\beta})$, and consider $h \in \mathcal{U}_{ad}$. Since $\mathcal{U}_{ad}$ is open, there exists a sufficiently small radius $r > 0$ such that $\bar{\beta} + h \in \mathcal{U}_{ad}$ and $\|h\|_{\mathcal{U}_{ad}} \leq r$. Without loss of generality, we can assume $h \in \mathcal{U}_{ad}$ is sufficiently small so that $(S_h, E_h, I_{1_h}, I_{2_h}, R_h) := \mathcal{S}(\bar{\beta} + h)$, and let $(\xi_s^h, \xi_e^h, \xi_1^h, \xi_2^h, \xi_r^h)$ be the unique solution of the linearized system (3.13)-(3.17) associated with h.

First, note that the map $h := (h_e, h_1, h_2) \mapsto (\xi_s^h, \xi_e^h, \xi_1^h, \xi_2^h, \xi_r^h)$ defined in the statement indeed belongs to $\mathcal{L}(\mathcal{U}_{ad}, \mathbb{Y})$ thanks to Theorem 3.2. Thus, we only need to show that

$$\|\mathcal{S}(\bar{\beta} + h) - \mathcal{S}(\bar{\beta}) - \left(\xi_s^h, \xi_e^h, \xi_1^h, \xi_2^h, \xi_r^h\right)\|_{\mathbb{Y}} \leq \|h\|_{\mathcal{U}_{ad}}\, \varepsilon\left(\|h\|_{\mathcal{U}_{ad}}\right) \tag{3.32}$$

where $\bar{\beta} + h$ belongs to $\mathcal{U}_{ad}$, and $\varepsilon : [0, +\infty) \to \mathbb{R}$ is a function that tends to zero as its argument goes to zero.

Applying Theorem 2.1, we obtain the following estimation for the corresponding solutions:

$$\|\left(\bar{S}, \bar{E}, \bar{I}_1, \bar{I}_2, \bar{R}\right)\|_{H^1(0,T;V^*)\cap C^0([0,T];H)\cap L^2(0,T;V)\cap L^\infty(Q)} \leq K_1,$$
$$\|(S_h, E_h, I_{1_h}, I_{2_h}, R_h)\|_{H^1(0,T;V^*)\cap C^0([0,T];H)\cap L^2(0,T;V)\cap L^\infty(Q)} \leq K_1.$$

We also define

$$\left(\varphi_s^h, \varphi_e^h, \varphi_1^h, \varphi_2^h, \varphi_r^h\right) := \left(S_h - \bar{S} - \xi_s^h,\, E_h - \bar{E} - \xi_e^h,\, I_{1_h} - \bar{I}_1 - \xi_1^h,\, I_{2_h} - \bar{I}_2 - \xi_2^h,\, R_h - \bar{R} - \xi_r^h\right)$$

and note that $(\varphi_s^h, \varphi_e^h, \varphi_1^h, \varphi_2^h, \varphi_r^h)$ belongs to $\mathbb{Y}$. Now if we set

$$\begin{aligned}\Phi^h = {} & \bar{\beta}_1\left(S_h - \bar{S}\right)\left(I_{1_h} - \bar{I}_1\right) + \bar{\beta}_2\left(S_h - \bar{S}\right)\left(I_{2_h} - \bar{I}_2\right) + \bar{\beta}_e\left(S_h - \bar{S}\right)\left(E_h - \bar{E}\right) \\ & + h_1 I_{1_h}\left(S_h - \bar{S}\right) + h_1\bar{S}\left(I_{1_h} - \bar{I}_1\right) + h_2 I_{2_h}\left(S_h - \bar{S}\right) + h_2\bar{S}\left(I_{2_h} - \bar{I}_2\right) \\ & + h_e E_h\left(S_h - \bar{S}\right) + h_e\bar{S}\left(E_h - \bar{E}\right)\end{aligned} \tag{3.33}$$

and note that $(\varphi_s^h, \varphi_e^h, \varphi_1^h, \varphi_2^h, \varphi_r^h)$ belongs to $\mathbb{Y}$. Now, we can observe that $(\varphi_s^h, \varphi_e^h, \varphi_1^h, \varphi_2^h, \varphi_r^h)$ is a solution to the following problem:

$$\left\langle \partial_t\varphi_s^h, v\right\rangle + \int_\Omega \left[(A_1 + \eta + \mu_s)\varphi_s^h + B_1\varphi_e^h + C_1\varphi_1^h + D_1\varphi_2^h - \delta_r\varphi_r^h\right] v + \int_\Omega k_s\nabla\varphi_s^h\cdot\nabla v = -\int_\Omega \Phi^h v \tag{3.34}$$

$$\left\langle \partial_t\varphi_e^h, v\right\rangle - \int_\Omega \left[A_1\varphi_s^h + (B_1 - (\gamma_1 + \gamma_2 + \mu_e))\varphi_e^h + C_1\varphi_1^h + D_1\varphi_2^h\right] v + \int_\Omega k_e\nabla\varphi_e^h\cdot\nabla v = \int_\Omega \Phi^h v \tag{3.35}$$

$$\left\langle \partial_t\varphi_2^h, v\right\rangle - \int_\Omega \left[\gamma_2\varphi_e^h - (\rho + \delta_2 + \mu_2)\varphi_2^h\right] v + \int_\Omega k_2\nabla\varphi_2^h\cdot\nabla v = 0 \tag{3.36}$$

$$\left\langle \partial_t \varphi_1^h, v \right\rangle - \int_\Omega \left[\gamma_1 \varphi_e^h + \rho \varphi_2^h - (\alpha + \delta_1 + \mu_1)\varphi_1^h \right] v + \int_\Omega k_1 \nabla \varphi_1^h \cdot \nabla v = 0 \tag{3.37}$$

$$\left\langle \partial_t \varphi_r^h, v \right\rangle - \int_\Omega \left[\delta_1 \varphi_1^h + \delta_2 \varphi_2^h - (\delta_r + \mu_r)\varphi_r^h \right] v + \int_\Omega k_r \nabla \varphi_r^h \cdot \nabla v = 0 \tag{3.38}$$

almost everywhere in $(0, T)$ and for all $v \in V$, along with the initial condition,

$$\left(\varphi_s^h, \varphi_e^h, \varphi_1^h, \varphi_2^h, \varphi_r^h \right)(0) = (0, 0, 0, 0, 0)$$

