Asia Pac. J. Math. 2025 **12**:21

CONTROLLABILITY OF A STOCHASTIC MODEL OF LYMPHATIC FILARIASIS

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Received Jan. 8, 2025

ABSTRACT. This paper analyzes the controllability of a stochastic model of lymphatic filariasis. Like other diseases, the spread of lymphatic filariasis is subject to a degree of randomness due to fluctuations in the natural environment. This provides an opportunity to formulate a mathematical model of lymphatic filariasis that accounts for as much external stochasticity as possible. First, we use stochastic optimal control theory to formulate the control problem associated with the model. We then study the existence of an adapted optimal control and characterize the stochastic optimal control. Finally, we present some numerical simulations to validate our results.

2020 Mathematics Subject Classification. 92D30; 34H05; 60H10.

Key words and phrases. lymphatic filariasis; optimal control; stochastic model.

1. INTRODUCTION

Lymphatic filariasis is a neglected tropical disease that primarily affects subtropical regions, where poverty is particularly prevalent. The main vector of this well-known disease is the mosquito. In this context, mathematical models are valuable tools for providing a clearer understanding of the disease's prevalence dynamics. As a result, numerous researchers have proposed mathematical models related to lymphatic filariasis [1–5]. Among these works, the study by C. P. Bhunu and S. Mushayabasa [1] is particularly noteworthy, as it presents a model for lymphatic filariasis. They further explore the local stability of equilibrium points using central manifold theory [6] and Theorem 4.1 from the paper by C. Castillo-Chavez and B. Song [7]. Additionally, they consider the transition from the exposed to the infectious state as a reinfection process, using a function λ_m dependent on I_m and a constant ρ.

However, while deterministic models of lymphatic filariasis are useful for analyzing certain dynamics, they are limited in accurately predicting the disease's future progression. This limitation can be

DOI: 10.28924/APJM/12-21

addressed by stochastic models, which offer a higher degree of realism. Many authors have studied stochastic systems and models in this context [8–20]. To our knowledge, few studies have focused on applying stochastic control to lymphatic filariasis models. In this work, optimal control theory is applied to the stochastic model developed by Fourtoua Victorien KONANE and Ragnimwendé SAWADOGO in [23]. These authors have also studied in depth the persistence and extinction dynamics of the disease.

First, the existence and uniqueness of the global positive solution will be analyzed based on the theory of stochastic differential equations [21]. Next, the controllability of the proposed stochastic model for lymphatic filariasis will be assessed, starting with the introduction of the stochastic optimal control problem, followed by the formulation of the control problem, the demonstration of the existence of an optimal control adapted to the states of lymphatic filariasis, and the characterization of the control. Finally, numerical simulations will be presented to evaluate and validate the theoretical results.

2. Stochastic model

2.1. **Formulation of the stochastic model.** This section is dedicated to the presentation of the stochastic model. Let N_h and N_m represent the population sizes of humans and mosquitoes, respectively. The compartments S, E, and I correspond to the susceptible, exposed, and infectious individuals during the epidemic. The following tables provide a detailed description of these classes. As mentioned earlier in the introduction, the study of the persistence and extinction of the disease has already been addressed in the article by Fourtoua Victorien KONANE and Ragnimwendé SAWADOGO [23]. In the context of our work, we focus exclusively on the control aspect of the model.

Table 1. Parameters for humans hosts.

Notations	**Biological description**
S_h	compartment of susceptible humans
E_h	compartment of latent humans
I_h	compartment of infectious humans

Table 2. Parameters for vectors hosts.

Notations	**Biological description**
S_m	compartment of susceptible mosquitoes
I_m	compartment of infectious mosquitoes

The compartmental diagram describing the progression of infection in the different compartments is given by Figure 1.

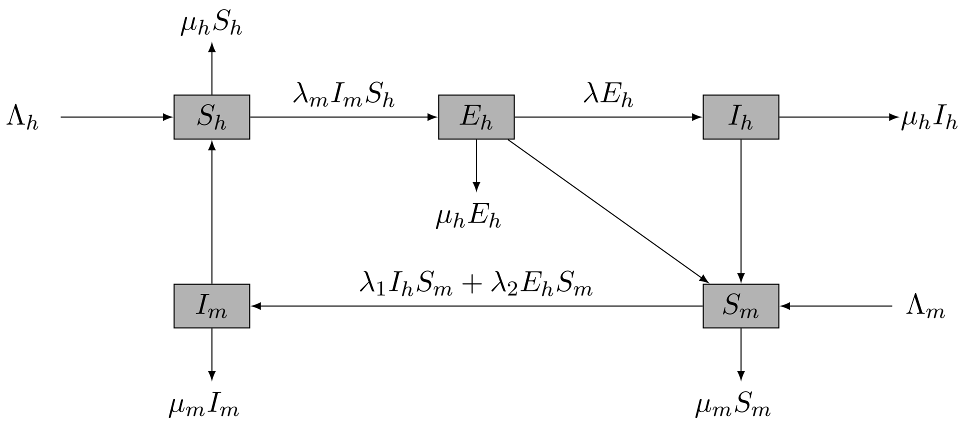

FIGURE 1. Transfer diagram of the deterministic model

Let us now introduce the proportions

$$\begin{aligned} s_h(t) &= \frac{S_h(t)}{N_h},\ e_h(t) = \frac{E_h(t)}{N_h}, \\ i_h(t) &= \frac{I_h(t)}{N_h},\ s_m(t) = \frac{S_m(t)}{N_m}, \\ i_m(t) &= \frac{I_m(t)}{N_m}. \end{aligned}$$

Then, we following the equalities $s_h(t) + e_h(t) + i_h(t) = 1$ and $s_m(t) + i_m(t) = 1$.

2.2. **Preliminary.** We give some basic theory on the differentials stochastic equations (see [22]). Throughout this paper, unless otherwise specified, let $(\Omega, \mathcal{F}, \mathbf{P})$ a complete probability space with $(\mathcal{F}_t)_{t\geq 0}$ filtration that satisfying the usual conditions. We note
$\mathbf{R}^5_+ = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{R}^5 : x_1 > 0, x_2 > 0, x_3 > 0, x_4 > 0, x_5 > 0\}$ and
$\chi^+ = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{R}^5_+ : x_1 + x_2 + x_3 < 1, x_4 + x_5 < 1\}$.

The following stochastic system is considered:

$$dX(t) = f(t, X(t))dt + g(t, X(t))dB(t), \tag{1}$$

for $t \geq t_0$ with$X(t_0) = X_0 \in \mathbf{R}^n$, $B(t)$ denotes n dimensional standard Brownian motion defined on the above probability space. Define the differential operator $\mathcal{L}$ associated to (1) by:

$$\mathcal{L}V(t, X) = \frac{\partial V(t, X)}{dt} + f^T \frac{\partial V(t, X)}{dX} + \frac{1}{2} Tr\left[g^T \frac{\partial^2 V(t, X)}{dX^2} g\right] \text{ où } V(t, X) \in \mathcal{C}^{1,2}\left(\mathbf{R} \times \mathbf{R}^m\right). \tag{2}$$

The description leads to the following system of differential stochastic equations:

$$\begin{cases} ds_h(t) = [\mu_h - \beta_1 i_m(t)s_h(t) - \mu_h s_h(t)]\,dt - \eta_1 i_m(t)s_h(t)dB_1(t), \\ de_h(t) = [\beta_1 i_m(t)s_h(t) - (\lambda+\mu_h)e_h(t)]\,dt + \eta_1 i_m(t)s_h(t)dB_1(t) - \eta_2 e_h(t)dB_2(t), \\ di_h(t) = [\lambda e_h - \mu_h i_h(t)]\,dt + \eta_2 e_h(t)dB_2(t), \\ ds_m(t) = [\mu_m - (\beta_3 i_h(t) + \beta_2 e_h(t))s_m(t) - \mu_m s_m(t)]\,dt - \eta_3 i_h(t)s_m(t)dB_3(t) - \eta_4 e_h s_m(t)dB_4(t), \\ di_m(t) = [(\beta_3 i_h(t) + \beta_2 e_h(t))s_m(t) - \mu_m i_m(t)]\,dt + \eta_3 i_h(t)s_m(t)dB_3(t) + \eta_4 e_h s_m(t)dB_4(t), \end{cases} \tag{3}$$

where $B_j, j = \overline{1,4}$ are mutually independent Brownians and $\eta_j, j = \overline{1,4}$ are their respective intensities and $\beta_1 = \beta_m$, $\beta_2 = \theta_h\beta_h$, $\beta_3 = \beta_h$. The study of the existence and uniqueness of solutions to the system 3 has already been analyzed in [23].

