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CONVERGENCE ANALYSIS OF A MODIFIED PROXIMAL POINT ALGORITHM FOR A NEARLY ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPING

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Abstract. In this paper, we introduce a new modified proximal point algorithm to approximate a common element of the set of solutions of convex minimization problems and the set of fixed points of nearly asymptotically quasi-nonexpansive mapping in CAT(0) space. We also provide convergence guarantees for solving convex optimization problems and fixed-point problems involving total asymptotically nonexpansive mappings, while assuming only mild and practical conditions.

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1. Introduction

Let $\mathcal{C}$ be a nonempty subset of a metric space $(\mathcal{X}, d)$ and $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ be a nonlinear mapping. The fixed point set of $\mathcal{T}$ is denoted by $F(\mathcal{T})$, that is, $F(\mathcal{T}) = \{x \in \mathcal{C} : x = \mathcal{T}x\}$.

Fixed point theory in CAT(0) spaces was proposed by Kirk [20], which subsequently drew the attention of many researchers in this subject and has been a fascinating area of study for the past few years. Kirk demonstrated that a nonexpansive mapping constructed on a bounded convex closed subset of a complete CAT(0) space has a fixed point (see [23]).

Recently, Asifa et al. [1] proposed a four step iterative algorithm to solve the fixed point problem, and they also used a numerical example to understand the effectiveness of the new four step iteration procedure, which is given as follows:

Let $\mathcal{C}$ be a nonempty closed convex subset of complete CAT(0) space and $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ be a mapping. Let $x_1 \in \mathcal{T}$ be arbitrary and the sequence generated iteratively by

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$$
\begin{aligned}
c_n &= \mathcal{T}^n((1-\rho_n)x_n \oplus \rho_n \mathcal{T}^n x_n),\\
b_n &= \mathcal{T}^n(\mathcal{T}^n c_n),\\
a_n &= \mathcal{T}^n(\mathcal{T}^n b_n),\\
x_{n+1} &= \mathcal{T}^n a_n,\ n \geq 1.
\end{aligned}
\tag{1.1}
$$

Let $(\mathcal{X}, d)$ be a metric space and $f : \mathcal{X} \to (-\infty, \infty]$ be a proper and convex function. One of the major problems in optimization is to find $x \in \mathcal{X}$ such that

$$
f(x) = \min_{y \in \mathcal{X}} f(y). \tag{1.2}
$$

The set of minimizers of f is denoted by $\arg\min_{y\in\mathcal{X}} f(y)$. The proximal point algorithm (PPA) was developed by Martinet [2] in 1970 as a technique for locating minimizers of convex lower semicontinuous (lsc) functions constructed on Hilbert spaces. Since then, the PPA has grown incredibly popular among the different scholars' interest in optimization theory and also exposed a number of difficult mathematical problems. Optimization problems on manifolds are solved by a wide range of applications in computer vision, machine learning, electronic structure computation, system balance, and robot manipulation (see [4–7]).

In 2014, Bačák [8] attained some results using the proximal point algorithm in CAT(0) spaces. Also, he extended the findings of Bertsekas [9] into Hadamard spaces by using a splitting version of the PPA to determine the minimizer of a sum of convex functions. Since then, a great deal of mathematician have produced a number of findings pretaining to the proximal point methods within the context of CAT(0) spaces (see [10–16,28]).

We provide the following modified proximal point approach for nearly asympotically quasi-nonexpansive mappings in CAT(0) spaces utilising the iteration process defined by Asifa et al. [1]:

$$
\begin{aligned}
& x_1 \in \mathcal{C}\\
v_n &= \arg\min_{c\in\mathcal{C}} \Big(\mathcal{L}(c) + \frac{1}{2\lambda_n} d^2(c, x_n)\Big),\\
u_n &= \arg\min_{b\in\mathcal{C}} \Big(\mathcal{G}(b) + \frac{1}{2\varphi_n} d^2(b, v_n)\Big),\\
c_n &= ((1-\rho_n)x_n \oplus \rho_n \mathcal{T}^n u_n),\\
b_n &= \mathcal{T}^n(\mathcal{T}^n c_n),\\
a_n &= \mathcal{T}^n(\mathcal{T}^n b_n),\\
x_{n+1} &= \mathcal{T}^n a_n,\ n \geq 1.
\end{aligned}
\tag{1.3}
$$

We illustrate convergence outcomes of the proposed process under several moderate conditions based on previous work.

2. Preliminaries

This section reiterates a few commonly employed lemmas and ideas that are frequently used in our main findings.

If every geodesic triangle in a metric space $\mathcal{X}$ is at least as thin as its corresponding triangle in the Euclidean plane and the space is geodesically connected, it is referred to as a CAT(0) space (more information may be found in [17]).

When a subset $\mathcal{C}$ of a CAT(0) space $\mathcal{X}$ contains every geodesic segment linking two of its points, it is said to be convex; that is, for every pair of points $u, v \in \mathcal{C}$, we obtain $[u, v] \subset \mathcal{C}$, where $[u, v] := \{\varrho u \oplus (1-\varrho)v : 0 \leq \varrho \leq 1\}$ is the unique geodesic joining u and v.

If, for every $u, v \in \mathcal{C}$, $d(\mathcal{T}u, \mathcal{T}v) \leq d(u, v)$, then single-valued mapping $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ is referred as nonexpansive mapping.

