Asia Pac. J. Math. 2025 **12**:22

# RESULTS ON ISOLATE RINGS DOMINATION IN SOME GRAPHS

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Received Aug. 21, 2024

Abstract. The set $S \subseteq V(G)$ is said to be an isolate rings dominating set of a graph $G$ if it is both an isolate dominating set of $G$ and a rings dominating set of $G$. The minimum cardinality of an isolate rings dominating set is called the isolate rings domination number of $G$ and is denoted by $\gamma_{0ri}(G)$. An isolate rings dominating set $S$ of $G$ is called $\gamma_{0ri}$-set of $G$. This study examined the isolate rings dominating set of certain graphs, including those derived from the join and corona of two graphs. In addition, the isolate rings domination numbers for these graphs were determined, and graphs without isolate rings dominating sets were also explored.

2020 Mathematics Subject Classification. 05C69; 05C38.

Key words and phrases. domination; isolate domination; rings domination; isolate rings domination.

## 1. Introduction

In 1958, Claude Berge [4] introduced the concept of the coefficient external stability, which eventually became known as domination. From then, the concept of domination in graphs has expanded and became phenomenal until today due to its richness in research investigations and application in various physical models. Due to the numerous applications and opportunities, the science of domination began to grow for research purposes.

When Ariola [2] established the binary operation of isolate domination in graphs, Hamid and Balamurugan [13] instantly developed and generalized the isolate domination. This idea opened up a wide range of study opportunities, and as a result, many graph theorists have emerged and made some interesting studies about the variants of isolate domination. Rajesekar and Kani Bala in [16] presented the k-isolate domination number of total graphs, and Armada and Hamja in [3] explored the perfect isolate domination in graphs where they introduced the dominating set to be both perfect and isolate

DOI: 10.28924/APJM/12-22

at the same time. Caay and Hernandez [6] studied the Perfect equitable isolate domination of some graphs where they gave characterizations of a dominating set to be perfect, equitable and isolate at the same time as an extension of the work of Caay and Arugay in [7] about the perfect equitable domination in graphs. Additionally, Caay, et.al in [8], studied the notion of equitable isolate domination in graphs.

On the other hand, a new concept of domination, known as Rings domination in graphs, has been introduced by Abed and Al-Here [1]. As one of the latest developments in the field, there is currently limited literature available on this topic. However, interest in this concept is growing. For instance, Caay [5] recently introduced the notion of Equitable rings domination in graphs, where the dominating set is defined to be both equitable and rings at the same time, and the same author together with Dondoyano in [9] studied the notion of perfect rings domination in graphs where a dominating set needs to be perfect and rings at the same time. Additionally, Caay and Necessito [15] have recently presented a study that explores and characterizes dominations in graphs as rings convex.

This research aims to introduce the concept of Isolate rings domination in graphs, where the set serves as both an isolate dominating set and a rings dominating set. This study aims to investigate the presence of this concept in different graphs and provides characterizations of these graphs. Furthermore, the paper intends to extend the results to binary operations, including the join and corona product of graphs.

## Preliminaries and Working Definitions

Unless otherwise stated, this study will consider a graph $G = (V(G), E(G))$ to be connected simple graph. In other words, the graphs have no loops and multiple edges. The set of vertices of $G$ and the set of edges of $G$ are denoted by $V(G)$ and $E(G)$, respectively.

Given $v \in V(G)$, the neighborhood of $v$ is the set $N_G(v) = \{u \in V(G) : uv \in E(G).\}$. Given $D \subseteq V$, the set $N_G(D) = N(D) = \bigcup_{v\in D} N_G(v)$ and the set $N_G[D] = N[D] = D \bigcup N(D)$ are the *open neighborhood* and the *closed neighborhood* of $D$, respectively.

In this paper, we denote $\Delta(G)$ and $\delta(G)$ to be the minimum and maximum degree of $G$, respectively.