2.3. **Statement of the control problem and preliminaries.** Let $(\Omega, \mathcal{F}, \mathbf{P})$ a probability space, $(\mathcal{F}_t)_{t\geq 0}$ a filtration. Let $B(t) = (B_1(t), B_2(t), 0, B_3(t), B_4(t))$ a standard Brownian motion (in particular $B(t)$ is a martingale of $\mathcal{F}_t$) with real values. Assume that

$$\mathcal{F}_t = \sigma\left\{B(\xi), 0 \leq \xi \leq t\right\}. \tag{4}$$

Let us consider the system of stachastic differential equations (3). The resulting controlled system is given by

$$\begin{cases} dX(t) = F\left(t, X(t), U(t)\right)dt + M\left(t, X(t)\right)dB(t), \\ X(0) = X_0, \end{cases} \tag{5}$$

where

$$F(\cdot,\cdot,\cdot) \quad : \quad [0,T]\times(0,1)^5\times\mathbf{U} \to (0,1)^5, (t, X_t, U_t) \mapsto F(t, X_t, U_t) \tag{6}$$

(7)

$$M\left(t, X(t)\right) = \begin{pmatrix} -\eta_1 i_m(t)s_h(t) & 0 & 0 & 0 & 0 \\ \eta_1 i_m(t)s_h(t) & -\eta_2 e_h(t) & 0 & 0 & 0 \\ 0 & \eta_2 e_h(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & -\eta_3 i_h(t)s_m(t) & -\eta_4 e_h(t)s_m(t) \\ 0 & 0 & 0 & \eta_3 i_h(t)s_m(t) & \eta_4 e_h(t)s_m(t) \end{pmatrix} \tag{8}$$

with

$$U_t = \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix}, X_t = \begin{pmatrix} s_h(t) \\ e_h(t) \\ i_h(t) \\ s_m(t) \\ i_m(t) \end{pmatrix} \tag{9}$$

and

$$F(t, X(t), U(t)) = \begin{pmatrix} \mu_h - v_1 i_m(t) s_h(t) - \mu_h s_h(t) \\ v_1 i_m(t) s_h(t) - (\lambda + \mu_h) e_h(t) \\ \lambda e_h(t) - \mu_h i_h(t) \\ \mu_m - v_2 i_h(t) s_m(t) - \beta_2 e_h(t) s_m(t) - \mu_m s_m(t) \\ v_2 i_h(t) s_m(t) + \beta_2 e_h(t) s_m(t) - \mu_m i_m(t) \end{pmatrix} \tag{10}$$

We assume that $\mathbf{U} \subseteq (0,1)^2$ and $T \in]0, +\infty[$ is fixed. The function $U(.)$ is called the control, which represents the action or decision. At any given moment, the controller is well informed about certain information in the system, as specified by the information field $\{\mathcal{F}_t\}_{t\geq 0}$, but is not in a position to say what will happen in the future, given the uncertainty of the system. Mathematically, this restriction to non-anticipation can be represented by $U(.)$ is $\{\mathcal{F}_t\}_{t\geq 0}$-adapted.

Definition 2.1. *An admissible control $U(.)$ is a process $\{\mathcal{F}_t\}_{t\geq 0}$-adapted with values in $\mathbf{U}$ such that*

$$\sup_{0\leq t\leq T} E\left[\mid U(t) \mid^m\right] < \infty, \forall m = 1, 2, ... \tag{11}$$

The set of all admissible controls is denoted by U_{ad}.

Definition 2.2. *The following problem*

$$J(\overline{u}(.)) = \inf_{u\in U_{ad}} J(u(.)) \tag{12}$$

is finite if the right-hand side is finite.

We denote by $\mathcal{L}^P_{\mathcal{F}}$ the set of real processes $X(.)$, $\{\mathcal{F}_t\}_{t\geq 0}$-adapted such that

$$E\left[\int_0^T \mid X(t) \mid^P\right] < \infty. \tag{13}$$

Let $S(t) : [0, T] \longrightarrow 2^{\mathbf{R}^5}$ a given multifunction. State constraints can be given by

$$X_t \in S(t), \forall t \in [0, T], \mathbf{P} - p.s. \tag{14}$$

The cost function we introduce is

$$J(U(.)) = E\left[\int_0^T f(t, X_t, U_t) dt\right] \tag{15}$$

with

$$f : [0;T] \times (0,1)^5 \times \mathbf{U} \longrightarrow (0,1) \tag{16}$$

define by

$$f(t, X_t, U_t) = v_1^2 i_m(t) s_h(t) + v_2^2 i_h(t) s_m(t). \tag{17}$$

Our problem is to find the control vector $U^\star$ that minimizes the cost function (15):

$$J(U^\star(.)) = \inf_{U \in U_{ad}} J(U(.)). \tag{18}$$

Given that

i) $U(.) \in U_{ad}$;

ii) (5) has a unique solution thanks to the section 1;

iii) The constraints verify (14);

iv) $f(., X(.), U(.)) \in \mathcal{L}^1_{\mathcal{F}}(0, T, (0,1))$, because $f(t, X_t, U_t) = v_1^2 i_m(t) s_h(t) + v_2^2 i_h(t) s_m(t) < \infty$ thanks to the fact that $v_1, v_2, i_h, s_h, i_m, s_m \in (0,1)$ and therefore

$$\int_0^T \mid f(t, X_t, U_t) \mid dt < \infty, \tag{19}$$

i.e.

$$E\left[\int_0^T \mid f(t, X_t, U_t) \mid dt\right] < \infty. \tag{20}$$

We can say that (18) is a strong formulation.

2.4. Existence of optimal control adapted to filariasis states.

Theorem 2.1. *Let $X(0) \in (0,1)^5$ be such that there exists a control $U(.)$ satisfying (5). If the problem (18) is finite then there exists an optimal control $U^\star$ on $[0,T]$ such that the associated trajectory $X_{U^\star}$ satisfies (5) and minimizes the cost $J(.)$ defined by (15).*

Proof. It is obvious that the set $\mathbf{U} \subseteq [0,1]^2$ is convex because

$$D_f = \begin{vmatrix} \dfrac{\partial^2 f}{\partial v_1^2} & \dfrac{\partial^2 f}{\partial v_1 \partial v_2} \\ \dfrac{\partial^2 f}{\partial v_2 \partial v_1} & \dfrac{\partial^2 f}{\partial v_2^2} \end{vmatrix} = \begin{vmatrix} 2 i_m(t) s_h(t) & 0 \\ 0 & 2 i_h(t) s_m(t) \end{vmatrix} \tag{21}$$

$$D_f = 4 i_h(t) s_h(t) i_m(t) s_m(t) > 0 \text{ and } \frac{\partial^2 f}{\partial v_1^2} = 2 i_m(t) s_h(t) > 0 \tag{22}$$

car $i_h(t), i_m(t), s_h(t), s_m(t) \in (0,1)$. □

In addition, for any $U \in \mathbf{U}_{ad}$,

$$\sup_{0\le t\le T} E \mid U(t) \mid^m < \infty \text{ for all } m = 1, 2, ... \tag{23}$$

That is, $J(U(.))$ is finite, so the problem (15) is finite. According to the theorem on the existence of optimal control, we have the result.