If any sequence $\{u_k\}$ in $\mathcal{C}$ fulfilling $\lim_{k\to\infty} d(\mathcal{T}u_k, u_k) = 0$, has a convergent subsequence, then the single-valued mapping $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ is referred to as semi-compact.

$F(\mathcal{T})$ represents the set of all fixed points of $\mathcal{T}$. Now, we talk about the following lemma, which turns out to be helpful later on.

Lemma 2.1. ([18]) *Given CAT(0) space $(\mathcal{X}, d)$, the subsequent claims hold:*

(i) *A unique $z \in [u, v]$ exists for $u, v \in \mathcal{X}$ and $p \in [0, 1]$ such that*

$$d(u, z) = pd(u, v) \ \ \textit{and} \ \ d(v, z) = (1-p)d(u, v).$$

(ii) *For $u, v, z \in \mathcal{X}$ and $p \in [0, 1]$, we have*

$$d((1-p)u \oplus pv, z) \leq (1-p)d(u, z) + pd(v, z)$$

and

$$d^2((1-p)u \oplus pv, z) \leq (1-p)d^2(u, z) + pd^2(v, z) - p(1-p)d^2(u, v).$$

In the preceding Lemma, for the unique point z, we use the notation $(1-p)u \oplus py$.

Now, we gather some fundamental geometric properties that will be useful in the article.

Let $\{u_k\}$ be a bounded sequence in a complete CAT(0) space $\mathcal{X}$. For $u \in \mathcal{X}$ we write:

$$r(x, \{u_k\}) = \limsup_{k\to\infty} d(u, u_k).$$

The asymptotic radius $r(\{u_k\})$ is provided by

$$r(\{u_k\}) = \inf\{r(u, u_k) : x \in \mathcal{X}\}$$

and the asymptotic center $A(\{u_k\})$ of $\{u_k\}$ is characterized as:

$$A(\{u_k\}) = \{x \in X : r(u, u_k) = r(\{u_k\})\}.$$

The fact that $A(\{u_k\})$ is composed of exactly one point in a complete CAT(0) space is widely known [19]. Now, in order to help with our explanation that follows, we provide the definition and some basic properties of the Δ-convergence.

Definition 2.1. ([20]) *For every subsequence $\{s_k\}$ of $\{u_k\}$, if u is the unique asymptotic center of $\{s_k\}$, then $\{u_k\}$ in a CAT(0) space $\mathcal{X}$ is said to be Δ-convergent to a point $u \in \mathcal{X}$. In this instance, we denote u the Δ-limit of $\{u_k\}$ and write $\Delta - \lim_{k\to\infty} u_k = u$.*

Lemma 2.2. ([20]) *There exists a Δ-convergent subsequence for every bounded sequence in a complete CAT(0) space.*

Lemma 2.3. ([21]) *If $\mathcal{C}$ is a closed convex subset of a complete CAT(0) space $\mathcal{X}$ and $\{u_k\}$ is a bounded sequence in $\mathcal{C}$, then the asymptotic center of $\{u_k\}$ is in $\mathcal{C}$.*

Lemma 2.4. ([18]) *In a complete CAT(0) space $(\mathcal{X}, d)$, let $\mathcal{C}$ be a nonempty closed convex subset, and let $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ be a nonexpansive mapping. Then x is a fixed point of $\mathcal{T}$ if $\{u_k\}$ is a bounded sequence in $\mathcal{C}$ such that $\Delta - \lim_k u_k = x$ and $\lim\limits_{k\to\infty} d(\mathcal{T}u_k, u_k) = 0$.*

Lemma 2.5. ([18]) *If $\{u_k\}$ is a bounded sequence in a complete CAT(0) space with $A(\{u_k\}) = \{x\}$, $\{\mathcal{S}_k\}$ is a subsequence of $\{u_k\}$ with $A(\{\mathcal{S}_k\}) = \{u\}$ and the sequence $\{d(u_k, u)\}$ converges, then $x = u$.*

In this paper, we mainly study lower semi-continuous and convex functions on CAT(0) spaces. Note that a function $\mathcal{L} : \mathcal{C} \to (-\infty, \infty]$ defined on a convex subset $\mathcal{C}$ of a CAT(0) space is convex if and only if the function $\mathcal{L}o\gamma$ is convex for any geodesic $\gamma : [a, b] \to \mathcal{C}$. In other words, $\mathcal{L}(\varrho u \oplus (1-\varrho)v) \leq \varrho\mathcal{L}(u) + (1-\varrho)\mathcal{L}(v)$ for all $u, v \in \mathcal{C}$. See [24] for a few noteworthy examples.

Further, a function $\mathcal{L}$ defined on $\mathcal{C}$ is considered as lower semi-continuous at $u \in \mathcal{C}$ if

$$\mathcal{L}(u) \leq \liminf_{k\to\infty} \mathcal{L}(u_k)$$

for each sequence $\{u_k\}$ such that $u_k \to u$ as $k \to \infty$. A function $\mathcal{L}$ is considered as a lower semi-continuous on $\mathcal{C}$ if it is lower semi-continuous at any point in $\mathcal{C}$.