Now, we test the above equations with $\varphi_s^h, \varphi_e^h, \varphi_1^h, \varphi_2^h$, and φ_r^h, respectively, then sum them and integrate over $(0, t)$. We obtain

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \left|\varphi_s^h(t)\right|^2 + \frac{1}{2}\int_\Omega \left|\varphi_e^h(t)\right|^2 + \frac{1}{2}\int_\Omega \left|\varphi_1^h(t)\right|^2 + + \frac{1}{2}\int_\Omega \left|\varphi_2^h(t)\right|^2 \frac{1}{2}\int_\Omega \left|\varphi_r^h(t)\right|^2 \\
&\quad + \int_{Q_t} k_s \left|\nabla \varphi_s^h\right|^2 + \int_{Q_t} k_e \left|\nabla \varphi_e^h\right|^2 + \int_{Q_t} k_1 \left|\nabla \varphi_1^h\right|^2 + \int_{Q_t} k_2 \left|\nabla \varphi_2^h\right|^2 + \int_{Q_t} k_r \left|\nabla \varphi_r^h\right|^2 \\
&\quad = \int_{Q_t} A_1 \varphi_s^h \left(\varphi_e^h - \varphi_s^h\right) + \int_{Q_t} \left(B_1 \varphi_e^h + C_1 \varphi_1^h + D_1 \varphi_2^h\right)\left(\varphi_e^h - \varphi_s^h\right) + \int_{Q_t} \delta_r \varphi_r^h \left(\varphi_s^h - \varphi_r^h\right) \\
&\qquad - (\eta + \mu_s)\int_{Q_t} |\varphi_s^h|^2 - (\gamma_1 + \gamma_2 + \mu_e)\int_{Q_t} |\varphi_e^h|^2 - (\alpha + \delta_1 + \mu_1)\int_{Q_t} |\varphi_1^h|^2 \\
&\qquad - (\rho + \delta_2 + \mu_2)\int_{Q_t} |\varphi_2^h|^2 - \mu_r \int_{Q_t} |\varphi_r^h|^2 + \int_{Q_t} \left(\varphi_e^h + \varphi_r^h\right)\left(\gamma_1 \varphi_1^h + \gamma_2 \varphi_2^h\right) + \rho \int_{Q_t} \varphi_1^h \varphi_2^h \\
&\qquad + \int_{Q_t} \Phi^h \left(\varphi_e^h - \varphi_s^h\right) \\
&\quad \le \int_{Q_t} A_1 \varphi_s^h \varphi_e^h + \int_{Q_t} \left(B_1 \varphi_e^h + C_1 \varphi_1^h + D_1 \varphi_2^h\right)\left(\varphi_e^h - \varphi_s^h\right) + \int_{Q_t} \Phi^h \left(\varphi_e^h - \varphi_s^h\right) \\
&\qquad + \int_{Q_t} \left(\varphi_e^h + \varphi_r^h\right)\left(\gamma_1 \varphi_1^h + \gamma_2 \varphi_2^h\right) + \delta_r \int_{Q_t} \varphi_r^h \varphi_s^h + \rho \int_{Q_t} \varphi_1^h \varphi_2^h.
\end{aligned} \tag{3.39}$$

Thus, by applying Young's inequality, we obtain that

$$\sum_{i \in \{s,e,1,2,r\}} \|\varphi_i^h(t)\|_H^2 + k_* \sum_{i \in \{s,e,1,2,r\}} \|\nabla \varphi_i^h\|_{L^2(Q_t)}^2 \le C \sum_{i \in \{s,e,1,2,r\}} \|\varphi_i^h\|_{L^2(Q_t)}^2 + \int_{Q_t} \Phi^h \left(\varphi_e^h - \varphi_s^h\right) \tag{3.40}$$

Now, we need to estimate the last integral on the right-hand side of (3.40). To do this, we will only focus on estimating two terms from the development of this integral, as the others are analogous.

By first applying Hölder's, Young's, Sobolev's, and compactness inequalities (see (2.9) and (2.10)), and then using the continuous dependence estimate (2.21),

$$\begin{aligned}
&- \int_{Q_t} \bar{\beta}_1 \left(S_h - \bar{S}\right)\left(I_{1_h} - \bar{I}\right)\varphi_e^h \\
&\le \int_0^t \|\bar{\beta}_1\|_{L^\infty(\Omega)} \left\|\left(S_h - \bar{S}\right)(\tau)\right\|_H \left\|\left(I_{1_h} - \bar{I}\right)(\tau)\right\|_{L^4(\Omega)} \left\|\varphi_e^h(\tau)\right\|_{L^4(\Omega)} d\tau \\
&\le \int_0^t \left\|\varphi_e^h(\tau)\right\|_{L^4(\Omega)}^2 d\tau + c_1 \int_0^t \left\|\left(S_h - \bar{S}\right)(\tau)\right\|_H^2 \left\|\left(I_{1_h} - \bar{I}\right)(\tau)\right\|_V^2
\end{aligned}$$