2.5. **Characterisation of optimal control for filariasis.** To characterise the control which minimises the cost function (15), we use the stochastic maximum principle (for more details, see [24], [25] and [27] theorem 3.2 p.118) .

Theorem 2.2. *Let $U^\star$ be a control vector, solution of the optimal control problem (18). There are two applications $\{\mathcal{F}_t\}_{t\ge0}$-adapted $p(.) = (p_{s_h}(.), p_{e_h}(.), p_{i_h}(.), p_{s_m}(.), p_{i_m}(.)) : [0,T] \to \mathbf{R}^5$ et $q(.) = (q_{s_h}(.), q_{e_h}(.), q_{i_h}(.), q_{s_m}(.), q_{i_m}(.)) : [0,T] \to \mathcal{M}_5((0,1))$ such as*
$U^\star = \left(\dfrac{p_{e_h(t)} - p_{s_h(t)}}{2}, \dfrac{p_{i_m(t)} - p_{s_m(t)}}{2}\right)$.

Proof. Through the theorem (2.1), we have shown that the optimal control problem. If $U(.) = (v_1(.), v_2(.)) \in \mathbf{U}_{ad}$ is the optimal control on $[0,T]$ and X_U the trajectory associated with the solution of the equation (5), then by application of the maximum stochastic principle, there exist two $\{\mathcal{F}_t\{_{t\ge0}$-adapted $p(.) = (p_{s_h}(.), p_{e_h}(.), p_{i_h}(.), p_{s_m}(.), p_{i_m}(.)) : [0,T] \to \mathbf{R}^5$ and $q(.) = (q_{s_h}(.), q_{e_h}(.), q_{i_h}(.), q_{s_m}(.), q_{i_m}(.)) : [0,T] \to \mathcal{M}_5((0,1))$ absolutely continuous called adjoint vectors. For all $tin[0,T]$, the latter satisfy the first-order adjoint equations below and in this case, since the matrix function $M(t,X_t)$ does not depend on the control vector, we do not need to introduce second-order adjoint equations. That is, the assumption about the second-order differential of the functions F, M and f with respect to x is not necessary. To learn more about and formulate the first-order adjoint equation, see [28], [29] and [27].

$$dp(t) = -\frac{\partial H}{\partial X}\left(t, X_t, U_t, p_{s_h}, p_{e_h}, p_{i_h}, p_{s_m}, p_{i_m}\right) dt + q(t)dB(t), \tag{24}$$

where

$$\begin{aligned}
H\left(t, X_t, U_t, p_{s_h}, p_{e_h}, p_{i_h}, p_{s_m}, p_{i_m}\right) &= p(t).F(t, X_t, U_t) - f(t, X_t, U_t) \text{ i.e.} \\
H\left(t, X_t, U_t, p_{s_h}, p_{e_h}, p_{i_h}, p_{s_m}, p_{i_m}\right) &= p_{s_h}(t)\left[\mu_h - v_1 i_m(t)s_h(t) - \mu_h s_h(t)\right] \\
&\quad + p_{e_h}(t)\left[v_1 i_m(t)s_h(t) - (\lambda + \mu_h)e_h(t)\right] \\
&\quad + p_{i_h}(t)\left[\lambda e_h(t) - \mu_h i_h(t)\right] \\
&\quad + p_{s_m}(t)\left[\mu_m - v_2 i_h(t)s_m(t) - \beta_2 e_h(t)s_m(t) - \mu_m s_m(t)\right] \\
&\quad + p_{i_m}(t)\left[v_2 i_h(t)s_m(t) + \beta_2 e_h(t)s_m(t) - \mu_m i_m(t)\right] \\
&\quad - v_1^2 i_m(t)s_h(t) - v_2^2 i_h(t)s_m(t),
\end{aligned}$$

Thus,

$$\begin{cases} dp_{s_h}(t) = -\dfrac{\partial H}{\partial s_h}(t, X_t, U_t, p_{s_h}, p_{e_h}, p_{i_h}, p_{s_m}, p_{i_m})\,dt + q_{s_h}(t)dB_1(t) \\ dp_{e_h}(t) = -\dfrac{\partial H}{\partial e_h}(t, X_t, U_t, p_{s_h}, p_{e_h}, p_{i_h}, p_{s_m}, p_{i_m})\,dt + q_{e_h}(t)dB_2(t) \\ dp_{i_h}(t) = -\dfrac{\partial H}{\partial i_h}(t, X_t, U_t, p_{s_h}, p_{e_h}, p_{i_h}, p_{s_m}, p_{i_m})\,dt + q_{i_h}(t)dB_3(t) \\ dp_{s_m}(t) = -\dfrac{\partial H}{\partial s_m}(t, X_t, U_t, p_{s_h}, p_{e_h}, p_{i_h}, p_{s_m}, p_{i_m})\,dt + q_{s_m}(t)dB_4(t) \\ dp_{i_m}(t) = -\dfrac{\partial H}{\partial i_m}(t, X_t, U_t, p_{s_h}, p_{e_h}, p_{i_h}, p_{s_m}, p_{i_m})\,dt + q_{i_m}(t)dB_5(t) \\ p_{s_h}(T) = p_{e_h}(T) = p_{i_h}(T) = p_{s_m}(T) = p_{i_m}(T) = 0, \end{cases} \tag{25}$$

Set $\chi_t = (t, X_t, U_t, p_{s_h}, p_{e_h}, p_{i_h}, p_{s_m}, p_{i_m})$ for all $t \in [0, T]$.

$$\frac{\partial H}{\partial s_h}(\chi_t) = -p_{s_h}(t)\left[v_1 i_m(t) + \mu_h\right] + v_1 p_{e_h}(t) i_m(t) - v_1^2(t) i_m(t) \tag{26}$$

$$\frac{\partial H}{\partial e_h}(\chi_t) = -(\lambda + \mu_h) p_{e_h}(t) + \lambda p_{i_h}(t) - \beta_2 s_m(t) p_{s_m}(t) + \beta_2 s_m(t) p_{s_m}(t) \tag{27}$$

$$\frac{\partial H}{\partial i_h}(\chi_t) = -\mu_h p_{i_h}(t) - v_2 s_m(t) p_{s_m}(t) + v_2 s_m(t) p_{i_m}(t) - v_2^2 s_m(t) \tag{28}$$

$$\frac{\partial H}{\partial s_m}(\chi_t) = -p_{s_m}(t)\left[v_2 i_h(t) + \beta_2 e_h(t) + \mu_m\right] + p_{i_m}\left[v_2(t) i_h(t) + \beta_2 e_h(t)\right] - v_2^2 i_h(t) \tag{29}$$

$$\frac{\partial H}{\partial i_m}(\chi_t) = -v_1 s_h(t) p_{s_h}(t) + p_{e_h}(t) v_1 s_h(t) - \mu_m p_{i_m}(t) - v_1^2 s_h(t). \tag{30}$$