For any $\lambda > 0$, define the Moreau-Yosida resolvent of $\mathcal{L}$ in CAT(0) space as follows:

$$J_\lambda(u) = \arg\min_{v\in\mathcal{C}} [\mathcal{L}(v) + \frac{1}{2\lambda} d^2(v, u)]$$

for all $u \in \mathcal{C}$. For any $\lambda \geq 0$, the mapping J_λ is clearly defined; see [3]. The set $\mathfrak{U}(J_\lambda)$ of the fixed point of the resolvent J_λ associated with $\mathcal{L}$ coincides with the set $\arg\min\limits_{v\in\mathcal{C}} \mathcal{L}(v)$ of minimizers of $\mathcal{L}$ if $\mathcal{L}$ is a proper, convex, and lower semi-continuous function; see [24]. Moreover, the resolvent J_λ of $\mathcal{L}$ is nonexpansive for every $\lambda > 0$; see [25].

Lemma 2.6. ([26]) *For a given complete CAT(0) space $(\mathcal{X}, d)$, consider that $\mathcal{L} : \mathcal{X} \to (-\infty, \infty]$ is a proper, convex and lower semi-continuous function, then for all $u, v \in \mathcal{X}$ and $\lambda > 0$, we have*

$$\frac{1}{2\lambda}d^2(J_\lambda u, v) - \frac{1}{2\lambda}d^2(u, v) + \frac{1}{2\lambda}d^2(y, J_\lambda u) + \mathcal{L}(J_\lambda u) \leq \mathcal{L}(v).$$

Lemma 2.7. ([25,27]) *Assume that $(\mathcal{X}, d)$ is a complete CAT(0) space and that $\mathcal{L} : \mathcal{X} \to (-\infty, \infty]$ is a lower semi-continuous, proper, convex function. Then, the subsequent identity is valid:*

$$J_\lambda u = J_\mu(\frac{\lambda - \mu}{\lambda} J_\lambda u \oplus \frac{\mu}{\lambda} u)$$

for all $u \in \mathcal{X}$ and $\lambda > \mu > 0$.

Lemma 2.8. [29] *(Ariza-Ruiz et al. 2014). Let $(\mathcal{X}, d)$ be a complete CAT(0) space and $\mathcal{G} : \mathcal{X} \to (-\infty, \infty]$ a proper convex and lower semi-continuous. Then, for any $\chi > 0$,*
(a) the proximal operator $prox_{\chi\mathcal{G}}$ of $\mathcal{G}$ is firmly nonexpansive, i.e.,

$$d\left(\mathrm{prox}_{\chi\mathcal{G}}(a), \mathrm{prox}_{\chi\mathcal{G}}(b)\right) \leq d\left((1-t)a \oplus \mathrm{tprox}_{\mathcal{G}}(a), (1-t)b \oplus \mathrm{tprox}_{\chi\mathcal{G}}(b)\right)$$

for all $a, b \in \mathcal{X}$ and $t \in (0, 1)$;
(b) the set $Fix(prox_{\chi\mathcal{G}})$ of fixed points of $prox_{\chi\mathcal{G}}$ coincides with the set $\mathrm{argmin}_{b \in \mathcal{X}}\, \mathcal{G}(b)$ of minimizers of $\mathcal{G}$.

It is well known that every firmly nonexpansive mapping is nonexpansive.

Lemma 2.9. [30] *Let $\{s_n\}$ be the sequence of nonnegative numbers such that*

$$s_{n+1} \leq \gamma_n s_n + \xi_n$$

where $\{\gamma_n\}$ and $\{\xi_n\}$ are sequences of non negative numbers such that $\gamma_n \subseteq [1, \infty)$ and $\sum_{n=1}^{\infty}(\gamma_n - 1) < \infty$ and $\sum_{n=1}^{\infty}\xi_n < \infty$. Then $\lim_{n\to\infty} s_n$ exists.

3. Main results

We begin this section by the following useful theorem:

Theorem 3.1. *Let $\mathcal{C}$ be a nonempty closed convex subset of a complete CAT (0) space $(\mathcal{X}, d)$. Let $\mathcal{L}, \mathcal{G} : \mathcal{C} \to (-\infty, \infty]$ be proper convex and lower semi-continuous functions and $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ be uniformly continuous mapping satisfying the following:*
(a) $\mathcal{T}$ is nearly asymptotically quasi-nonexpansive mapping with sequence $\{(\kappa_n, \gamma_n)\}$ such that $\sum_{n=1}^{\infty}\kappa_n < \infty$ and $\sum_{n=1}^{\infty}(\gamma_n - 1) < \infty$;
(b) $\mathcal{T}$ is nearly uniformly Υ-Lipschitzian mapping with sequence $\{(\theta_n, \Upsilon)\}$.

Let $P = \mathrm{Fix}(\mathcal{T}) \cap \mathrm{argmin}_{b\in\mathcal{C}}\, \mathcal{L}(b) \cap \mathrm{argmin}_{c\in\mathcal{C}}\, \mathcal{G}(c) \neq \emptyset$. Let $\{\rho_n\}$ be the sequence in $(0, 1)$ such that $0 < \rho \leq \rho_n < 1$ for all $n \in \mathbb{N}$ and for some ρ and let $\{\lambda_n\}$ and $\{\varphi_n\}$ be sequences in $(0, \infty)$ such that $0 < \lambda \leq \lambda_n$ and $0 < \varphi \leq \varphi_n$ for all $n \in \mathbb{N}$. For $x_1 \in \mathcal{C}$, let $\{x_n\}$ be a sequence in $\mathcal{C}$ defined by (1.3). Then we have the *following:*

$(D1)$ $\lim_{n\to\infty} d(x_n, p)$ *exists for each* $p \in P$;

$(D2)$ $\lim_{n\to\infty} d(x_n, \mathcal{T}x_n) = 0$.