**Theorem 1.** [10] *A graph $G$ is a cyclic graph if and only if every vertex of $G$ is adjacent to two vertex.*

**Definition 1.** [10] *A **spanning subgraph** of a graph $G$ is a subgraph obtained by deleting some edges of $G$ with the same vertex set.*

**Example 1.** *A cycle $C_n$ is a spanning subgraph of a complet graph $K_n$.*

**Definition 2.** [14] *The **join** $G + H$ of the two graphs $G$ and $H$ is the graph with vertex.*

$$V(G + H) = V(G) + V(H)$$

*and the edge set*

$$E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$$

**Definition 3.** [11] *The **corona** $G \circ H$ of two graphs $G$ and $H$ is the graph obtained by taking one copy of $G$ of order $n$ and $n$ copies of $H$, and then joining the $i$th vertex of $G$ to every vertex in the $i$th copy of $H$.*

*In [4], a subset $S$ of $V(G)$ is a dominating set of $G$ if for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $uv \in E(G)$. That is, $N[S] = V(G)$. The minimum cardinality of the dominating set $S$ of $G$ is called a domination number of $G$ and is denoted by $\gamma(G)$. In this case, $S$ is called **$\gamma$**-set of $G$. In this paper, if $u$ is in a dominating set $S$, and $v \in V(G) \setminus S$ such that $uv \in E(G)$, then we say $u$ dominates $v$.*

**Theorem 2.** [3] *Let $G_1$ and $G_2$ be any connected graphs. Then*

$$\gamma(G_1 + G_2) = \begin{cases} 1, & \gamma(G_1) = 1 \text{ or } \gamma(G_2) = 1 \\ 2, & \gamma(G_1) \neq 1 \text{ and } \gamma(G_2) \neq 1. \end{cases}.$$

In [13], a dominating set $S \subseteq V(G)$ is said to be an *isolate dominating set* of $G$ if there exists $u \in S$ such that $uv \notin E(G)$ for all $v \in S$. The minimum cardinality of an equitable dominating set $S$ of $G$ is called an *isolate domination number* of $G$ and is denoted by $\gamma_0(G)$. In this case, we say $S$ a $\gamma_0$-set of $G$. In this paper, if $u$ is in an isolate dominating set $S$, and $v \in V(G) \setminus S$ such that $uv \in E(G)$, then we say $u$ *isolate dominates* $v$.

**Theorem 3.** [13] *A dominating set $S$ of $G$ is a minimal dominating set if and only if for every $u \in S$, $u$ is an isolate of $\langle S \rangle$. In particular, $S = \{u\}$ is a $\gamma$-set of $G$ if and only if $S$ is a $\gamma_0$-set of $G$.*

In [1], a dominating $S$ of $G$ is a *rings dominating set* of $G$ if every $v_1 \in V(G) \setminus S$ is adjacent to at least two other vertices in $V(G) \setminus S$. The minimum cardinality of the rings dominating set $S$ of $G$ is called a *rings domination number* of $G$ and is denoted by $\gamma_{ri}(G)$. In this case, $S$ is called **$\gamma$**-*set* of $G$. In this paper, if $u$ is in a rings dominating set $S$, and $v \in V(G) \setminus S$ such that $uv \in E(G)$, then we say $u$ *rings dominates* $v$.

**Remark 1.** [1] *For a $\gamma_{ri}$-set $S$ of $G$ of order $n$, we have*

i. *the order of $G$ is $n \geq 4$.*
ii. *for each $v \in V(G) \setminus S$, the $\deg(v) \geq 3$.*
iii. $1 \leq |S| \leq n-1$.
iv. $3 \leq |V(G) \setminus S| \leq n-1$.
v. $1 \leq \gamma_{ri}(G) \leq |S| \leq n-3$.

In the context of domination, there are some graphs at which the isolate domination number and the rings domination number coincide. Consider a complete graph of order 5 in Figure 3. Note $\{a\}$ is a $\gamma_0$-set of $K_5$ and it is also a $\gamma_{ri}$-set of $K_5$. Thus, $\gamma_0(K_5) = \gamma_{ri}(K_5) = 1$.

Now, the working definition is established in the event that $\gamma_0(G) = \gamma_{ri}(G)$ for any graph $G$.