This results in the following adjoint system:

$$\begin{cases} dp_{s_h}(t) = \left[-p_{s_h}(t)(\mu_h + v_1 i_m(t)) + v_1 i_m(t) p_{s_h}(t) - v_1^2 i_m(t)\right] dt + q_{s_h}(t)dB_1(t) \\ dp_{e_h}(t) = \left[-(\lambda + \mu_h) p_{e_h}(t) + \lambda p_{i_h}(t) - \beta_2 s_m(t) p_{s_m}(t) + \beta_2 s_m(t) p_{i_m}(t)\right] dt + q_{e_h}(t)dB_2(t) \\ dp_{i_h}(t) = \left[-\mu_h p_{i_h}(t) - v_2 s_m(t) p_{s_m}(t) + v_2 s_m(t) p_{i_m}(t) - v_2^2 s_m(t)\right] dt + q_{i_h}(t)dB_3(t) \\ dp_{s_m}(t) = \left[p_{s_m}(t)\left(-v_2 i_h(t) - \beta_2 e_h(t) - \mu_m\right) + p_{i_m}(t)\left(v_2 i_h(t) + \beta_2 e_h(t)\right) - v_2^2 i_h(t)\right] dt + q_{s_m}(t)dB_4(t) \\ dp_{i_m}(t) = \left[-p_{i_m}(t) v_1 s_h(t) + p_{e_h}(t) v_1 s_h(t) - \mu_m p_{i_m}(t) - v_1^2 s_h(t)\right] dt + +q_{i_m}(t)dB_5(t) \\ p_{s_h}(T) = p_{e_h}(T) = p_{i_h}(T) = p_{s_m}(T) = p_{i_m}(T) = 0. \end{cases}$$

Optimal control theory allows us to obtain the control $U^*(.) = (v_1^*(..), v_2^*(..)) \in U_{ad}$ satisfying the sufficient optimality condition which holds almost everywhere on $[0, T]$.

$$H\left(t, X_t, U_t^*, p_{s_h}, p_{e_h}, p_{i_h}, p_{s_m}, p_{i_m}\right) = \max_{U_t \in [0, \beta_1, max] \times [0, \beta_2, max]} H\left(t, X_t, U_t, p_{s_h}, p_{e_h}, p_{i_h}, p_{s_m}, p_{i_m}\right)$$

This means finding the maxima at the points:

$$(0, v_2(.)), (1, v_2(.)), (v_1(.), 0), (v_1(.), 1) \text{ and } (v_1, v_2(.)) \text{ where } v_1(.), v_2(.) \in (0, 1) \tag{31}$$

and then compare them to find the overall maximum.

Searching for local maxima

For the point $U_t = (0, v_2(.))$, we have:

$$\begin{aligned} H(\chi_t) &= p_{s_h}(t)\left[\mu_h - \mu_h s_h(t)\right] - p_{e_h}(t)\left(\lambda+\mu_h\right)e_h(t) + p_{i_h}(t)\left(\lambda e_h(t) - \mu_h i_h(t)\right) && (32)\\ &+p_{s_m}(t)\left[\mu_m - v_2 i_h(t)s_m(t) - \beta_2 e_h(t)s_m(t) - \mu_m s_m(t)\right] && (33)\\ &+p_{i_m}(t)\left[v_2 i_h(t)s_m(t) + \beta_2 e_h(t)s_m(t) - \mu_m i_m(t)\right] - v_2^2 i_h(t)s_m(t). && (34)\end{aligned}$$

So

$$\begin{aligned} \frac{\partial H}{\partial v_2}(\chi_t) &= -p_{s_m}(t)i_h(t)s_m(t) + p_{i_m}(t)i_h(t)s_m(t) - 2v_2 i_h(t)s_m(t) && (35)\\ &= \left(p_{i_m}(t) - p_{s_m}(t)\right)i_h(t)s_m(t) - 2v_2(t)s_m(t)i_h(t). && (36)\end{aligned}$$

Therefore

$$\frac{\partial H}{\partial v_2}(\chi_t) = 0 \Leftrightarrow v_2 = \frac{p_{i_m}(t) - p_{s_m}(t)}{2}. \tag{37}$$

Hence $U_t^1 = \left(0, \dfrac{p_{i_m}(t) - p_{s_m}(t)}{2}\right)$ is a local maximum of H.
For the point $U_t = (1, v_2(.))$, we have:

$$\begin{aligned} H(\chi_t) &= p_{s_h}(t)\left[\mu_h - i_m(t)s_h(t) - \mu_h s_h(t)\right] + p_{e_h}(t)\left[i_m(t)s_h(t) - (\lambda+\mu_h)e_h(t)\right]\\ &+p_{i_h}(t)\left[\lambda e_h(t) - \mu_h i_h(t)\right] + p_{s_m}(t)\left[\mu_m - v_2 i_h(t)s_m(t) - \beta_2 e_h(t)s_m(t) - \mu_m s_m(t)\right]\\ &+p_{i_m}(t)\left[v_2 i_h(t)s_m(t) + \beta_2 e_h(t)s_m(t) - \mu_m i_m(t)\right] - i_m(t)s_h(t) - v_2^2 i_h(t)s_m(t).\end{aligned}$$

So

$$\frac{\partial H}{\partial v_2}(\chi_t) = -p_{s_m}(t)i_h(t)s_m(t) + p_{i_m}(t)i_h(t)s_m(t) - 2v_2 i_h(t)s_m(t). \tag{38}$$

It follows that

$$\frac{\partial H}{\partial v_2}(\chi_t) = 0 \Leftrightarrow v_2 = \frac{p_{i_m}(t) - p_{s_m}(t)}{2}. \tag{39}$$

Hence $U_t^2 = \left(1, \dfrac{p_{i_m}(t) - p_{s_m}(t)}{2}\right)$ is a local maximum of H.
For the point $U_t = (v_1(.), 0)$, we have:

$$\begin{aligned} H(\chi_t) &= p_{s_h}\left[\mu_h - v_1 i_m(t)s_h(t) - \mu_h s_h(t)\right] + p_{e_h}(t)\left[v_1 i_m(t)s_h(t) - (\lambda+\mu_h)e_h(t)\right] && (40)\\ &+p_{i_h}(t)\left[\lambda e_h(t) - \mu_h i_h(t)\right] + p_{s_m}\left[\mu_m - \beta_2 e_h(t)s_m(t) - \mu_m s_m(t)\right] && (41)\\ &+p_{i_m}(t)\left[\beta_2 e_h(t)s_m(t) - \mu_m i_m(t)\right] - v_1^2 i_m(t)s_h(t). && (42)\end{aligned}$$

So

$$\frac{\partial H}{\partial v_1}(\chi_t) = -p_{s_h}(t)i_m(t)s_h(t) + p_{e_h}(t)i_m(t)s_h(t) - 2v_1 i_m(t)s_h(t). \tag{43}$$

Therefore

$$\frac{\partial H}{\partial v_1}(\chi_t) = 0 \Leftrightarrow v_1 = \frac{p_{e_h}(t) - p_{s_h}(t)}{2}. \tag{44}$$

Hence $U_t^3 = \left(\frac{p_{e_h}(t) - p_{s_h}(t)}{2}, 0\right)$ is a local maximum of H.
For the point $U_t = (v_1(.), 1)$, we have:

$$\begin{aligned} H(\chi_t) &= p_{s_h}(t)\left[\mu_h - v_1 i_m(t)s_h(t) - \mu_h s_h(t)\right] + p_{e_h}(t)\left[v_1 i_m(t)s_h(t) - (\lambda+\mu_h)e_h(t)\right] && (45)\\ &+ p_{i_h}\left[\lambda e_h(t) - \mu_h i_h(t)\right] + p_{s_m}(t)\left[\mu_m - i_h(t)s_m(t) - \beta_2 e_h(t)s_m(t) - \mu_m i_m(t)\right] && (46)\\ &+ p_{i_m}(t)\left[i_h(t)s_m(t) + \beta_2 e_h(t)s_m(t) - \mu_m i_m(t)\right] - v_1^2 i_m(t)s_h(t) - i_h(t)s_m(t). && (47)\end{aligned}$$