Proof. Let $p \in P$. Then $p = \mathcal{T}p$ and $\mathcal{L}(p) \leq \mathcal{L}(b)$ and $\mathcal{G}(p) \leq \mathcal{G}(c)$ for all $b, c \in \mathcal{C}$. Since, $\mathcal{L}(p) \leq \mathcal{L}(b)$, it follows that

$$\mathcal{L}(p) + \frac{1}{2\lambda_n} d^2(p, p) \leq \mathcal{L}(b) + \frac{1}{2\lambda_n} d^2(b, p)$$

for all $b \in \mathcal{C}$ and hence $p = \operatorname{prox}_{\lambda_n \mathcal{L}}(p)$ for all $n \in \mathbb{N}$. Similarly, we have $p = \operatorname{prox}_{\varphi_n \mathcal{G}}(p)$ for all $n \in \mathbb{N}$.

First, we prove that $\lim_{n\to\infty} d(x_n, p)$ exists. Note that $u_n = \operatorname{prox}_{\lambda_n \mathcal{L}}(v_n)$ and $v_n = \operatorname{prox}_{\varphi_n \mathcal{G}}(x_n)$ for all $n \in \mathbb{N}$. By Lemma 2.8, it follows that

$$d(u_n, p) = d\left(\operatorname{prox}_{\lambda_n \mathcal{L}}(v_n), \operatorname{prox}_{\lambda_n \mathcal{L}}(p)\right) \leq d(v_n, p)$$

and

$$d(v_n, p) = d\left(\operatorname{prox}_{\varphi_n \mathcal{G}}(x_n), \operatorname{prox}_{\varphi_n \mathcal{G}}(p)\right) \leq d(x_n, p).$$

Hence,

$$d(u_n, p) \leq d(x_n, p) \tag{3.1}$$

Using the definition of nearly asymptotically quasi-nonexpansive mapping and (1.3), we have

$$\begin{aligned}
d(c_n, p) &= d\big((1-\rho_n) x_n \oplus \rho_n \mathcal{T}^n u_n, p\big) \\
&\leq \big((1-\rho_n) d(x_n, p) + \rho_n d(\mathcal{T}^n u_n, p)\big) \\
&\leq (1-\rho_n) d(x_n, p) + \rho_n [\gamma_n d(u_n, p) + \kappa_n] \\
&\leq (1-\rho_n) d(x_n, p) + \rho_n [\gamma_n d(v_n, p) + \kappa_n] \\
&\leq (1-\rho_n) d(x_n, p) + \rho_n [\gamma_n d(x_n, p) + \kappa_n] \\
&\leq \gamma_n d(x_n, p) + \kappa_n,
\end{aligned} \tag{3.2}$$

Also, we attain

$$\begin{aligned}
d(b_n, p) &= d(\mathcal{T}^n(\mathcal{T}^n c_n), p) \\
&\leq \gamma_n d(\mathcal{T}^n c_n, p) + \kappa_n \\
&\leq \gamma_n [\gamma_n d(c_n, p) + \kappa_n] + \kappa_n \\
&\leq \gamma_n^2 d(c_n, p) + (1+\gamma_n) \kappa_n \\
&\leq \gamma_n^2 [\gamma_n d(x_n, p) + \kappa_n] + (1+\gamma_n) \kappa_n \\
&\leq \gamma_n^3 d(x_n, p) + \left(\gamma_n^2 + \gamma_n + 1\right) \kappa_n,
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
d\left(x_{n+1}, p\right) &= d\left(\mathcal{T}^{n} a_{n}, p\right) \\
&\leq \gamma_{n} d\left(a_{n}, p\right)+\kappa_{n} \\
&\leq \gamma_{n} d\left(\mathcal{T}^{n} b_{n}, p\right)+\kappa_{n} \\
&\leq \gamma_{n} d\left[\gamma_{n} d\left(b_{n}, p\right)+\kappa_{n}\right]+\kappa_{n} \\
&\leq \gamma_{n}^{2} d\left(b_{n}, p\right)+\gamma_{n} \kappa_{n}+\kappa_{n} \\
&\leq \gamma_{n}^{2}\left[\gamma_{n}^{3} d\left(x_{n}, p\right)+\gamma_{n}^{2} \kappa_{n}+\gamma_{n} \kappa_{n}+\kappa_{n}\right]+\gamma_{n} \kappa_{n}+\kappa_{n} \\
&\leq \gamma_{n}^{5} d\left(x_{n}, p\right)+\left(\gamma_{n}^{4}+\gamma_{n}^{3}+\gamma_{n}^{2}+\gamma_{n}+1\right) \kappa_{n} \\
&\leq (1+(\gamma_{n}-1)(\gamma_{n}^{4}+\gamma_{n}^{3}+\gamma_{n}^{2}+\gamma_{n}+1)) d\left(x_{n}, p\right) \\
&\quad +\left(\gamma_{n}^{4}+\gamma_{n}^{3}+\gamma_{n}^{2}+\gamma_{n}+1\right) \kappa_{n}.
\end{aligned} \tag{3.4}$$