$$\leq \frac{k_*}{2}\|\nabla\varphi_e^h\|^2_{L^2(Q_t)} + C_{\Omega,k_*}\|\varphi_e^h\|^2_{L^2(Q_t)} + c_1\left\|S_h - \bar{S}\right\|^2_{L^\infty(0,T;H)}\left\|I_{1_h} - \bar{I}\right\|^2_{L^2(0,T;V)}$$
$$\leq \frac{k_*}{2}\|\nabla\varphi_e^h\|^2_{L^2(Q_t)} + C_{\Omega,k_*}\|\varphi_e^h\|^2_{L^2(Q_t)} + c_1\|h\|^4_{\mathcal{U}_{ad}}$$

The second term we consider is estimated as follows:

$$-\int_{Q_t} h_e\bar{S}\left(E_h - \bar{E}\right)\varphi_e^h \leq \int_{Q_t}\left|\varphi_e^h\right|^2 + c\left\|h_e\right\|^2_{L^\infty(Q)}\left\|E_h - \bar{E}\right\|^2_{L^2(0,T;H)} \leq \|\varphi_e^h\|^2_{L^2(Q_t)} + C\|h\|^4_{\mathcal{U}_{ad}}$$

By treating the other terms in a similar manner and applying Gronwall's lemma, we conclude that

$$\left\|\left(\varphi_s^h, \varphi_e^h, \varphi_1^h, \varphi_2^h, \varphi_r^h\right)\right\|_{\mathbb{Y}} \leq c\|h\|^2_{\mathcal{U}_{ad}}$$

Since this inequality implies (3.32), the proof is complete. □

3.3. **First-order optimality conditions.** We set $\beta = (\beta_e, \beta_1, \beta_2)$ such that $(S, E, I_1, I_2, R) = \mathcal{S}(\beta)$. We now introduce the functionals $\mathcal{J}_1 : \left(C^0([0,T]; H)\right)^5 \to \mathbb{R}$ and $\mathcal{J}_2 : (L^\infty(Q))^3 \to \mathbb{R}$ such that

$$\mathcal{J}\left(\mathcal{S}\left(\beta\right);\beta\right) = \mathcal{J}_1\left(\mathcal{S}\left(\beta\right)\right) + \mathcal{J}_2\left(\beta\right) \quad \text{for all} \quad \beta \in \mathcal{U}_{ad},$$

with

$$\mathcal{J}_1(\mathcal{S}\left(\beta\right)) := \frac{\theta_e}{2}\int_\Omega |E - e|^2 + \frac{\theta_1}{2}\int_\Omega |I_1 - i_1|^2 + \frac{\theta_2}{2}\int_\Omega |I_2 - i_2|^2$$
$$\mathcal{J}_2\left(\beta\right) := \frac{1}{2}\int_Q\left(\varpi_1|\beta_1|^2 + \varpi_2|\beta_2|^2 + \varpi_e|\beta_e|^2\right),$$

Thanks to the Fréchet differentiability result above (Theorem 3.3), we can compute the derivative of $\mathcal{J}$ at points of $\mathcal{U}_{ad}$ by applying the chain rule to the composed mapping

$$\mathcal{U}_{ad} \ni \beta \mapsto (\mathcal{S}(\beta);\beta) \mapsto \mathcal{J}\left(\mathcal{S}(\beta);\beta\right)$$

Noting also that $\mathcal{U}_{ad}$ is a closed and convex subset of $L^2(Q)$, we immediately see that a necessary condition for $(\beta_e^*, \beta_1^*, \beta_2^*)$ to be an optimal control is given by the following result.

Corollary 3.1. *Under the assumptions (2.1)-(2.5) and (2.6)-(2.7), suppose that $(\beta_e^*, \beta_1^*, \beta_2^*)$, an optimal control and $(S^*, E^*, I_1^*, I_2^*, R^*) := \mathcal{S}\left(\beta_e^*, \beta_1^*, \beta_2^*\right)$, the corresponding optimal state. Then the variational inequality*

$$\theta_e\int_\Omega\left(E^*(T) - e\right)\xi_e(T) + \theta_1\int_\Omega\left(I_1^*(T) - i_1\right)\xi_1(T) + \theta_2\int_\Omega\left(I_2^*(T) - i_2\right)\xi_2(T)$$
$$+ \int_Q\left(\varpi_e\beta_e^*h_e + \varpi_1\beta_1^*h_1 + \varpi_2\beta_2^*h_2\right) \geq 0 \tag{3.41}$$

is satisfied for all $(\beta_e, \beta_1, \beta_2) \in \mathcal{U}_{ad}$, where ξ_e, ξ_1, and ξ_2 are the components of the solution $(\xi_s, \xi_e, \xi_1, \xi_2, \xi_r)$ of the linearized system (3.13) - (3.17) corresponding to the control triplet $(\beta_e^*, \beta_1^*, \beta_2^*)$ and to the variation (h_e, h_1, h_2) given by $(h_e, h_1, h_2) = (\beta_e - \beta_e^*, \beta_1 - \beta_1^*, \beta_2 - \beta_2^*)$.

