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RIESZ SPACE-VALUED FUNCTIONS AND THEIR INTEGRATION VIA THE MCSHANE INTEGRAL ON TIME SCALES

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Abstract. In this paper, we introduce and study the McShane integral for Riesz space-valued functions defined on time scales, a unifying framework that generalizes both continuous and discrete analysis. We establish key properties and results of the McShane delta (MΔ) integral for Riesz space-valued functions, providing a deeper understanding of its behavior on time scales. Additionally, we prove a uniform convergence theorem, which offers valuable insights into the convergence of sequences of functions in this setting. These results contribute to advancing the theory of integration on time scales and its applications in various fields of mathematical analysis.

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1. Introduction

The concept of integration has evolved significantly, with various types of integrals developed to handle different classes of functions. The McShane integral, which is widely regarded as a Riemann-like integral for real-valued functions, is fundamentally equivalent to the Lebesgue integral. In addition, McShane's novel approach to integration, initially based on Lebesgue's theory, has been extended to accommodate vector-valued functions, with an in-depth study of their properties. When dealing with functions defined on Banach spaces, the Bochner integral is typically preferred, as it offers a more comprehensive framework compared to the McShane integral. The McShane integral, in turn, is more general than the Henstock, Pettis, and Dunford integrals [7,14].

A time scale T is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$ with the subspace topology inherited from the standard topology of $\mathbb{R}$. The theory of time scales was introduced in

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1988 in the Ph.D. thesis of Hilger [17]. Its goal is to unify various definitions and results from the theories of discrete and continuous dynamical systems, and to extend these theories to more general types of dynamical systems. This theory has been widely studied from different perspectives by many researchers [8,10,13]. Peterson and Thompson [4] introduced a more general version of the Henstock-Kurzweil delta integral on time scales, which includes the Riemann delta and the Lebesgue delta integrals as special cases. The theory of integration for real-valued, Banach space-valued, and Riesz space-valued functions on time scales has also developed significantly in recent years [6,9,14,15]. A key challenge in integration theory is the task of integrating functions with values in general spaces. The Henstock and McShane integrals for Riesz-space-valued functions have been studied in [1–3,5,11,12,16]. Surprisingly, the McShane integral for Riesz-space-valued functions has not been explored in the context of time scales. The main objective of this paper is to generalize the results mentioned above by constructing the McShane integral for Riesz-space-valued functions on time scales.

2. Materials and Methods

Let N, R, and R^+ denote the sets of all natural numbers, real numbers, and positive real numbers, respectively, and let Y be a Banach lattice. A decreasing sequence $(a_n)_n$ in Y, such that $\bigwedge_n a_n = 0$, is called an (o)-sequence. A sequence $(a_n)_n$ is said to be order-convergent (or (o)-convergent) to a, if there exists a sequence $(p_n)_n \in R$, such that $p_n \downarrow 0$ and $|a_n - a| \leq p_n$ for all $n \in N$. We will write (o)-$\lim_n a_n = a$.

Let S be a time scale, i.e., a nonempty closed subset of R. For $c, d \in S$, we define the closed interval $[c, d]_S = \{x \in S : c \leq x \leq d\}$. For $x \in S$, we define the forward jump operator $\tau(x) = \inf\{r > x : r \in S\}$, where $\inf \emptyset = \sup\{S\}$, while the backward jump operator $\omega(x)$ is defined by $\omega(x) = \sup\{r < x : r \in S\}$, where $\sup \emptyset = \inf\{S\}$.

If $\tau(x) > x$, then we say that x is right-scattered, while if $\omega(x) < x$, then we say that x is left-scattered. If $\tau(x) = x$, then we say that x is right-dense, while if $\omega(x) = x$, then we say that x is left-dense. A point $x \in S$ is dense if it is both right and left dense; isolated if it is both right and left scattered. The forward graininess function $\theta(x)$ and the backward graininess function $\mu(x)$ are defined by $\theta(x) = \tau(x) - x$ and $\mu(x) = x - \omega(x)$ for all $x \in S$, respectively. If $\sup S$ is finite and left-scattered, then we define $S^k = S \setminus \sup S$; otherwise, $S^k = S$. If $\inf S$ is finite and right-scattered, then $S_K = S \setminus \inf S$; otherwise, $S_K = S$. We set $S_K^K = S_K \cap S^K$. Throughout this paper, all considered intervals will be intervals in S. A McShane partition P of $[c, d]_S$ is a finite collection of pairs $\{([x_{i-1}, x_i]_S, t_i), i = 1, \ldots, n\}$, where $c = x_0 < x_1 < \cdots < x_{n-1} < x_n = d$, $t_i \in [c, d]_S$, and t_i is not necessarily in $[x_{i-1}, x_i]_S$ for all $i = 1, \ldots, n$, and