where $M_1 = 1+\sup_{n\in\mathbb{N}}\left(\gamma_{n}^{4}+\gamma_{n}^{3}+\gamma_{n}^{2}+\gamma_{n}+1\right)\gamma_n$. By Lemma 2.9, $\lim_{n\to\infty} d\left(x_n, p\right)$ exists. (ii) Next we will prove that $\lim_{n\to\infty} d\left(x_n, v_n\right)=0$ and $\lim_{n\to\infty} d\left(x_n, u_n\right)=0$. Assume that

$$\lim_{n\to\infty} d\left(x_n, p\right)=r \tag{3.5}$$

for some $r>0$. By Lemma 2.6, we have

$$\frac{1}{2\varphi_n} d^2\left(\operatorname{prox}_{\varphi_n \mathcal{G}}\left(x_n\right), p\right)-\frac{1}{2\varphi_n} d^2\left(x_n, p\right)+\frac{1}{2\varphi_n} d^2\left(x_n, \operatorname{prox}_{\varphi_n \mathcal{G}}\left(x_n\right)\right) \leq \mathcal{G}(p)-\mathcal{G}\left(x_n\right).$$

Since $\mathcal{G}(p) \leq \mathcal{G}\left(x_n\right)$ for all $n\in\mathbb{N}$, it follows that

$$d^2\left(x_n, v_n\right) \leq d^2\left(x_n, p\right)-d^2\left(v_n, p\right) \tag{3.6}$$

and

$$\frac{1}{2\lambda_n} d^2\left(\operatorname{prox}_{\lambda_n \mathcal{L}}\left(v_n\right), p\right)-\frac{1}{2\lambda_n} d^2\left(v_n, p\right)+\frac{1}{2\lambda_n} d^2\left(v_n, \operatorname{prox}_{\lambda_n \mathcal{L}}\left(v_n\right)\right) \leq \mathcal{L}(p)-\mathcal{L}\left(v_n\right).$$

Since $\mathcal{L}(p) \leq \mathcal{L}\left(v_n\right)$ for all $n\in\mathbb{N}$, it follows that

$$d^2\left(v_n, u_n\right) \leq d^2\left(v_n, p\right)-d^2\left(u_n, p\right) \tag{3.7}$$

From (3.4), we have

$$d\left(x_{n+1}, p\right) \leq \gamma_n^2 d\left(b_n, p\right)+\gamma_n \kappa_n+\kappa_n.$$

By taking $\lim_{n\to\infty}$ both sides, we have

$$r=\liminf_{n\to\infty} d\left(x_{n+1}, p\right) \leq \liminf_{n\to\infty} d\left(b_n, p\right).$$

From (3.3), we have $\limsup_{n\to\infty} d\left(b_n, p\right) \leq \limsup_{n\to\infty} d\left(x_n, p\right)=r$. Thus,

$$\lim_{n\to\infty} d\left(b_n, p\right)=r \tag{3.8}$$

From (3.3), we also have,

$$d(b_n,p) \le \gamma_n^2 d(c_n,p) + \gamma_n\kappa_n + \kappa_n$$

By taking $\lim_{n\to\infty}$ both sides, we have

$$r = \liminf_{n\to\infty} d(b_n,p) \le \liminf_{n\to\infty} d(c_n,p$$

From (3.2), we have $\limsup_{n\to\infty} d(c_n,p) \le \limsup_{n\to\infty} d(x_n,p) = r$. Thus,

$$\lim_{n\to\infty} d(c_n,p) = r \tag{3.9}$$

From (3.2), we also get

$$\begin{aligned}
d(c_n,p) &\le (1-\rho_n)\, d(x_n,p) + \rho_n\left[d(u_n,p) + \kappa_n\right] \\
d(c_n,p) &\le d(x_n,p) - \rho_n d(x_n,p) + \rho_n d(u_n,p) + \rho_n\kappa_n \\
\rho_n d(x_n,p) &\le d(x_n,p) - d(c_n,p) + \rho_n d(u_n,p) + \rho_n\kappa_n \\
d(x_n,p) &\le \frac{1}{\rho_n}\left(d(x_n,p) - d(c_n,p)\right) + d(v_n,p) + \kappa_n.
\end{aligned} \tag{3.10}$$

Using (3.5) and (3.9), we get $r = \liminf_{n\to\infty} d(x_n,p) \le \liminf_{n\to\infty} d(v_n,p)$ which together with $d(v_n,p) \le d(x_n,p)$ gives us that

$$\lim_{n\to\infty} d(v_n,p) = r \tag{3.11}$$

Hence, from (3.6), we obtain

$$\lim_{n\to\infty} d(x_n,v_n) = 0 \tag{3.12}$$

From (3.2), we have

$$\begin{aligned}
d(c_n,p) &\le (1-\rho_n)\, d(x_n,p) + \rho_n\left[d(u_n,p) + \kappa_n\right] \\
d(c_n,p) &\le d(x_n,p) - \rho_n d(x_n,p) + \rho_n d(u_n,p) + \rho_n\kappa_n \\
\rho_n d(x_n,p) &\le d(x_n,p) - d(c_n,p) + \rho_n d(u_n,p) + \rho_n\kappa_n \\
d(x_n,p) &\le \frac{1}{\rho_n}\left(d(x_n,p) - d(c_n,p)\right) + d(u_n,p) + \kappa_n
\end{aligned} \tag{3.13}$$