Proof. Let $(\beta_e^*, \beta_1^*, \beta_2^*)$ be an optimal control corresponding to the optimal state $(S^*, E^*, I_1^*, I_2^*, R^*)$. Consider an arbitrary admissible control $(\beta_e, \beta_1, \beta_2)$ and its corresponding state (S, E, I_1, I_2, R). For any $\lambda \in (0,1)$, we introduce the incremented control

$$\left(\beta_e^\lambda, \beta_1^\lambda, \beta_2^\lambda\right) := (\beta_e^*, \beta_1^*, \beta_2^*) + \lambda\left[(\beta_e, \beta_1, \beta_2) - (\beta_e^*, \beta_1^*, \beta_2^*)\right] \tag{3.42}$$

which belongs to $\mathcal{U}_{\text{ad}}$ since $\mathcal{U}_{\text{ad}}$ is convex. We also define

$$\left(S^\lambda, E^\lambda, I_1^\lambda, I_2^\lambda, R^\lambda\right) \text{ as the state corresponding to } \left(\beta_e^\lambda, \beta_1^\lambda, \beta_2^\lambda\right) \tag{3.43}$$

and define the quotients

$$\xi_s^\lambda := \frac{S^\lambda - S^*}{\lambda}, \quad \xi_e^\lambda := \frac{E^\lambda - E^*}{\lambda}, \quad \xi_1^\lambda := \frac{I_1^\lambda - I_1^*}{\lambda}, \quad \xi_2^\lambda := \frac{I_2^\lambda - I_2^*}{\lambda}, \quad \text{and} \quad \xi_r^\lambda := \frac{R^\lambda - R^*}{\lambda}. \tag{3.44}$$

We prove that $\left(\xi_s^\lambda, \xi_e^\lambda, \xi_1^\lambda, \xi_2^\lambda, \xi_r^\lambda\right)$ converges to the solution $(\xi_s, \xi_e, \xi_1, \xi_2, \xi_r)$ of the linearized system (3.13)-(3.14) in an appropriate topology as λ tends to zero. First, we observe the following regularity:

$$\left(\xi_s^\lambda, \xi_e^\lambda, \xi_1^\lambda, \xi_2^\lambda, \xi_r^\lambda\right) \in \left(H^1\left(0,T;V^*\right) \cap L^2(0,T;V)\right)^5 \tag{3.45}$$

Next, we write the system that these quotients solve. This is obtained as follows: first, we write the equations (2.14)-(2.18) for the incremented control (3.42) and the corresponding state in (3.43); then we do the same for the optimal control $(\beta_e^*, \beta_1^*, \beta_2^*)$ with the corresponding state $(S^*, E^*, I_1^*, I_2^*, R^*)$; finally, we take the differences and divide by λ. Noting that

$$\frac{\beta_i^\lambda - \beta_i^*}{\lambda} = \beta_i - \beta_i^* := h_i \quad \text{for } i = e, 1, 2, \tag{3.46}$$

we obtain

$$\begin{aligned}\left\langle \partial_t \xi_s^\lambda, v\right\rangle + \int_\Omega \left[(A_1^* + \eta + \mu_s)\,\xi_s^\lambda + B_1^*\xi_e^\lambda + C_1^*\xi_1^\lambda + D_1^*\xi_2^\lambda - \delta_r\xi_r^\lambda\right] v + \int_\Omega k_s \nabla\xi_s^\lambda \cdot \nabla v \\ = -\int_\Omega \left(K_1^* h_1 + L_1^* h_2 + M_1^* h_e\right) v\end{aligned} \tag{3.47}$$

$$\begin{aligned}\left\langle \partial_t \xi_e^\lambda, v\right\rangle - \int_\Omega \left[A_1^*\xi_s^\lambda + \left(B_1^* - (\gamma_1 + \gamma_2 + \mu_e)\right)\xi_e^\lambda + C_1^*\xi_1^\lambda + D_1^*\xi_2^\lambda\right] v + \int_\Omega k_e \nabla\xi_e^\lambda \cdot \nabla v \\ = \int_\Omega \left(K_1^* h_1 + L_1^* h_2 + M_1^* h_e\right) v\end{aligned} \tag{3.48}$$

$$\left\langle \partial_t \xi_2^\lambda, v\right\rangle - \int_\Omega \left[\gamma_2\xi_e^\lambda - (\rho + \delta_2 + \mu_2)\xi_2^\lambda\right] v + \int_\Omega k_2 \nabla\xi_2^\lambda \cdot \nabla v = 0 \tag{3.49}$$

$$\left\langle \partial_t \xi_1^\lambda, v\right\rangle - \int_\Omega \left[\gamma_1\xi_e^\lambda + \rho\xi_2^\lambda - (\alpha + \delta_1 + \mu_1)\xi_1^\lambda\right] v + \int_\Omega k_1 \nabla\xi_1^\lambda \cdot \nabla v = 0 \tag{3.50}$$

$$\left\langle \partial_t \xi_r^\lambda, v\right\rangle - \int_\Omega \left[\delta_1\xi_1^\lambda + \delta_2\xi_2^\lambda - (\delta_r + \mu_r)\xi_r^\lambda\right] v + \int_\Omega k_r \nabla\xi_r^\lambda \cdot \nabla v = 0 \tag{3.51}$$

a.e. in $(0,T)$ and for all $v \in V$ where A_1^*, B_1^*, C_1^*, D_1^*, K_1^*, L_1^*, and M_1^* are the constants defined in (3.12) related to the optimal control $(\beta_e^*, \beta_1^*, \beta_2^*)$ with state $(S^*, E^*, I_1^*, I_2^*, R^*)$. Furthermore, the initial condition

$$\left(\xi_s^\lambda, \xi_e^\lambda, \xi_1^\lambda, \xi_2^\lambda, \xi_r^\lambda\right) = (0,0,0,0,0) \tag{3.52}$$