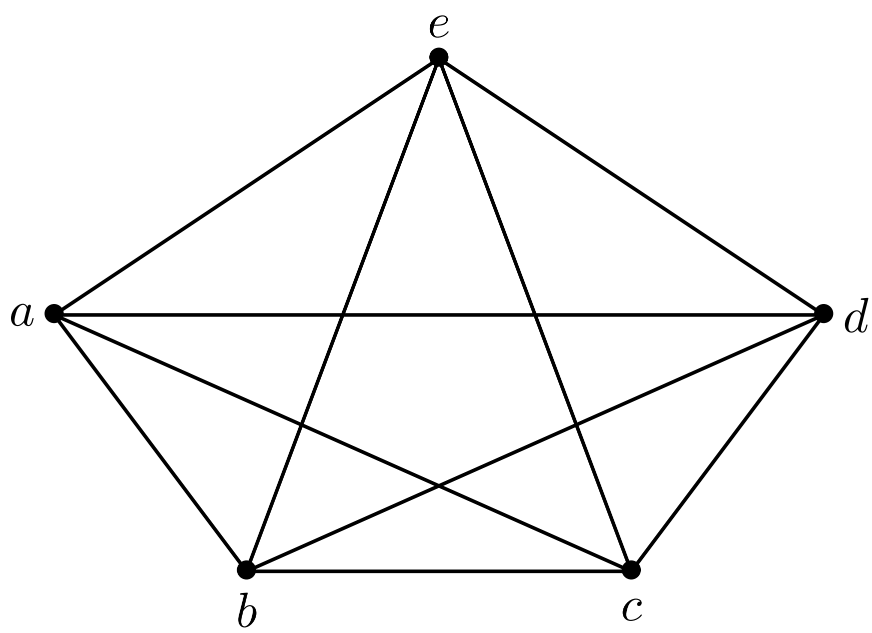

FIGURE 1. A complete graph of order 5.

**Definition 4.** *A dominating set $S \subsetneq V(G)$ is an* ***isolate rings dominating set*** *of $G$ if there exists $u \in S$ such that $uv \notin E(G)$ for all $v \in S$, and for every $w_i \in V(G) \setminus S$, there exists $w_j, w_k \in V(G) \setminus S$ such that $w_iw_j, w_iw_k \in E(G)$, $i \neq j \neq k$.*

**Remark 2.** *If $u$ is an isolate rings dominating set $S$, and $v \in V(G) \setminus S$ such that $uv \in E(G)$, then we say $u$* ***isolate rings dominates*** *$v$.*

**Example 2.** *Consider the graph $G$ in Figure 2. Note that $\{a, g\}$ is a. $\gamma_0$-set of $G$. Also, $V(G) \setminus \{a, g\} = \{b, c, d, e, g\}$, and each element of $V(G) \setminus \{a, g\}$ is adjacent to at least two elements in $V(G) \setminus \{a, g\}$. Thus $\{a, g\}$ is also a $\gamma_{ri}$-set of $G$. Hence, $\{a, g\}$ is a $\gamma_{0ri}$-set of $G$. Therefore, $\gamma_{0ri}(G) = 2$.*

## 2. Isolate Rings Domination in Graphs

The following results are easy to prove.

**Theorem 4.** *Let $G$ be a graph of order $n$. Then $\gamma_{0ri}(G) = 1$ if and only if there exists $u \in V(G)$ such that $deg(u) = n - 1$ and $deg(v) \geq 3$, for all $v \in V(G)$ with $u \neq v$.*

**Example 3.** *Consider the graph $G$ in Figure 3.*

*Note that $deg(b) = 4$ and $deg(a) = deg(c) = deg(d) = deg(e) = 3$. Thus, $\{b\}$ is a $\gamma_{0ri}$-set and so, $\gamma_{0ri}(G) = 1$.*