So

$$\frac{\partial H}{\partial v_1}(\chi_t) = -p_{s_h}(t)i_m(t)s_h(t) + p_{e_h}(t)i_m(t)s_h(t) - 2v_1 i_m(t)s_h(t). \tag{48}$$

Therefore

$$\frac{\partial H}{\partial v_1}(\chi_t) = 0 \Leftrightarrow v_1 = \frac{p_{e_h}(t) - p_{s_h}(t)}{2}. \tag{49}$$

Hence $U_t^4 = \left(\frac{p_{e_h}(t) - p_{s_h}(t)}{2}, 1\right)$ is a local maximum of H.
For the point $U_t = (v_1(.), v_2(.))$, we have:

$$\begin{aligned} H(\chi_t) &= p_{s_h}(t)\left[\mu_h - v_1 i_m(t)s_h(t) - \mu_h s_h(t)\right] + p_{e_h}(t)\left[v_1 i_m(t)s_h(t) - (\lambda+\mu_h)e_h(t)\right]\\ &+ p_{i_h}(t)\left[\lambda e_h(t) - \mu_h i_h(t)\right] + p_{s_m}(t)\left[\mu_m - v_2 i_h(t)s_m(t) - \beta_2 e_h(t)s_m(t) - \mu_m s_m(t)\right]\\ &+ p_{i_m}(t)\left[v_2 i_h(t)s_m(t) + \beta_2 e_h(t)s_m(t) - \mu_m i_m(t)\right] - v_1^2 i_m(t)s_h(t) - v_2^2 i_h(t)s_m(t).\end{aligned}$$

So

$$\begin{aligned} \frac{\partial H}{\partial v_1}(\chi_t) &= -p_{s_h}(t)i_m(t)s_h(t) + p_{e_h}(t)i_m(t)s_h(t) - 2v_1 i_m(t)s_h(t),\\ \frac{\partial H}{\partial v_2}(\chi_t) &= -p_{s_m}(t)i_h(t)s_m(t) + p_{i_m}(t)i_h(t)s_m(t) - 2v_2 i_h(t)S_m(t).\end{aligned}$$

Therefore

$$\begin{aligned} \frac{\partial H}{\partial v_1}(\chi_t) = 0 \Leftrightarrow v_1 &= \frac{p_{e_h}(t) - p_{s_h}(t)}{2}, && (50)\\ \frac{\partial H}{\partial v_2}(\chi_t) = 0 \Leftrightarrow v_2 &= \frac{p_{i_m}(t) - p_{s_m}(t)}{2}. && (51)\end{aligned}$$

Hence $U_t^5 = \left(\frac{p_{e_h}(t) - p_{s_h}(t)}{2}, \frac{p_{i_m}(t) - p_{s_m}(t)}{2}\right)$ is a local maximum of H.

Searching for the global maximum

We look for the global maximum by comparing all the maxima. To do this, we calculate

$$H\left(t, X_t, U_t, p_{s_h}, p_{e_h}, p_{i_h}, p_{s_m}, p_{i_m}\right) \tag{52}$$

Set $\chi_t^i = \big(t, X_t, U_t^i, p_{s_h}, p_{e_h}, p_{i_h}, p_{s_m}, p_{i_m}\big)$ for $i = \overline{1,5}$. We have

$$\begin{aligned} H(\chi_t^5) &= p_{s_h}(t)\left[\mu_h - \frac{p_{e_h} - p_{s_h}}{2} i_m(t)s_h(t) - \mu_h s_h(t)\right] + p_{e_h}(t)\left[\frac{p_{e_h} - p_{s_h}}{2} i_m(t)s_h(t) - (\lambda + \mu_h)e_h(t)\right] \\ &\quad + p_{i_h}\left[\lambda e_h(t) - \mu_h i_h(t)\right] + p_{s_m}(t)\left[\mu_m - \frac{p_{i_m} - p_{s_m}}{2} i_h(t)s_m(t) - \beta_2 e_h(t)s_m(t) - \mu_m s_m(t)\right] \\ &\quad + p_{i_m}\left[\frac{p_{i_m} - p_{s_m}}{2} i_h(t)s_m(t) + \beta_2 e_h(t)s_m(t) - \mu_m i_m(t)\right] \\ &\quad - \frac{(p_{e_h} - p_{s_h})^2}{4} i_m(t)s_h(t) - \frac{(p_{i_m}(t) - p_{s_m}(t))^2}{4} i_h(t)s_m(t). \end{aligned}$$

Let's compare $H(\chi_t^5)$ and $H(\chi_t^1)$.

$$\begin{aligned} H(\chi_t^5) &= H(\chi_t^1) - \frac{p_{e_h}(t) - p_{s_h}(t)}{2} i_m(t)s_h(t)p_{s_h}(t) \\ &\quad + \frac{p_{e_h}(t) - p_{s_h}(t)}{2} i_m(t)s_h(t)p_{e_h}(t) - \frac{(p_{e_h} - p_{s_h})^2}{4} i_m(t)s_h(t) \\ &= H(\chi_t^1) + \frac{(p_{e_h}(t) - p_{s_h}(t))^2}{2} i_m(t)s_h(t)p_{s_h}(t) - \frac{(p_{e_h} - p_{s_h})^2}{4} i_m(t)s_h(t) \\ &= H(\chi_t^1) + \left[\frac{(p_{e_h}(t) - p_{s_h}(t))^2}{2} - \frac{(p_{e_h} - p_{s_h})^2}{4}\right] i_m(t)s_h(t) \\ &= H(\chi_t^1) + \frac{(p_{e_h} - p_{s_h})^2}{4} i_m(t)s_h(t). \end{aligned}$$

So

$$H(\chi_t^1) \leq H(\chi_t^5). \tag{53}$$

Let's compare $H(\chi_t^5)$ and $H(\chi_t^2)$.

$$\begin{aligned} H(\chi_t^5) &= H(\chi_t^2) - \frac{p_{e_h}(t) - p_{s_h}(t)}{2} i_m(t)s_h(t)p_{s_h}(t) \\ &\quad + \frac{p_{e_h}(t) - p_{s_h}(t)}{2} i_m(t)s_h(t)p_{e_h}(t) - \frac{(p_{e_h} - p_{s_h})^2}{4} i_m(t)s_h(t) \\ &\quad + i_m(t)s_h(t)p_{s_h}(t) - i_m(t)s_h(t)p_{e_h}(t) + i_m(t)s_h(t) \\ &= H(\chi_t^2) + \left[1 + p_{s_h}(t) - p_{e_h}(t) - \frac{p_{e_h}(t) - p_{s_h}(t)}{2} p_{s_h}(t)\right] i_m(t)s_h(t) \\ &\quad + \left[\frac{p_{e_h}(t) - p_{s_h}(t)}{2} p_{e_h}(t) - \frac{(p_{e_h} - p_{s_h})^2}{4}\right] i_m(t)s_h(t) \\ &= H(\chi_t^2) + \left[1 - (p_{e_h}(t) - p_{s_h}(t)) + \frac{(p_{e_h} - p_{s_h})^2}{4}\right] i_m(t)s_h(t) \\ &= H(\chi_t^2) + \left[1 - 2\frac{(p_{e_h}(t) - p_{s_h}(t))}{2} + \frac{(p_{e_h} - p_{s_h})^2}{4}\right] i_m(t)s_h(t) \\ &= H(\chi_t^2) + \left(\frac{p_{e_h} - p_{s_h} + 2}{2}\right)^2 i_m(t)s_h(t). \end{aligned}$$