is also satisfied. To pass to the limit as λ tends to zero, an estimation is required. To achieve this, first note that the stability estimate (2.20) holds for $(S^*, E^*, I_1^*, I_2^*, R^*)$ and $\left(S^\lambda, E^\lambda, I_1^\lambda, I_2^\lambda, R^\lambda\right)$. By applying the continuous dependence estimate (2.21) and recalling (3.42), we have

$$\begin{aligned}\left\|\left(S^\lambda, E^\lambda, I_1^\lambda, I_2^\lambda, R^\lambda\right) - (S^*, E^*, I_1^*, I_2^*, R^*)\right\|_{H^1(0,T;V^*)\cap L^2(0,T;V)} \\ \leq c\left\|\left(\beta_e^\lambda, \beta_1^\lambda, \beta_2^\lambda\right) - (\beta_e^*, \beta_1^*, \beta_2^*)\right\|_{L^\infty(Q)} \\ = c\lambda \left\|(\beta_e, \beta_1, \beta_2) - (\beta_e^*, \beta_1^*, \beta_2^*)\right\|_{L^\infty(Q)} \leq c\lambda\end{aligned} \tag{3.53}$$

from which

$$\left\|\left(\xi_s^\lambda, \xi_e^\lambda, \xi_1^\lambda, \xi_2^\lambda, \xi_r^\lambda\right)\right\|_{H^1(0,T;V^*)\cap L^2(0,T;V)} \leq c$$

Therefore, thanks to standard compactness results, we obtain by taking the limit that

$$\left(\xi_s^\lambda, \xi_e^\lambda, \xi_1^\lambda, \xi_2^\lambda, \xi_r^\lambda\right) \rightharpoonup (\xi_s, \xi_e, \xi_1, \xi_2, \xi_r) \quad \text{weakly in } \left(H^1(0,T;V^*) \cap L^2(0,T;V)\right)^5$$

for some quintuplet $(\xi_s, \xi_e, \xi_1, \xi_2, \xi_r)$ satisfying (3.45).

However, thanks to the uniqueness of the solution of the linearized problem (3.13)-(3.17) (Theorem 3.2), up to a subsequence, we prove that the limit quintuplet $(\xi_s, \xi_e, \xi_1, \xi_2, \xi_r)$ solves the linearized problem (3.13)-(3.17).

Thanks to (3.53) and the Aubin-Lions lemma (see, for example, [8], Thm. 5.1, p. 58), we have the following convergences:

$$\left(S^\lambda, E^\lambda, I_1^\lambda, I_2^\lambda, R^\lambda\right) \to (S^*, E^*, I_1^*, I_2^*, R^*) \text{ and } \left(\xi_s^\lambda, \xi_e^\lambda, \xi_1^\lambda, \xi_r^\lambda\right) \to (\xi_s, \xi_e, \xi_1, \xi_2, \xi_r) \tag{3.54}$$
$$\text{strongly in } \left(L^2(0,T;H)\right)^5 \text{ and a.e. in } Q.$$

At this stage, we are ready to prove (3.41). Due to optimality, we have that

$$\frac{\mathcal{J}\left(\mathcal{S}(\beta^\lambda); \beta^\lambda\right) - \mathcal{J}\left(\mathcal{S}(\beta^*); \beta^*\right)}{\lambda} \geq 0 \quad \text{for all } \lambda \in (0,1) \tag{3.55}$$

where $\beta^\lambda = \left(\beta_e^\lambda, \beta_1^\lambda, \beta_2^\lambda\right)$ and $\left(S^\lambda, E^\lambda, I_1^\lambda, I_2^\lambda, R^\lambda\right) = \mathcal{S}(\beta^\lambda)$. The quantities β^* and $\mathcal{S}(\beta^*)$ are defined similarly.

We aim to let λ tend to zero in the inequality (3.55). We will consider only two of the terms involved in (3.55), namely

$$\frac{\theta_e}{2}\int_\Omega \frac{|E^\lambda - e|^2 - |E^* - e|^2}{\lambda} \quad \text{and} \quad \frac{\varpi_e}{2}\int_Q \frac{|\beta_e^\lambda|^2 - |\beta_e^*|^2}{\lambda} \tag{3.56}$$

since the other terms are analogous. We have that

$$\begin{aligned}\frac{\theta_e}{2}\int_\Omega \frac{|E^\lambda(T)-e|^2-|E^*(T)-e|^2}{\lambda} &= \theta_e \int_\Omega \frac{(E^\lambda(T)-E^*(T))}{\lambda}\frac{(E^\lambda(T)+E^*(T)-2e)}{2} \\ &= \theta_e \int_\Omega \xi_e^\lambda \frac{E^\lambda(T)+E^*(T)-2e}{2} \\ &\longrightarrow \theta_e \int_\Omega (E^*(T)-e)\,\xi_e(T) \quad \text{as } \lambda \to 0. \end{aligned} \tag{3.57}$$

The convergence (3.57) is due to the convergence results (3.54). Similarly

$$\begin{aligned}\frac{\varpi_e}{2}\int_Q \frac{\left|\beta_e^\lambda\right|^2-|\beta_e^*|^2}{\lambda} &= \varpi_e \int_Q \frac{\beta_e^\lambda-\beta_e^*}{\lambda}\frac{\beta_e^\lambda+\beta_e^*}{2} \\ &= \varpi_e \int_Q (\beta_e-\beta_e^*)\,\frac{\beta_e^\lambda+\beta_e^*}{2} \\ &\longrightarrow \int_Q \varpi_e \beta_e^* h_e \quad \text{as} \quad \lambda \to 0 \quad \text{where} \quad h_e=(\beta_e-\beta_e^*) \end{aligned} \tag{3.58}$$

For the convergence (3.58), thanks to (3.42), we can note that $\beta_e^\lambda \to \beta_e^*$ as $\lambda \to 0$. Therefore, (3.41) follows immediately. □

The result just proved is not satisfactory. Indeed, the linearized problem (3.13)-(3.17) is involved infinitely many times since $(\beta_e, \beta_1, \beta_2)$ is arbitrary in $\mathcal{U}_{\text{ad}}$. This issue is circumvented by introducing a suitable adjoint problem, which we immediately present in its variational form.