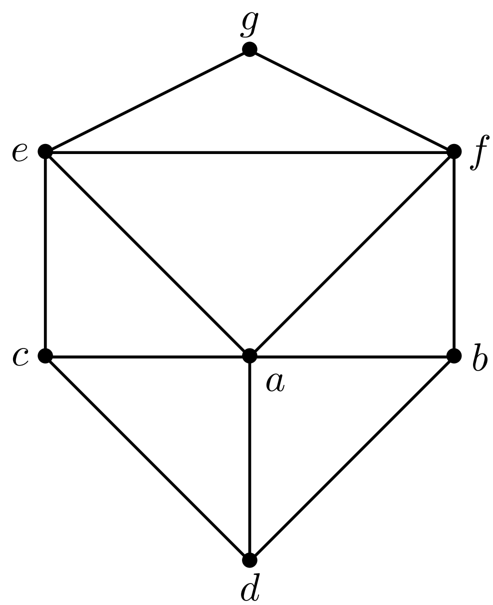


FIGURE 2. Graph $G$

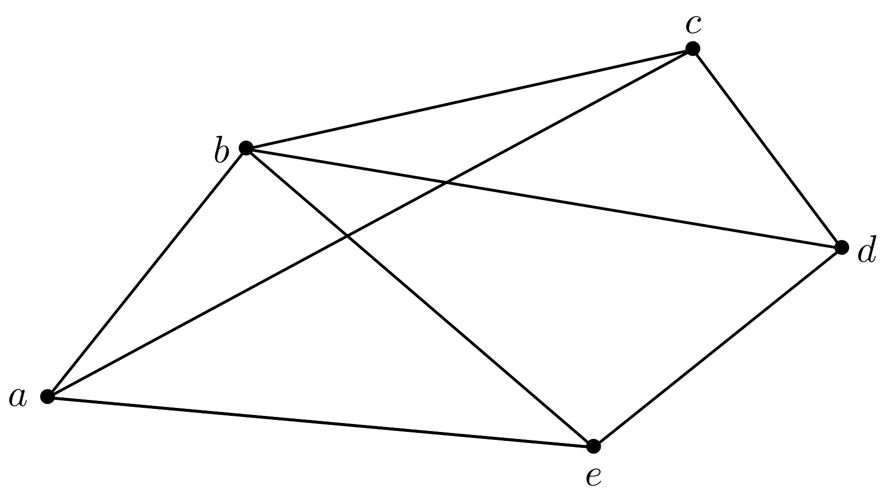


FIGURE 3. Graph $G$

**Theorem 5.** *If $S$ is a dominating set, or an isolate dominating set, or a rings dominating set of $G$ with $|S| = 1$, then $S$ is an isolate rings dominating set of $G$. In particular, $\gamma(G) = \gamma_0(G) = \gamma_{ri}(G) = 1$ if and only if $\gamma_{0ri}(G) = 1$.*

**Theorem 6.** *There does not exist a $\gamma_{0ri}$-set of a tree graph $T_n$ of order $n$. In particular, no $\gamma_{0ri}$-set exists in path $P_n$.*

**Proposition 1.** *There does not exist a $\gamma_{0ri}$-set of a cycle $C_n$*

**Proposition 2.** *Given a wheel graph $W_n$, $n \geq 4$, $\gamma_{0ri}(W_n) = 1$.*

The next results follow directly from the definition of perfect isolate dominating set.

**Theorem 7.** *Let $S \subseteq V(G)$ be a rings dominating set of $G$. Then $S$ is an isolate rings dominating set if and only if there exists a partition $S_i \subset S$ such that $S_i \cap S_j = \varnothing$, for all partitions $S_j \subset S$.*

**Proposition 3.** *If $\gamma_{ri}(G) = c$ where $c \in \mathbb{Z}^+$ and if $S$ is a $\gamma_{ri}$-set of $G$ such that $\langle S \rangle$ has an isolated vertex, then $\gamma_{0ri}(G) = c$.*

**Proposition 4.** *If $\gamma_0(G) = c$ where $c \in \mathbb{Z}^+$ and if $S$ is a $\gamma_0$-set of $G$ such that $V(G) \setminus S$ has an induced subgraph $G' \cong C_k$, then $\gamma_{0ri}(G) = c$.*

**Theorem 8.** *Let $G = K_{P_1,\cdots,P_k}$ be a complete $k$-partite graphs such that $k \geq 4$. Then*

$$\gamma_{0ri}(G) = \min\{|P_i| : P_i \textit{ is the vertex partition of } G\}.$$

*Proof.* Let $P_t$ be a partition of $G = K_{P_1,\cdots,P_k}$, $k \geq 4$, such that