Using (3.5) and (3.9), we get $r = \liminf_{n\to\infty} d(x_n,p) \le \liminf_{n\to\infty} d(u_n,p)$ which together with $d(u_n,p) \le d(x_n,p)$ gives us that

$$\lim_{n\to\infty} d(u_n,p) = r. \tag{3.14}$$

Hence, from (3.7), (3.11) and (3.13), we have

$$\lim_{n\to\infty} d(v_n,u_n) = 0 \tag{3.15}$$

From (3.5) and (3.14), we get

$$\lim_{n\to\infty} d(x_n, u_n) = 0 \tag{3.16}$$

From (3.1), we have

$$\begin{aligned}
d^2(c_n, p) =& d^2((1-\rho_n)x_n \oplus \rho_n \mathcal{T}^n u_n, p) \\
&\leq (1-\rho_n) d^2(x_n, p) + \rho_n d^2(\mathcal{T}^n u_n, p) - \rho_n(1-\rho_n) d^2(x_n, \mathcal{T}^n u_n) \\
&\leq (1-\rho_n) d^2(x_n, p) + \rho_n (\gamma_n d(u_n, p) + \kappa_n)^2 - \rho_n(1-\rho_n) d^2(x_n, \mathcal{T}^n u_n) \\
&= (1-\rho_n) d^2(x_n, p) + \rho_n \left(\gamma^2 d^2(u_n, p) + (\kappa_n + 2\gamma_n d(u_n, p))\kappa_n\right) \\
&\quad - \rho_n(1-\rho_n) d^2(x_n, \mathcal{T}^n u_n) \\
&\leq \gamma_n^2 (1-\rho_n) d^2(x_n, p) + \rho_n \left(\gamma_n^2 d^2(u_n, p) + M_2\kappa_n\right) - \rho_n(1-\rho_n) d^2(x_n, \mathcal{T}^n u_n) \\
&\leq \gamma_n^2 (1-\rho_n) d^2(x_n, p) + \rho_n^2 \gamma_n^2 d^2(u_n, p) + M_2\kappa_n - \rho_n(1-\rho_n) d^2(x_n, \mathcal{T}^n u_n) \\
&= \gamma_n^2 d^2(x_n, p) + M_2\kappa_n - \rho_n(1-\rho_n) d^2(x_n, \mathcal{T}^n u_n),
\end{aligned} \tag{3.17}$$

where $M_2 = \sup_{n\in\mathbb{N}} (\kappa_n + 2d(u_n, p))$. From (3.17),

$$\rho_n(1-\rho_n) d^2(x_n, \mathcal{T}^n u_n) \leq \left(\gamma_n^2 d^2(x_n, p) - d^2(c_n, p)\right) + M_2\kappa_n$$

Hence from (3.5) and (3.9), we have

$$\lim_{n\to\infty} d(x_n, \mathcal{T}^n u_n) = 0.$$

Moreover, we get

$$\begin{aligned}
d(x_n, \mathcal{T}^n x_n) &\leq d(x_n, \mathcal{T}^n u_n) + d(\mathcal{T}^n u_n, \mathcal{T}^n x_n) \\
&\leq d(x_n, \mathcal{T}^n u_n) + d(u_n, x_n) + \kappa_n \to 0 \text{ as } n \to \infty.
\end{aligned} \tag{3.18}$$

Using (3.18), we have

$$\begin{aligned}
d(x_n, c_n) &= d(x_n, (1-\rho_n)x_n \oplus \rho_n \mathcal{T}^n u_n) \\
&\leq (1-\rho_n) d(x_n, x_n) + \rho_n d(x_n, \mathcal{T}^n u_n) \\
&\to 0 \text{ as } n \to \infty.
\end{aligned} \tag{3.19}$$

Next, we get

$$\begin{aligned}
d(x_n, x_{n+1}) &= d(x_n, \mathcal{T}^n a_n) \\
&\leq d(x_n, \mathcal{T}^n x_n) + d(\mathcal{T}^n x_n, \mathcal{T}^n a_n) \\
&\leq d(x_n, \mathcal{T}^n x_n) + \gamma_n d(x_n, a_n) + \kappa_n \\
&\leq d(x_n, \mathcal{T}^n x_n) + \gamma_n d(x_n, \mathcal{T}^n(\mathcal{T}^n c_n)) + \kappa_n \\
&\leq d(x_n, \mathcal{T}^n x_n) + \gamma_n [\gamma_n d(x_n, \mathcal{T}^n c_n) + \kappa_n] + \kappa_n
\end{aligned} \tag{3.20}$$

$$
\begin{aligned}
&\leq d\left(x_n, \mathcal{T}^n x_n\right)+\gamma_n\left[\gamma_n\left(d\left(x_n, \mathcal{T}^n x_n\right)+d\left(\mathcal{T}^n x_n, c_n\right)\right)+\kappa_n\right]+\kappa_n \\
&\leq d\left(x_n, \mathcal{T}^n x_n\right)+\gamma_n^2 d\left(x_n, \mathcal{T}^n x_n\right)+\gamma_n^2 d\left(\mathcal{T}^n x_n, c_n\right)+\gamma_n \kappa_n+\kappa_n \\
&\leq\left(\gamma_n^2+1\right) d\left(x_n, \mathcal{T}^n x_n\right)+\gamma_n^2\left[\gamma_n d\left(x_n, c_n\right)+\kappa_n\right]+\gamma_n \kappa_n+\kappa_n \\
&\leq\left(\gamma_n^2+1\right) d\left(x_n, \mathcal{T}^n x_n\right)+\gamma_n^3 d\left(x_n, c_n\right)+\gamma_n^2 \kappa_n+\gamma_n \kappa_n+\kappa_n \\
&\leq\left(\gamma_n^2+1\right) d\left(x_n, \mathcal{T}^n x_n\right)+\gamma_n^3 d\left(x_n, c_n\right)+\kappa_n\left(\gamma_n^2+\gamma_n+1\right) \\
&\rightarrow 0 \text{ as } n \rightarrow \infty .
\end{aligned}
$$