Given an optimal control $(\beta_e^*, \beta_1^*, \beta_2^*)$ and the corresponding state $(S^*, E^*, I_1^*, I_2^*, R^*)$, the associated adjoint problem consists of finding a quintuplet (p, q, w, y, z) that satisfies the regularity requirement

$$p, q, w, y, z \in H^1\left(0, T; V^*\right) \cap L^2(0, T; V) \hookrightarrow C^0([0, T]; H) \tag{3.59}$$

and the variational equations,

$$-\int_\Omega \partial_t p v + \int_\Omega k_s \nabla p \cdot \nabla v + \int_\Omega A_1^*(p-q)v + \int_\Omega (\eta+\mu_s)pv = 0 \tag{3.60}$$

$$-\int_\Omega \partial_t q v + \int_\Omega k_e \nabla q \cdot \nabla v + \int_\Omega B_1^*(p-q)v + \int_\Omega \left[\gamma_1(q-y)+\gamma_2(q-w)+\mu_e q\right] v = 0 \tag{3.61}$$

$$-\int_\Omega \partial_t w v + \int_\Omega k_2 \nabla w \cdot \nabla v + \int_\Omega D_1^*(p-q)v + \int_\Omega \left[\rho(w-y)+\delta_2(w-z)+\mu_2 w\right] v = 0 \tag{3.62}$$

$$-\int_\Omega \partial_t y v + \int_\Omega k_1 \nabla y \cdot \nabla v + \int_\Omega C_1^*(p-q)v + \int_\Omega \left[\delta_1(y-z)+(\alpha+\mu_1)y\right] v = 0 \tag{3.63}$$

$$-\int_\Omega \partial_t z v + \int_\Omega k_r \nabla z \cdot \nabla v + \int_\Omega \left[\delta_r(z-p)+\mu_r z\right] v = 0 \tag{3.64}$$

$$\partial_\nu p = \partial_\nu q = \partial_\nu w = \partial_\nu y = \partial_\nu z = 0 \tag{3.65}$$

$$p(T) = z(T) = 0, \quad q(T) = \theta_e(E^*(T)-e),\ y(T) = \theta_1(I_1^*(T)-i_1),\ w(T) = \theta_2(I_2^*(T)-i_2) \tag{3.66}$$

where A_1^*, B_1^*, C_1^*, and D_1^* are the constants defined in (3.12) with the optimal control $(\beta_e^*, \beta_1^*, \beta_2^*)$ and the corresponding state $(S^*, E^*, I_1^*, I_2^*, R^*)$. We have the following existence and uniqueness result.

Theorem 3.4. *Assume that the hypotheses (2.1)-(2.5) and (2.6)-(2.7) are satisfied. Then the adjoint system (3.60) - (3.66) has a unique solution.*

Proof. Since all coefficients are bounded, this is a standard parabolic problem. Therefore, by a Galerkin scheme and the Gronwall lemma (as in the proof of Theorem 3.2), this problem admits a unique solution (p, q, w, y, z) with the regularity given by (3.59). □

At this stage, we are ready to prove a necessary optimality condition. Our statement concerns the following closed convex sets:

$$\mathcal{U}_e := \left\{ v \in L^2(Q) : \quad \beta_e^{\min} \le v \le \beta_e^{\max} \text{ a.e. in } Q \right\} \tag{3.67}$$

$$\mathcal{U}_1 := \left\{ v \in L^2(Q) : \quad \beta_1^{\min} \le v \le \beta_1^{\max} \text{ a.e. in } Q \right\} \tag{3.68}$$

$$\mathcal{U}_2 := \left\{ v \in L^2(Q) : \quad \beta_2^{\min} \le v \le \beta_2^{\max} \text{ a.e. in } Q \right\} \tag{3.69}$$

Theorem 3.5. *Assume the hypotheses (2.1)-(2.5) and (2.6)-(2.7). Let $(\beta_e^*, \beta_1^*, \beta_2^*)$ be an optimal control and $(S^*, E^*, I_1^*, I_2^*, R^*)$ the corresponding state, respectively, and let (p, q, w, y, z) be the solution of the associated adjoint problem. Then, the variational inequality*

$$\int_Q \left[\left(S^* E^* (q - p) + \varpi_e \beta_e^* \right) (\beta_e - \beta_e^*) \right.$$
$$\left. + \left(S^* I_1^* (q - p) + \varpi_1 \beta_1^* \right) (\beta_1 - \beta_1^*) + \left(S^* I_2^* (q - p) + \varpi_2 \beta_2^* \right) (\beta_2 - \beta_2^*) \right] \ge 0. \tag{3.70}$$

$$\int_Q \left(s^* i^* (q - p) + u_i^* \right) (u_i - u_i^*) + \int_Q \left(s^* e^* (q - p) + u_e^* \right) (u_e - u_e^*) \ge 0$$

is satisfied for all $(\beta_e, \beta_1, \beta_2) \in \mathcal{U}_{ad}$, and β_e^*, β_1^*, and β_2^* are the L^2 projections of $S^* E^* (p - q)$, $S^* I_1^* (p - q)$, and $S^* I_2^* (p - q)$ onto $\mathcal{U}_e$, $\mathcal{U}_1$, and $\mathcal{U}_2$, respectively.