$$|P_t| = \min\{|P_i| \ : \ P_i \text{ is a partition of } G\}.$$

Let $u_i^{(t)}$ be a vertex $u_i$ in the partition $P_t$. Then $u_i^{(t)}u_j^{(s)} \in E(G)$, for all $j = 1, 2, \cdots, |P_i|$, and for all $s = 1, \cdots, k$, $s \neq t$. Note the all vertices $u_j^{(s)} \neq u_i^{(t)}$ is adjacent to all vertices in $P_q$, $q \neq s$ and $q \neq t$. Since $k \geq 4$, it follows that $\deg\left(u_j^{(s)}\right) \geq 2$ in $V(G) \setminus P_t$. This means that $u_i^{(t)}$ rings dominates $u_j^{(s)}$, $s \neq t$. Since $u_i^{(t)}u_h^{(t)} \notin E(G)$, for all $u_h^{(t)} \in P_t$, $h \neq i$, it implies that $u_i^{(t)}$ isolates from all other vertices $u_h^{(t)}$. Hence, $P_t$ isolates rings dominates all vertices in $V(G) \setminus P_t$. Since $|P_t| \leq |P_s|$, for all $s \neq t$, $P_t$ is the minimal $\gamma_{0ri}$-set of $G$. Therefore, $\gamma_{0ri}(G) = |P_t|$. This proves the claim. □

**Corollary 1.** *Given $G = K_{P_1,\cdots,P_k}$ with $k \geq 4$, if there exists a vertex partition $P_i$ for some $i$ such that $|P_i| = 1$, then $\gamma_{0ri}(G) = 1$.*

**Corollary 2.** *There does not exist $\gamma_{0ri}$-set of $G = K_{P_1,\cdots,P_k}$ for $k \leq 3$, with at least two partitions contain one vertex.*

## 3. Binary Operations

**Theorem 9.** *Let $G_1$ and $G_2$ be any graphs with $\gamma(G_1) = 1$ or $\gamma(G_2) = 1$. Then $\gamma_{0ri}(G_1 + G_2) = 1$.*

*Proof.* By Theorem 2, $\gamma(G_1 + G_2) = 1$. By Theorem 5, $\gamma_{0ri}(G_1 + G_2) = 1$. □

**Corollary 3.** *Let $G_1$ and $G_2$ be arbitrary graphs such that they don't have $\gamma_{0ri}$-sets. Then $G_1 + G_2$ has a $\gamma_{0ri}$-set if and only if either $G_1$ is a trivial graph and $\delta(G_2) \geq 2$, or $G_2$ is a trivial graph and $\delta(G_1) \geq 2$.*

The following propositions are obvious results.

**Proposition 5.** *$\gamma_{0ri}(G) = 1$ for the following graphs $G$:*

i. *Wheel graph, $G = W_n = K_1 + C_{n-1}$, $n \geq 4$.*
ii. *Windmill graph $G = W_m^n = K_1 + nC_{n-1}$, $m \geq 4$.*

**Proposition 6.** *There does not exist a $\gamma_{0ri}$-set of a star graph, $S_n = K_1 + \overline{K_{n-1}}$, $n \geq 3$.*

*Proof.* Let $S_1$ be a $\gamma_{0ri}$-set of $G_1$. Then there exists $u_1 \in S_1$ such that $u_1u_i \notin E(G_1)$, for all $u_i \in S_1$ and $i \neq 1$. Furthermore, there exists $v_1 \in V(G_1) \setminus S_1$ such that $v_1v_k, v_1v_j \in E(G_1)$, for some $v_k, v_j \in V(G_1) \setminus S_1$, $j \neq k$. Similarly, if $S_2$ is a $\gamma_{0ri}$-set of $G_1$, then there exists $x_1 \in S_2$ such that $x_1x_i \notin E(G_2)$, for all $x_i \in S_2$ and $i \neq 1$. Moreover, there exists $y_1 \in V(G_2) \setminus S_2$ such that $y_1y_s, y_1y_t \in E(G_1)$, for some $y_s, y_t \in V(G_2) \setminus S_2$, $s \neq t$.

Consider $u_j \in S_1$. Then $u_j$ dominates $x_j \in V(G_2)$. If $x_j \in V(G_2) \setminus S_2$, then $x_j$ is adjacent to at least two vertices in $V(G_2) \setminus S_2$. In this case, $u_j$ isolate rings dominates $x_j$. If $x_j \in S_2$, then $x_j$ is adjacent to at least one vertex in $y_h \in V(G_2) \setminus S_2$. But there exists $v_k \in V(G_1) \setminus S1$ such that $v_ky_h \in E(G_1 + G_2)$, for some $k$. This means $x_j$ is adjacent to at least two vertices outside $S_1$. That is, $u_j$ isolate rings dominates $x_j$. Hence $S_1$ is enough to isolate rings dominate all other vertices in $G_1 + G_2$ outside $S_1$, whenever $|S_1| \leq |S_2|$. Therefore, $S_1$ is the minimum $\gamma_{0ri}$-set of $G_1 + G_2$. Consequently, $\gamma_{0ri}(G_1 + G_2) = \min\{|S_1|, |S_2|\}$. □