So

$$H(\chi_t^2) \leq H(\chi_t^5). \tag{54}$$

Let's compare $H(\chi_t^5)$ and $H(\chi_t^3)$.

$$\begin{aligned}
H(\chi_t^5) &= H(\chi_t^3) - \frac{p_{i_m}(t) - p_{s_m}(t)}{2} i_h(t) s_m(t) p_{s_m}(t) \\
&\quad + \frac{p_{i_m}(t) - p_{s_m}(t)}{2} i_h(t) s_m(t) p_{i_m}(t) - \frac{(p_{i_m} - p_{s_m})^2}{4} i_h(t) s_m(t) \\
&= H(\chi_t^3) + \frac{(p_{i_m}(t) - p_{s_m}(t))^2}{2} i_h(t) s_m(t) p_{s_m}(t) - \frac{(p_{i_m} - p_{s_m})^2}{4} i_h(t) s_m(t) \\
&= H(\chi_t^3) + \left[\frac{(p_{i_m}(t) - p_{s_m}(t))^2}{2} - \frac{(p_{i_m} - p_{s_m})^2}{4}\right] i_h(t) s_m(t) \\
&= H(\chi_t^3) + \frac{(p_{i_m} - p_{s_m})^2}{4} i_h(t) s_m(t).
\end{aligned}$$

So

$$H(\chi_t^3) \leq H(\chi_t^5). \tag{55}$$

Let's compare $H(\chi_t^5)$ and $H(\chi_t^4)$.

$$\begin{aligned}
H(\chi_t^5) &= H(\chi_t^4) - \frac{p_{i_m}(t) - p_{s_m}(t)}{2} i_h(t) s_m(t) p_{s_m}(t) \\
&\quad + \frac{p_{i_m}(t) - p_{s_m}(t)}{2} i_h(t) s_m(t) p_{i_m}(t) - \frac{(p_{i_m} - p_{s_m})^2}{4} i_h(t) s_m(t) \\
&\quad + i_h(t) s_m(t) p_{s_m}(t) - i_h(t) s_m(t) p_{i_m}(t) + i_h(t) s_m(t) \\
&= H(\chi_t^4) + \left[1 + p_{s_m}(t) - p_{i_m}(t) - \frac{p_{i_m}(t) - p_{s_m}(t)}{2} p_{s_m}(t)\right] i_h(t) s_m(t) \\
&\quad + \left[\frac{p_{i_m}(t) - p_{s_m}(t)}{2} p_{i_m}(t) - \frac{(p_{i_m} - p_{s_m})^2}{4}\right] i_h(t) s_m(t) \\
&= H(\chi_t^2) + \left[1 - (p_{i_m}(t) - p_{s_m}(t)) + \frac{(p_{i_m} - p_{s_m})^2}{4}\right] i_h(t) s_m(t) \\
&= H(\chi_t^4) + \left[1 - 2\frac{(p_{i_m}(t) - p_{s_m}(t))}{2} + \frac{(p_{i_m} - p_{s_m})^2}{4}\right] i_h(t) s_m(t) \\
&= H(\chi_t^4) + \left(\frac{p_{i_m} - p_{s_m} + 2}{2}\right)^2 i_h(t) s_m(t).
\end{aligned}$$

So

$$H(\chi_t^4) \leq H(\chi_t^5). \tag{56}$$

From (53), (54), (55) and (56) it follows that

$$H(\chi_t^i) \leq H(\chi_t^5)\ \forall i \in \{1, 2, 3, 4\}. \tag{57}$$

So the global maximum is

$$U_t^5 = \left(\frac{p_{e_h} - p_{s_h}}{2}, \frac{p_{i_m} - p_{s_m}}{2} \right). \tag{58}$$

□

2.6. **Numeric simulation.** The numerical simulations of the optimality system (5), along with the corresponding results obtained by varying the optimal controls v_1 and v_2, the choice of parameters, and the interpretations of the different cases, are now discussed. The numerical solutions are illustrated using MATLAB and the technique developed in [30]. The values of the white noise intensities are

$$\sigma_1 = 0.3,\ \sigma_2 = 0.125,\ \sigma_3 = 0.225,\ \sigma_4 = 0.225.$$

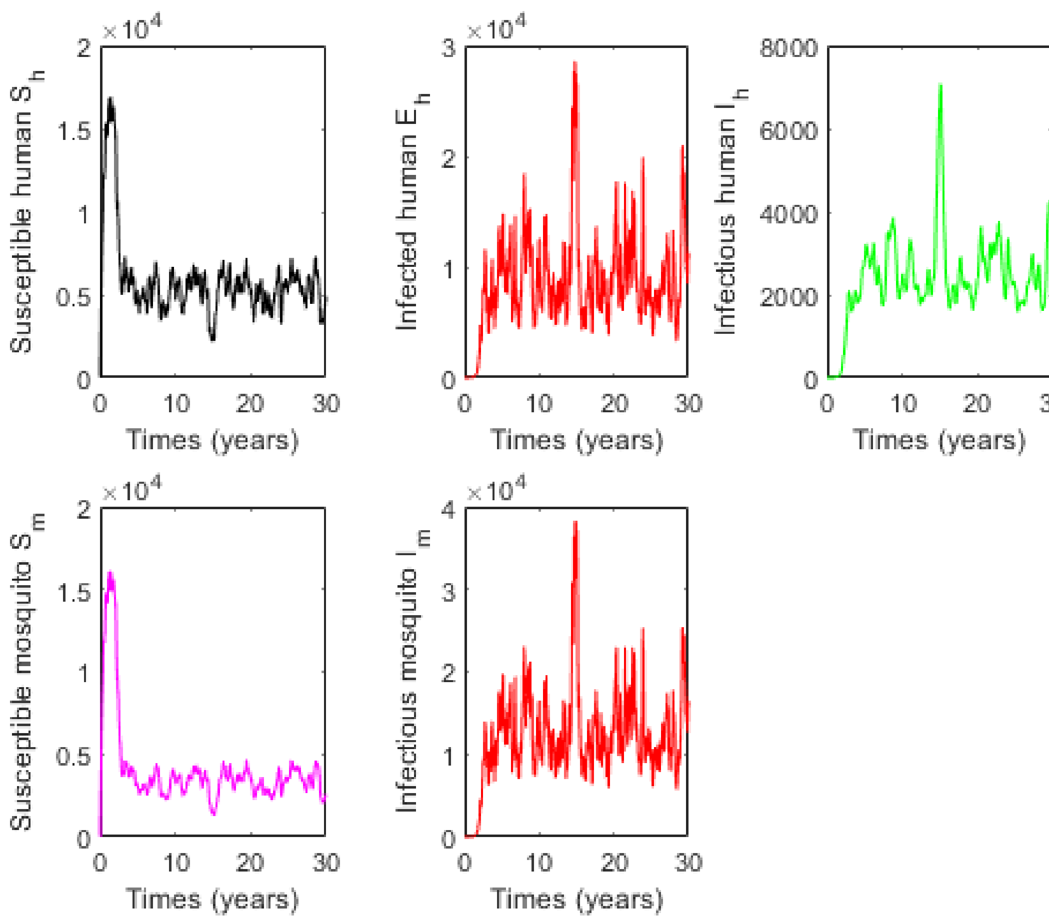

Figure 2. Graphs showing the transmission dynamics of Filarisis for $v_1 = 0.8$ and $v_2 = 0.08$.