Using the uniform continuity of $\mathcal{T}$ in (3.18) and the definition of nearly uniformly Υ-Lipschitzian mapping $\mathcal{T}$ in (3.19), we have $\lim_{n\to\infty} d\left(\mathcal{T}x_n, \mathcal{T}^{n+1}x_n\right)=0$ and $\lim_{n\to\infty} d\left(\mathcal{T}^{n+1}x_n, \mathcal{T}^{n+1}x_{n+1}\right)=0$.

$$
\begin{aligned}
d\left(x_n, \mathcal{T} x_n\right) \leq & d\left(x_n, x_{n+1}\right)+d\left(x_{n+1}, \mathcal{T}^{n+1} x_{n+1}\right)+d\left(\mathcal{T}^{n+1} x_{n+1}, \mathcal{T}^{n+1} x_n\right) \\
& +d\left(\mathcal{T}^{n+1} x_n, \mathcal{T} x_n\right) \qquad (3.21)\\
\rightarrow & 0 \text{ as } n \rightarrow \infty .
\end{aligned}
$$

This completes the proof. □

Now, we are in position to state the first main result of the paper

Theorem 3.2. *Let $\mathcal{C}$ be a nonempty closed convex subset of a complete CAT (0) space $(\mathcal{X}, d)$. Let $\mathcal{L}, \mathcal{G}: \mathcal{C} \rightarrow (-\infty, \infty]$ be proper convex and lower semi-continuous functions and let $\mathcal{T}: \mathcal{C} \rightarrow \mathcal{C}$ be uniformly continuous mapping satisfying the following:*
(a) $\mathcal{T}$ is nearly asymptotically quasi-nonexpansive mapping with sequence $\{(\kappa_n,)\}$ such that $\sum_{n=1}^{\infty} \kappa_n < \infty$ and $\sum_{n=1}^{\infty}(-1) < \infty$;
(b) $\mathcal{T}$ is nearly uniformly Υ-Lipschitzian mapping with sequence $\{(\theta_n, \Upsilon)\}$. Let $\mathcal{T}$ satisfies the demiclosedness principle and $P = \operatorname{Fix}(\mathcal{T}) \cap \operatorname{argmin}_{b\in\mathcal{C}} \mathcal{L}(b) \cap \operatorname{argmin}_{c\in\mathcal{C}} \mathcal{G}(c) \neq \emptyset$. Let $\{\rho_n\}$ be the sequence in $(0,1)$ such that $0<\rho \leq \rho_n<1$ for all $n \in \mathbb{N}$ and for some ρ and let $\{\lambda_n\}$ and $\{\varphi_n\}$ be sequences in $(0, \infty)$ such that $0<\lambda \leq \lambda_n$ and $0<\varphi \leq \varphi_n$ for all $n \in \mathbb{N}$. For $x_1 \in \mathcal{C}$, let $\{x_n\}$ be a sequence in $\mathcal{C}$ defined by (1.1). Then the sequence $\{x_n\}$ Δ - converges to an element of P.

Proof. Since $0<\lambda \leq \lambda_n$, therefore from Lemma 2.7 and (3.15), we have

$$
\begin{aligned}
d\left(\operatorname{prox}_{\lambda\mathcal{L}} x_n, x_n\right) & \leq d\left(\operatorname{prox}_{\lambda\mathcal{L}} x_n, u_n\right)+d\left(u_n, v_n\right)+d\left(v_n, x_n\right) \\
& =d\left(\operatorname{prox}_{\lambda\mathcal{L}} x_n, \operatorname{prox}_{\lambda_n\mathcal{L}} v_n\right)+d\left(u_n, v_n\right)+d\left(v_n, x_n\right) \\
& =d\left(\operatorname{prox}_{\lambda\mathcal{L}} x_n, \operatorname{prox}_{\lambda\mathcal{L}\mathcal{Y}}\left(\frac{\lambda_n-\lambda}{\lambda_n} \operatorname{prox}_{\lambda_n\mathcal{L}} v_n \oplus \frac{\lambda}{\lambda_n} v_n\right)\right)+d\left(u_n, v_n\right)+d\left(v_n, x_n\right) \\
& \leq d\left(x_n, \frac{\lambda_n-\lambda}{\lambda_n} \operatorname{prox}_{\lambda_n\mathcal{L}} v_n \oplus \frac{\lambda}{\lambda_n} v_n\right)+d\left(u_n, v_n\right)+d\left(v_n, x_n\right) \qquad (3.22)
\end{aligned}
$$

$$
\begin{aligned}
&= \left(1-\frac{\lambda}{\lambda_n}\right) d\left(x_n, \operatorname{prox}_{\lambda_n \mathcal{L}} v_n\right) + \frac{\lambda}{\lambda_n} d\left(x_n, v_n\right) + d\left(u_n, v_n\right) + d\left(v_n, x_n\right) \\
&= \left(1-\frac{\lambda}{\lambda_n}\right) d\left(x_n, u_n\right) + \frac{\lambda}{\lambda_n} d\left(x_n, v_n\right) + d\left(u_n, v_n\right) + d\left(v_n, x_n\right) \\
&= \left(1-\frac{\lambda}{\lambda_n}\right) d\left(x_n, u_n\right) + \left(1+\frac{\lambda}{\lambda_n}\right) d\left(x_n, v_n\right) + d\left(u_n, v_n\right) \\
&\to 0 \; as \; n \to \infty.
\end{aligned}
$$