**Theorem 10.** *Let $G_1$ and $G_2$ be any graphs, and let $S$ be a $\gamma_{0ri}$-set of $G_2$ such that $S \subsetneq V(G_2)$. Then there exists a $\gamma_{0ri}$-set of $G_1 \circ G_2$. Moreover, $\gamma_{0ri}(G_1 \circ G_2) \leq |V(G_2)||S|$.*

*Proof.* Let $S \subsetneq V(G_2)$ be a $\gamma_{0ri}$-set of $G_2$. Then there exists $u_i \in V(G_2) \setminus S$ such that $u_iv_j, u_iw_k \in E(G_2)$, for some $v_j, w_k \in V(G_2) \setminus S$. Let $g_i \in V(G_1)$. Then for some $i$, $g_iu_j(i) \in E(G_1 \circ G_2)$ for all vertices $u_j$ in some $i$th copy of $G_2$. This means that if $S^{(i)}$ is the $i$th copy of $\gamma_{0ri}$-set $S$ of $G_2$, then $g_i$ is dominated by $g' \in S^{(i)}$. Since $S^{(i)}$ is a $\gamma_{0ri}$-set of the $i$th copy of $S$, there exist $u_i, v_j, w_k \in \left[S^{(i)}\right]^c$. Hence $g'$ isolate rings dominates $g$. This means $S^{(i)}$ isolate rings dominates $g_i \in V(G_1)$, for all $i = 1, 2, \cdots, |V(G_1)|$. Hence, $\bigcup_{i=1}^{|V(G_1)|} S^{(i)}$ is a $\gamma_{0ri}$-set of $G_1 \circ G_2$. Consequently, we have

$$\gamma_{0ri}(G_1 \circ G_2) \leq \left| \bigcup_{i=1}^{|V(G_1)|} S^{(i)} \right| = |V(G_1)| \cdot |S|.$$

This proves the claim. □

**Theorem 11.** *Let $G_1$ and $G_2$ be any graphs of orders at least four such that $\Delta(G_2) = |V(G_2)| - 1$ and $\delta(G_2) \geq 2$. Then $G_1 \circ G_2$ has a $\gamma_{0ri}$-set. Moreover, $\gamma_{0ri}(G_1 \circ G_2) = |V(G_1)|$.*

*Proof.* If $\Delta(G_2) = |V(G_2)| - 1$, then there exists only one vertex $u \in V(G_2)$ such that $u_i \in E(G_2)$, for all $u_i$, $i = 1, 2, \cdots, |V(G_2)| - 1$. Since $\delta(G_2) \geq 2$, it follow that $\deg(u_i) \geq 2$, for all $u_i \neq u$. Hence, each

$u_i$ is adjacent to atleast one $u_j \in V(G_2) \setminus \{u\}$, $i \neq j$. Thus, each $u_i$ is adjacent to atleast two vertices in $V(G_1 \circ G_2) \setminus \{u\}$. If $w_k^{(i)} \in V(G_1)$, then $w_k^{(i)}$ is adjacent to all vertices in the $i$th copy of $G_2$. Hence, $u$ isolate rings dominates $u_i$ and $w_k$. Since there are $|V(G_1)|$ copies of $u$, by Theorem 10, $\bigcup_{i=1}^{|V(G_1)|} \{u\}^{(i)}$ is a $\gamma_{0ri}$-set of $G_1 \circ G_2$, where $\{u\}^{(i)}$ denotes the $i$th copy of single-ton subset containing vertex $u$ of $G_2$. Therefore, $\gamma_{0ri}(G_1 \circ G_2) = |V(G_1)| \cdot |\{u\}| = |V(G_1)|$. □