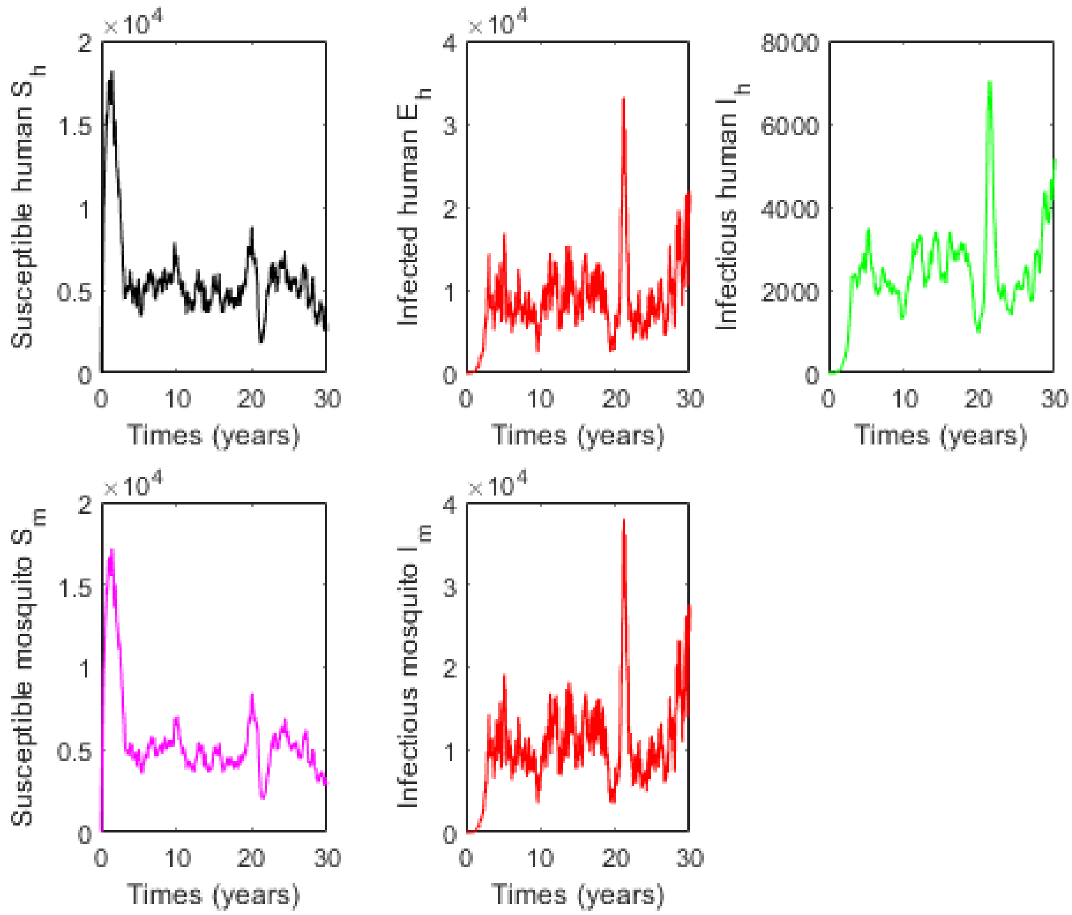

FIGURE 3. Graphs showing the transmission dynamics of Filarisis for $v_1 = 0.5$ and $v_2 = 0.1$.

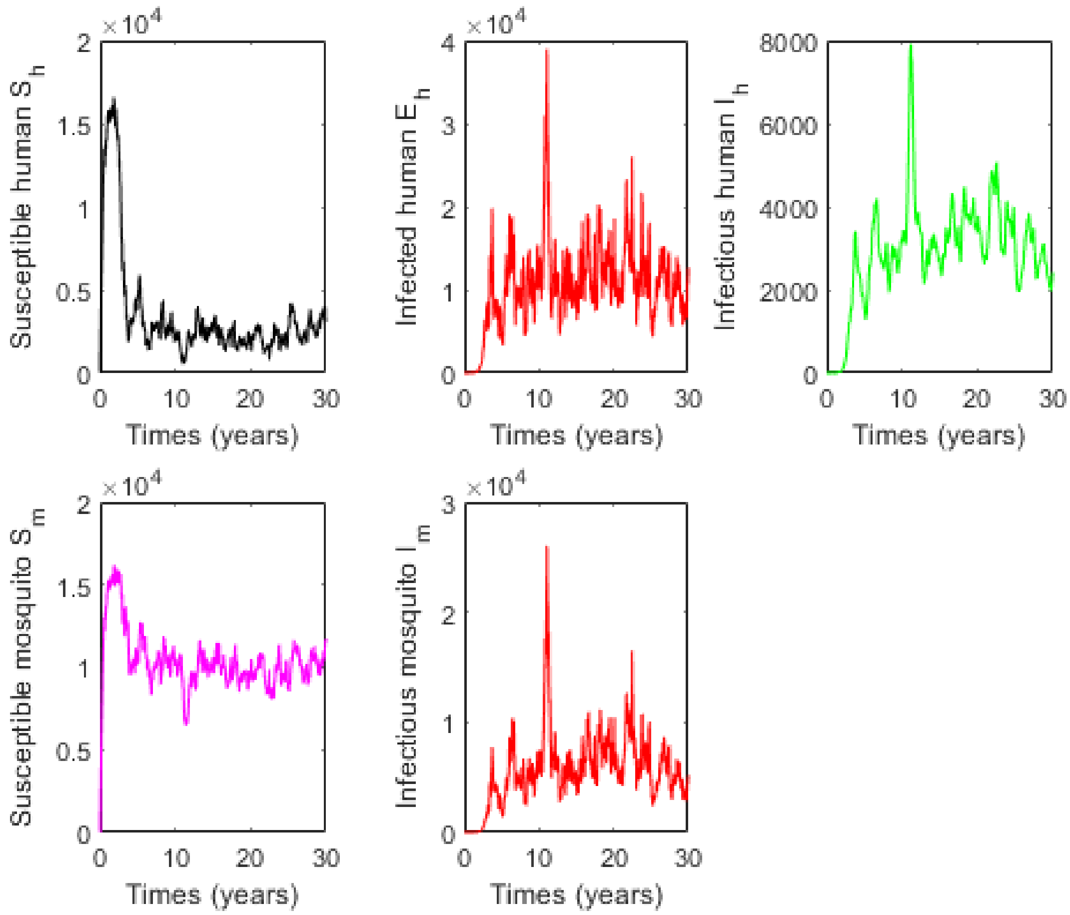

Figure 4. Graphs showing the transmission dynamics of Filarisis for $v_1 = 0.2$ and $v_2 = 0.2$.