Proof. We consider the linearized problem (3.13)-(3.17) where $(\bar{\beta}_e, \bar{\beta}_1, \bar{\beta}_2)$ and $(\bar{S}, \bar{E}, \bar{I}_1, \bar{I}_2, \bar{R})$ are interpreted as $(\beta_e^*, \beta_1^*, \beta_2^*)$ and $(S^*, E^*, I_1^*, I_2^*, R^*)$, respectively, and we set the variation $(h_e, h_1, h_2) := (\beta_e - \beta_e^*, \beta_1 - \beta_1^*, \beta_2 - \beta_2^*)$ for an arbitrary control $(\beta_e, \beta_1, \beta_2) \in \mathcal{U}_{\text{ad}}$.

$$\langle \partial_t \xi_s, v \rangle + \int_\Omega \left[\left(A_1^* + \eta + \mu_s \right) \xi_s + B_1^* \xi_e + C_1^* \xi_1 + D_1^* \xi_2 - \delta_r \xi_r \right] v + \int_\Omega k_s \nabla \xi_s \cdot \nabla v$$
$$= - \int_\Omega \left(K_1^* h_1 + L_1^* h_2 + M_1^* h_e \right) v \tag{3.71}$$

$$\langle \partial_t \xi_e, v \rangle - \int_\Omega \left[A_1^* \xi_s + \left(B_1^* - (\gamma_1 + \gamma_2 + \mu_e) \right) \xi_e + C_1^* \xi_1 + D_1^* \xi_2 \right] v + \int_\Omega k_e \nabla \xi_e \cdot \nabla v$$
$$= \int_\Omega \left(K_1^* h_1 + L_1^* h_2 + M_1^* h_e \right) v \tag{3.72}$$

$$\langle \partial_t \xi_2, v \rangle - \int_\Omega \left[\gamma_2 \xi_e - (\rho + \delta_2 + \mu_2) \xi_2 \right] v + \int_\Omega k_2 \nabla \xi_2 \cdot \nabla v = 0 \tag{3.73}$$

$$\langle \partial_t \xi_1, v \rangle - \int_\Omega \left[\gamma_1 \xi_e + \rho \xi_2 - (\alpha + \delta_1 + \mu_1) \xi_1 \right] v + \int_\Omega k_1 \nabla \xi_1 \cdot \nabla v = 0 \tag{3.74}$$

$$\langle \partial_t \xi_r, v\rangle - \int_\Omega [\delta_1\xi_1 + \delta_2\xi_2 - (\delta_r + \mu_r)\xi_r]\, v + \int_\Omega k_r \nabla\xi_r \cdot \nabla v = 0 \tag{3.75}$$

$$(\xi_s, \xi_e, \xi_1, \xi_2, \xi_r) = (0,0,0,0,0) \tag{3.76}$$

a.e. in $(0,T)$ and for all $v \in V$, with A_1^*, B_1^*, C_1^*, D_1^*, K_1^*, L_1^*, and M_1^* being the constants defined in (3.12) for the optimal control $(\beta_e^*, \beta_1^*, \beta_2^*)$ and the corresponding state $(S^*, E^*, I_1^*, I_2^*, R^*)$.

On the one hand, we fix an arbitrary element $(\beta_e, \beta_1, \beta_2) \in \mathcal{U}_{\text{ad}}$ and choose $(h_e, h_1, h_2) := (\beta_e - \beta_e^*, \beta_1 - \beta_1^*, \beta_2 - \beta_2^*)$ in the system above. We then test the equations (3.71)-(3.75) with p, q, w, y, and z, respectively, and sum them, then integrate over $(0,T)$. Thus, after rearranging the terms, we obtain

$$\begin{aligned}
&\int_0^T (\langle \partial_t\xi_s, p\rangle + \langle \partial_t\xi_e, q\rangle + \langle \partial_t\xi_2, w\rangle + \langle \partial_t\xi_1, y\rangle + \langle \partial_t\xi_r, z\rangle)\, dt \\
&\quad + \int_Q (k_s\nabla\xi_s\cdot\nabla p + k_e\nabla\xi_e\cdot\nabla q + k_2\nabla\xi_2\cdot\nabla w + +k_1\nabla\xi_1\cdot\nabla y + k_r\nabla\xi_r\cdot\nabla z) \\
&\quad + \int_Q (A_1^*\xi_s + B_1^*\xi_e + C_1^*\xi_1 + D_1^*\xi_2)(p-q) + \int_Q [(\eta+\mu_s)\xi_s - \delta_r\xi_r]\, p \\
&\quad + \int_Q (\gamma_1+\gamma_2+\mu_e)\xi_e q - \int_Q [\gamma_2\xi_e - (\rho+\delta_2+\mu_2)\xi_2]\, w - \int_Q [\gamma_1\xi_e\rho\xi_2 - (\alpha+\delta_1+\mu_1)\xi_1]\, y \\
&\quad - \int_Q [\delta_1\xi_1 + \delta_2\xi_2 - (\delta_r+\mu_r)\xi_r]\, z \\
&= \int_Q (K_1^*h_1 + L_1^*h_2 + M_1^*h_e)(q-p)
\end{aligned} \tag{3.77}$$