**Theorem 12.** *Let $G = K_{P_1,P_2,...,P_n}$ be a complete m-partite graph whose partitions are denoted by $P_j, j = 1, ..., n$. If $n \geq 4$, then*

$$\gamma_{0ri} = min\{|Pj|, j = 1, ..., n\}.$$

*Proof.* Let $n \geq 4$ and consider $v_i^{(j)}$ be the $ith$ vertex in $P_j$, where $j = 1, ..., n$. Then $v_i^{(j)} v_k^{(s)} \in E(G)$, for some $s \neq j$ and for all $k \in \{1, ..., |P_s|\}$. But $v_i^{(j)} v_t^{(j)} \notin E(G)$, for some $t \neq i$. Thus, we consider

$$S = \{v_i^{(j)} | i = 1, ..., t, ..., |P_j|\}.$$

Note that $v_t^{(s)}$ is adjacent to all other vertices except vertices in $P_j$. This means that $S$ is $\gamma$-set. Also, $v_k^{(s)}$ is adjacent to all other vertices in $P_w$, where $w \neq j$ and $m \geq 4$. It implies that $S$ is $\gamma_{0ri}$-set. Lastly, since $S$ is a minimal partition, therefore,

$$S = min\{|P_j|, j = 1, ..., m\}.$$

□

**Theorem 13.** *Let $G_1$ and $G_2$ be any graphs. If $S_1$ and $S_2$ are $\gamma_{0ri}$-sets of $G_1$ and $G_2$ respectively, then*

$$\gamma_{0ri}(G_1 + G_2) = min\{|S_1|, |S_2|\}.$$

*Proof.* Let $S_1$ and $S_2$ be the $\gamma_0 ri$-sets of $G_1$ and $G_2$, respectively. Since all vertices of $S_1$ are adjacent to all vertices of $S_2$ in $S_1 + S_2$, and all vertices of $S_2$ are adjacent to all vertices of $S_1$ in $S_1 + S_2$, it follows that $S_1$ and $S_2$ are isolate rings dominating sets of $S_1 + S_2$. Hence, $\gamma_{0ri}(G_1 + G_2) = min\{|S_1|, |S_2|\}$. □

**Theorem 14.** *Let $G_1$ and $G_2$ be any graphs such that $G_2$ has a $\gamma_{0ri}$-set. Then $G_1 \circ G_2$ has a $\gamma_{0ri}$-set.*

*Proof.* If $G_2$ has a $\gamma_{0ri}$-set, say $S_1$, then $V(G_2) \setminus S$ is a subgraph of some $C_k$, for some $k \in \mathbb{Z}^+$. Let $S_i$ be a $\gamma_{0ri}$-set of the $i$th copy of $G_2$ in $G_1 \circ G_2$. Since $S_i$ is a dominating set, $S_i$ dominates $u_i \in V(G_1)$, where $u_i$ is adjacent to all vertices in the $i$th copy of $G_2$. Since $V(G_2) \setminus S$ is a subgraph of some cycle $C_k$, for some $k \in \mathbb{Z}$, it follows that every vertices of the $i$th copy of $G_2$ excluding $S_i$, are adjacent to at least two vertices excluding $S_i$. Hence, $u_i$ is adjacent to at least two vertices of the $i$th copy of $G_2$ excluding $S_i$. Thus, $S_i$ is also $\gamma_{0ri}$-set. Since $S_i$ are mutually disjoint, $\bigcup_{i=1}^{|V(G_1)|} S_i$ is also a $\gamma_{0ri}$-set. Hence, the claim is proven. Consequently, $\gamma_{0ri}(G_1 \circ G_2) = |V(G_1)||S|$, where $S$ is $\gamma_{0ri}$-set of $G_2$. □

## 4. Conclusions and Recommendations

Building on the study of the binary operation of isolate domination in graphs by Ariola [2], the generalization of isolate domination introduced by Hamid and Balamarugan [13], and the introduction of the concept of rings domination in graphs by Abed and Al-Here [1], the authors developed and introduced the concept of isolate rings dominating sets in certain graphs. The characterizations of these sets under binary operations, such as the join and corona of two graphs, were also shown. In addition, the authors examined the existence of the isolate rings dominating sets of some graphs and some binary operations. For future research, the authors recommend to further investigate this parameter to identify exact values of some graphs and some binary operations in graphs other than join and corona. Furthermore, the authors also suggest exploring the relationship of isolate rings domination with other domination parameters.

**Authors' Contributions.** All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

**Conflicts of Interest.** The author declares that there are no conflicts of interest regarding the publication of this paper.

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