$$\bigcup_{i=1}^{n} [x_{i-1}, x_i]_S = [c, d]_S.$$

Let M be the family of all finite unions of closed subintervals of S, and define $\lambda([x_{i-1}, x_i]_S) = x_i - x_{i-1} = \Delta x_i$, where $c \leq x_{i-1} < x_i \leq d$. We say that $\gamma(t) = (\gamma_l(t), \gamma_r(t))$ is a Δ-gauge for $[c, d]_S$ provided that $\gamma_l(t) > 0$ on $]c, d]_S$, $\gamma_l(c) \geq 0$, $\gamma_r(t) > 0$ on $[c, d[_S$, $\gamma_r(d) \geq 0$, and $\gamma_r(t) \geq \theta(t)$ for all $t \in [c, d[_S$. Let $\gamma^1(t)$ and $\gamma^2(t)$ be Δ-gauges for $[c, d]_S$ such that $0 < \gamma_l^1(t) < \gamma_l^2(t)$ for $t \in]c, d]_S$ and $0 < \gamma_r^1(t) < \gamma_r^2(t)$ for $t \in [c, d[_S$. We say $\gamma^1(t)$ is finer than $\gamma^2(t)$ and write $\gamma^1(t) < \gamma^2(t)$.

A McShane partition (M-partition) $P = \{([x_{i-1}, x_i]_S, t_i) : i = 1, \ldots, n\}$ of $[c, d]_S$ is said to be γ-fine if $[x_{i-1}, x_i]_S \subset (t_i - \gamma_l(t_i), t_i + \gamma_r(t_i))_S$ for all $i = 1, \ldots, n$.

If $P = \{([x_{i-1}, x_i]_S, t_i) : i = 1, \ldots, n\}$ is a γ-fine McShane partition of $[c, d]_S$, then the integral sum over P is defined by the formula

$$\sigma(g, P) = \sum_{i=1}^{n} g(t_i)(x_i - x_{i-1})$$

when $g : [c, d]_S \to Y$.

Definition 2.1. The function $g : [c, d]_S \to Y$ is called McShane Δ-integrable (MΔ-integrable) on $[c, d]_S$, and $J \in Y$ is its $(M\Delta)$-McShane integral if for every (o)-sequence $(b_n)_n$ in Y, there is a Δ-gauge γ for $[c, d]_S$, such that for every n and γ-fine M-partition $P = \{([x_{i-1}, x_i]_S, t_i), i = 1, \ldots, n\}$ of $[c, d]_S$, the inequality

$$|\sigma(g, P) - J| < b_n$$

holds, where $\sigma(g, P) = \sum_{i=1}^{n} g(t_i)(x_i - x_{i-1})$. We denote the McShane Delta Integral:

$$(M\Delta) \int_c^d g(x) = J.$$

Lemma 2.2 *The element J from Definition 2.1 is uniquely determined.*

Proof. Let I and J be such elements. For every (o)-sequence $(a_n)_n$, $(b_n)_n$ in Y, there exist corresponding Δ-gauges γ_1 and γ_2 for $[c, d]_S$, such that for every n and for any γ_1-fine M-partition P_1 and γ_2-fine M-partition P_2 of $[c, d]_S$, the following inequalities hold:

$$|\sigma(g, P_1) - I| < a_n,$$

$$|\sigma(g, P_2) - J| < b_n.$$

Let $\gamma = \min\{\gamma_1, \gamma_2\}$. Then for every n and for any γ-fine M-partition $P = \{([x_{i-1}, x_i]_S, t_i), i = 1, \ldots, n\}$ of $[c, d]_S$, we have:

$$|I - J| < |I - \sigma(g, P)| + |\sigma(g, P) - J| < a_n + b_n,$$

$$0 \leq |I - J| \leq 0.$$

□

Definition 3.1. We say that $g_k \to g$ converges with a common (o)-sequence if there exists an (o)-sequence $(a_n)_n$ of elements of Y such that for every n and every $x \in [c,d]_S$, there exists $\omega = \omega(t)$ such that

$$|g_n(x) - g(x)| < a_n$$

for all $n \geq \omega$.