On the other hand, we consider the adjoint problem (3.60)-(3.66) and test the equations respectively with $-\xi_s$, $-\xi_e$, $-\xi_1$, $-\xi_2$, and $-\xi_r$. We then add them together and integrate the resulting equality over $(0,T)$. After rearranging the terms, we obtain

$$\begin{aligned}
&\int_0^T (\langle \partial_t p, \xi_s\rangle + \langle \partial_t q, \xi_e\rangle + \langle \partial_t w, \xi_2\rangle + \langle \partial_t y, \xi_1\rangle + \langle \partial_t z, \xi_r\rangle)\, dt \\
&\quad - \int_Q (k_s\nabla\xi_s\cdot\nabla p + k_e\nabla\xi_e\cdot\nabla q + k_2\nabla\xi_2\cdot\nabla w + +k_1\nabla\xi_1\cdot\nabla y + k_r\nabla\xi_r\cdot\nabla z) \\
&\quad - \int_Q (A_1^*\xi_s + B_1^*\xi_e + C_1^*\xi_1 + D_1^*\xi_2)(p-q) - \int_Q [(\eta+\mu_s)\xi_s - \delta_r\xi_r]\, p \\
&\quad - \int_Q (\gamma_1+\gamma_2+\mu_e)\xi_e q + \int_Q [\gamma_2\xi_e - (\rho+\delta_2+\mu_2)\xi_2]\, w + \int_Q [\gamma_1\xi_e\rho\xi_2 - (\alpha+\delta_1+\mu_1)\xi_1]\, y \\
&\quad + \int_Q [\delta_1\xi_1 + \delta_2\xi_2 - (\delta_r+\mu_r)\xi_r]\, z \\
&= 0
\end{aligned} \tag{3.78}$$

At this stage, we sum (3.77) and (3.78) and obtain that

$$\int_0^T [\langle \partial_t p, \xi_s\rangle + \langle \partial_t p, \xi_s\rangle + \langle \partial_t q \xi_e\rangle + \langle \partial_t q, \xi_e\rangle + \langle \partial_t w, \xi_2\rangle + \langle \partial_t w, \xi_2\rangle$$

$$+\langle \partial_t y, \xi_1\rangle + \langle \partial_t y, \xi_1\rangle + \langle \partial_t z, \xi_r\rangle + \langle \partial_t z, \xi_r\rangle]\, dt$$
$$= \int_Q \left(K_1^* h_1 + L_1^* h_2 + M_1^* h_e\right)(q-p) \tag{3.79}$$

Note that $\langle \partial_t p, \xi_s\rangle + \langle \partial_t q, \xi_e\rangle = \frac{d}{dt}\langle \xi_s, p\rangle$. Thus, by integrating by parts for functions in the space $H^1(0,T;V^*) \cap L^2(0,T;V)$ and taking into account the initial conditions (3.10) for $(\xi_s, \xi_e, \xi_1, \xi_2, \xi_r)$ and the final conditions (3.66) for (p, q, w, y, z), we deduce that

$$\theta_e \int_\Omega \left(E^*(T) - e\right)\xi_e(T) + \theta_1 \int_\Omega \left(I_1^*(T) - i_1\right)\xi_1(T) = \int_Q \left(K_1^* h_1 + L_1^* h_2 + M_1^* h_e\right)(q-p) \tag{3.80}$$

By adding to each side of the last equality the quantity

$$\theta_2 \int_\Omega \left(I_2^*(T) - i_2\right)\xi_2(T) + \int_Q \left(\varpi_e \beta_e^* h_e + \varpi_1 \beta_1^* h_1 + \varpi_2 \beta_2^* h_2\right),$$

we obtain, thanks to Corollary 3.41, the inequality

$$\int_Q \left[\left(M_1^*(q-p) + \varpi_e \beta_e^*\right) h_e + \left(K_1^*(q-p) + \varpi_1 \beta_1^*\right) h_1 + \left(L_1^*(q-p) + \varpi_2 \beta_2^*\right) h_2\right] \geq 0. \tag{3.81}$$

which can be rewritten as

$$\begin{aligned}\int_Q &\left[\left(S^* E^*(q-p) + \varpi_e \beta_e^*\right)(\beta_e - \beta_e^*)\right. \\ &\left. + \left(S^* I_1^*(q-p) + \varpi_1 \beta_1^*\right)(\beta_1 - \beta_1^*) + \left(S^* I_2^*(q-p) + \varpi_2 \beta_2^*\right)(\beta_2 - \beta_2^*)\right] \geq 0.\end{aligned} \tag{3.82}$$

for all $(\beta_e, \beta_1, \beta_2) \in \mathcal{U}_{ad}$. To prove the last part of the statement, we observe that $\mathcal{U}_{ad} = \mathcal{U}_e \times \mathcal{U}_1 \times \mathcal{U}_2$. Therefore, by taking $(\beta_e, \beta_1, \beta_2) = (\beta_e, \beta_1^*, \beta_2^*)$ in (3.70), we see that (3.70) is equivalent to the condition

$$\int_Q \left(S^* E^*(q-p) + \varpi_e \beta_e^*\right)(\beta_e - \beta_e^*) \geq 0 \quad \text{for all } \beta_e \in \mathcal{U}_e. \tag{3.83}$$

Similarly, it can also be shown that (3.70) is equivalent to the conditions

$$\int_Q \left(S^* I_1^*(q-p) + \varpi_1 \beta_1^*\right)(\beta_1 - \beta_1^*) \geq 0 \quad \text{for all } \beta_1 \in \mathcal{U}_1. \tag{3.84}$$
$$\int_Q \left(S^* I_2^*(q-p) + \varpi_2 \beta_2^*\right)(\beta_2 - \beta_2^*) \geq 0 \quad \text{for all } \beta_2 \in \mathcal{U}_2.. \tag{3.85}$$

and this concludes the proof of Theorem 3.5. □

Authors' Contributions

All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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