Proceeding in the same manner as above and using (3.12), we have

$$
\begin{aligned}
d\left(\operatorname{prox}_{\varphi \mathcal{G}} x_n, x_n\right) &\leq d\left(\operatorname{prox}_{\varphi \mathcal{G}} x_n, v_n\right) + d\left(v_n, x_n\right) \\
&= d\left(\operatorname{prox}_{\varphi \mathcal{G}} x_n, \operatorname{prox}_{\varphi_n \mathcal{G}} x_n\right) + d\left(v_n, x_n\right) \\
&= d\left(\operatorname{prox}_{\varphi_{\mathcal{G}}} x_n, \operatorname{prox}_{\varphi_{\mathcal{G}}}\left(\frac{\varphi_n-\varphi}{\varphi_n} \operatorname{prox}_{\varphi_n \mathcal{G}} x_n \oplus \frac{\varphi}{\varphi_n} x_n\right)\right) + d\left(v_n, x_n\right) \\
&\leq d\left(x_n, \frac{\varphi_n-\varphi}{\varphi_n} \operatorname{prox}_{\varphi_n \mathcal{G}} x_n \oplus \frac{\varphi}{\varphi_n} x_n\right) + d\left(v_n, x_n\right) \\
&= \left(1-\frac{\varphi}{\varphi_n}\right) d\left(x_n, \operatorname{prox}_{\varphi_n \mathcal{G}} x_n\right) + \frac{\varphi}{\varphi_n} d\left(x_n, x_n\right) + d\left(v_n, x_n\right) \\
&= \left(1-\frac{\varphi}{\varphi_n}\right) d\left(x_n, v_n\right) + \frac{\varphi}{\varphi_n} d\left(x_n, x_n\right) + d\left(v_n, x_n\right) \\
&= \left(1-\frac{\varphi}{\varphi_n}\right) d\left(x_n, v_n\right) + d\left(v_n, x_n\right) \\
&= \left(2-\frac{\varphi}{\varphi_n}\right) d\left(x_n, v_n\right) \\
&\to 0 \text{ as } n \to \infty
\end{aligned}
\tag{3.23}
$$

Next we show that $w_{\Delta}\left(x_n\right) = \bigcup_{\{\zeta_n\} \subset \{x_n\}} A\left(\{\zeta_n\}\right) \subset P$. Let $u \in w_{\Delta}\left(x_n\right)$. Then there exists a subsequence $\{\zeta_n\}$ of $\{x_n\}$ such that $A\left(\{\zeta_n\}\right) = \{\zeta_n\}$. Therefore, there exists a subsequence $\{\vartheta_n\}$ of $\{\zeta_n\}$ such that $\Delta - \lim_{n\to\infty} \vartheta_n = \vartheta$ for some $\vartheta \in P$.

In view of Theorem 3.1, we have

$$
\lim_{n\to\infty} d\left(\vartheta_n, \mathcal{T}\vartheta_n\right) = 0, \quad \lim_{n\to\infty} d\left(\operatorname{prox}_{\lambda \mathcal{L}} \vartheta_n, \vartheta_n\right) = 0, \quad \lim_{n\to\infty} d\left(\operatorname{prox}_{\varphi_n \mathcal{G}} \vartheta_n, \vartheta_n\right) = 0.
$$

Since $\mathcal{T}$ satisfies demiclosedness conditions, we have $\vartheta \in P$. Hence, by Theorem 3.1(D1), $\lim_{n\to\infty} d\left(x_n, \vartheta\right)$ exists and by Lemma 2.5, we have $\zeta = \vartheta$. This shows that $w_{\Delta}\left(x_n\right) \subset P$.

Finally, we show that the sequence $\{x_n\}$ generated by (1.3) Δ-converges to a point in P. To this end, it suffices to show that $w_{\Delta}\left(x_n\right)$ consists of exactly one point. Let $\{\zeta_n\}$ be a subsequence of $\{x_n\}$ and let $A\left(\{x_n\}\right) = \{a\}$. Since $\zeta \in w_{\Delta}\left(x_n\right) \subset P$ and $d\left(x_n, \zeta\right)$ converges, we have $a = \zeta$. Hence $w_{\Delta}\left(x_n\right) = \{a\}$. □

4. Conclusion

In this work, we presented a modified proximal point approach to approximate a shared point of the set of fixed points of nearly asymptotically quasi-nonexpansive mapping in CAT(0) space and the set of solutions to convex minimization problems. Assuming only mild and practical conditions, we also offered convergence guarantees for convex optimization issues and fixed-point problems using total asymptotically nonexpansive mappings.

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