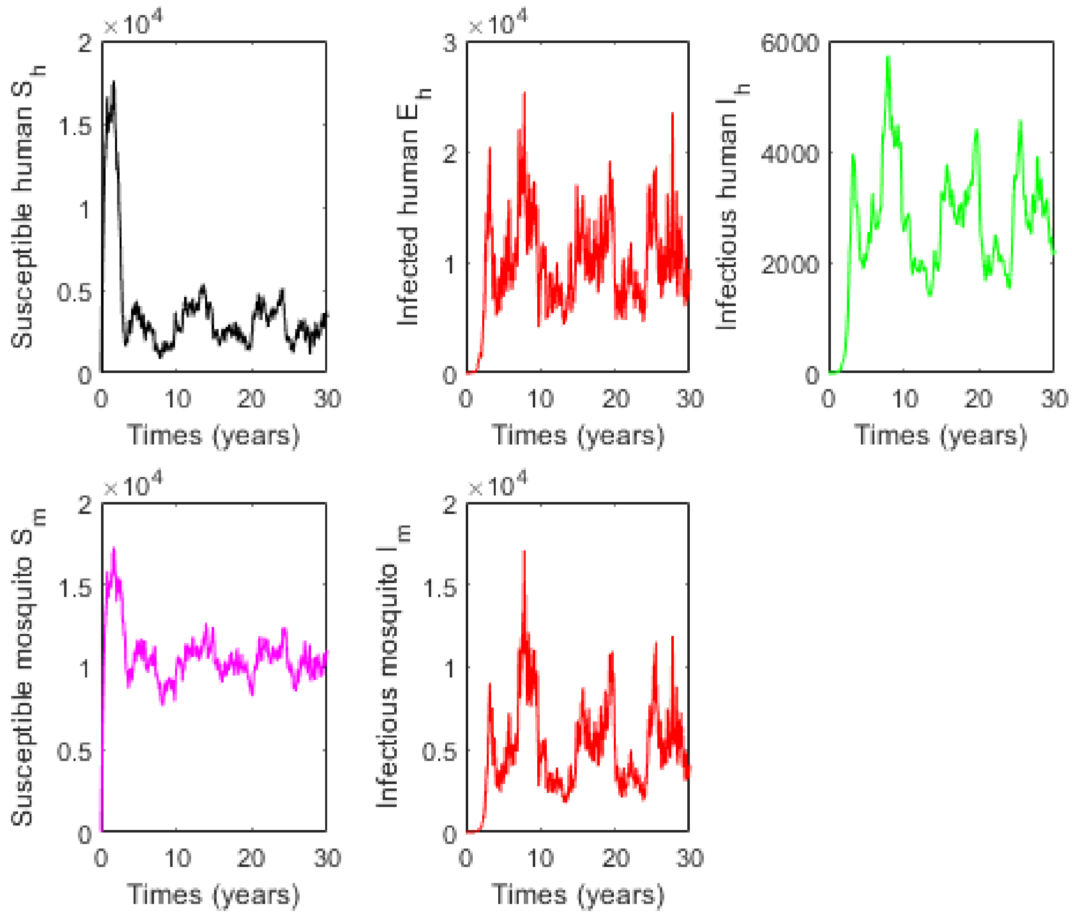

FIGURE 5. Graphs showing the transmission dynamics of Filarisis for $v_1 = 0.1$ and $v_2 = 0.5$.

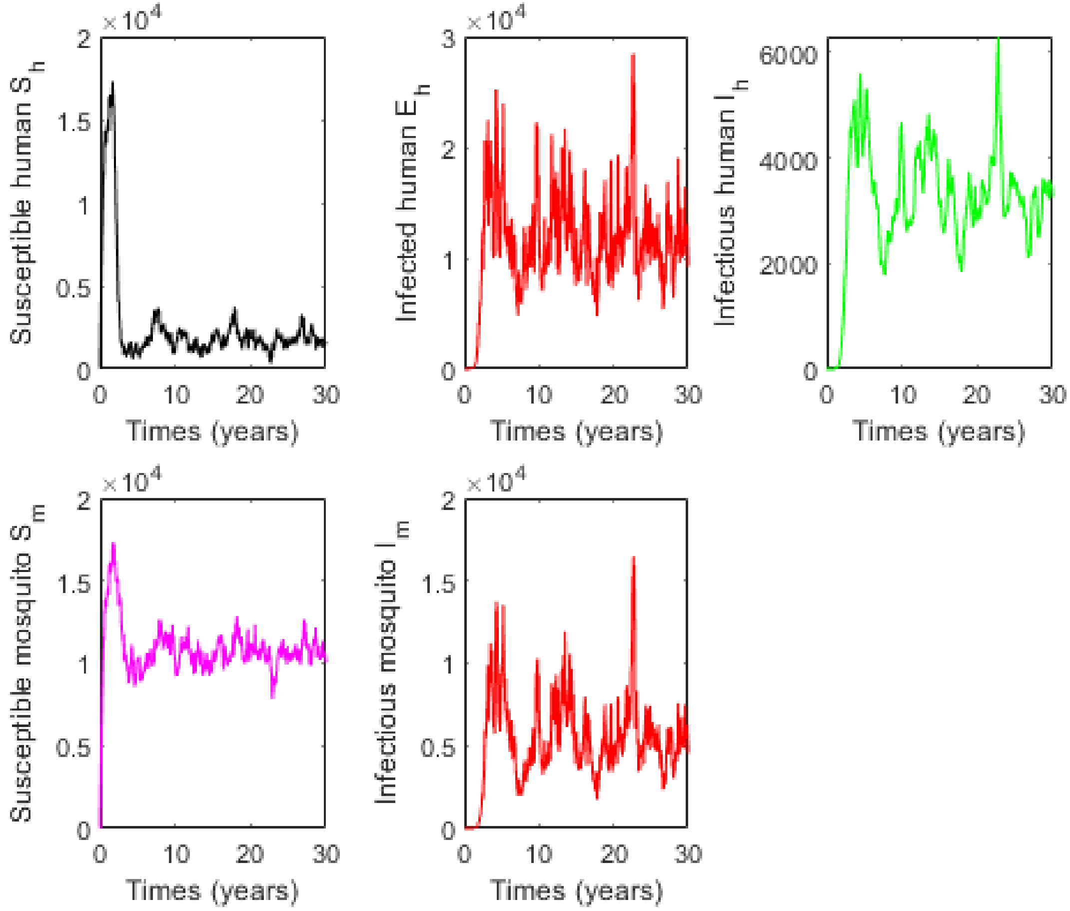

FIGURE 6. Graphs showing the transmission dynamics of Filarisis for $v_1 = 0.08$ and $v_2 = 0.8$.

As the v_1 control approaches 0 and the v_2 control approaches 1, Figures 2 − 6 show a significant increase in the number of infectious individuals in the human population. Therefore, we can conclude that control v_1 is more effective than control v_2 in the eradication of Lymphatic Filariasis.

Tables

Tables 3 give descriptions and values of the parameters used in the numerical simulation.

Table 3. Model parameters and their interpretations

Parameters	**Biological description**	**Values**	**Source**
Λ_h	constant recruitment rate of human (it also includes births).	$0.00242.10^4$	[1,31]
β_h	rate of passage from susceptible humans to infectious humans through blood transfusion	0.015	Assumed
μ_h	mortality rate on human	0.8	Assumed
λ	rate of passage from latent human to infectious human	0.01	[1,31]
θ_h	modification parameter	0.25	[1]
Λ_m	constant recruitment rate of mosquitoes	$4.227.10^4$	[1,31]
β_m	average number of mosquito bites that cause transmission of disease from infectious mosquito to susceptible human per mosquito	0.091	[1,31]
μ_m	Mosquito mortality rate	3.623	[1,31]

3. Conclusion

In this article, we have presented a stochastic model of lymphatic filariasis. Lymphatic filariasis is one of the most serious diseases in the world and still lacks adequate treatment. Therefore, using stochastic theory, we developed a model of lymphatic filariasis that incorporates random effects. We applied stochastic control theory to formulate the stochastic optimal control problem for the proposed model, establish the existence of an adapted optimal control, and characterize it. The originality of our article, compared to existing papers on lymphatic filariasis, lies in the addition of white noise and the introduction of control into the stochastic model. Finally, we performed numerical simulations to further illuminate our results. In the future, we plan to introduce controls to the parameters dependent on the diffusion part in order to reduce the number of people infected with lymphatic filariasis.

Authors' Contributions. All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Acknowledgements. The authors express their deepest thanks to two anonymous referees for their valuables comments which contributed to improve the paper.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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