Definition 3.2. We say that $G = \{g_k : S \to Y; k \in \mathbb{N}\}$ is uniformly $M\Delta$-integrable on $[c,d]_S$ if each g_k is $M\Delta$-integrable on $[c,d]_S$ and there exists an (o)-sequence $(a_n)_n$ of elements of Y such that for every n, there exists a Δ-gauge, γ, for $[c,d]_S$ such that

$$|\sigma(g_n, P) - (M\Delta)\int_c^d g_n(x)\,\Delta x| < a_n,$$

for each γ-fine M-partition P of $[c,d]_S$ and for all $n \in \mathbb{N}$.

For uniformly $M\Delta$-integrable sequences of integrable functions, we have the following convergence theorem.

3. Results

We present key results on the $M\Delta$-integral for Riesz space-valued functions on time scales. We prove fundamental properties such as additivity and homogeneity, and establish the Bolzano-Cauchy condition as a criterion for $M\Delta$-integrability. Additionally, we prove a uniform convergence theorem for sequences of $M\Delta$-integrable functions.

Proposition 2.3 If $f, g : [c,d]_S \to Y$ are $M\Delta$-integrable on $[c,d]_S$ and $\alpha \in \mathbb{R}$, then $f+g$ and αg are $M\Delta$-integrable on $[c,d]_S$ as well, and

$$(M\Delta)\int_c^d (f(x)+g(x))\,\Delta x = (M\Delta)\int_c^d f(x)\,\Delta x + (M\Delta)\int_c^d g(x)\,\Delta x,$$

$$(M\Delta)\int_c^d \alpha g(x)\,\Delta x = \alpha(M\Delta)\int_c^d g(x)\,\Delta x.$$

Proof. Suppose that f and g are $M\Delta$-integrable on $[c,d]_S$. For every two (o)-sequences $(a_n)_n$ and $(b_n)_n$ in Y, there exist corresponding Δ-gauges γ_1 and γ_2 for $[c,d]_S$, such that for every n, and for any γ_1-fine M-partition P_1 and γ_2-fine M-partition P_2 of $[c,d]_S$, the following inequalities hold:

$$|\sigma(f, P_1) - (M\Delta)\int_c^d f(x)\,\Delta x| < a_n,$$

$$|\sigma(g, P_2) - (M\Delta)\int_c^d g(x)\,\Delta x| < b_n.$$

Let $\gamma = \min\{\gamma_1, \gamma_2\}$. Then, for every n, and for any γ-fine M-partition $P = \{([x_{i-1}, x_i]_S, t_i), i = 1, \ldots, n\}$ of $[c,d]_S$, we have:

$$|\sigma(f+g,P)-(M\Delta)\int_c^d f(x)\,\Delta x-(M\Delta)\int_c^d g(x)\,\Delta x| =$$

$$|\sigma(f,P)-(M\Delta)\int_c^d f(x)\,\Delta x+\sigma(g,P)-(M\Delta)\int_c^d g(x)\,\Delta x| \leq$$

$$|\sigma(f,P)-(M\Delta)\int_c^d f(x)\,\Delta x|+|\sigma(g,P)-(M\Delta)\int_c^d g(x)\,\Delta x| < a_n+b_n.$$

Thus, $f(x)+g(x)$ is $M\Delta$-integrable on $[c,d]_S$, and

$$(M\Delta)\int_c^d (f(x)+g(x))\,\Delta x=(M\Delta)\int_c^d f(x)\,\Delta x+(M\Delta)\int_c^d g(x)\,\Delta x.$$

Next, for $\alpha\in\mathbb{R}$, the sequence $(|\alpha|a_n)_n$ is an (o)-sequence. Then, for each γ-fine M-partition P of $[c,d]_S$, we have:

$$|\sigma(\alpha g,P)-\alpha(M\Delta)\int_c^d g(x)\,\Delta x| \leq |\alpha||\sigma(g,P)-(M\Delta)\int_c^d g(x)\,\Delta x| < |\alpha|a_n.$$

This shows that αg is $M\Delta$-integrable, and

$$(M\Delta)\int_c^d \alpha g(x)\,\Delta x=\alpha(M\Delta)\int_c^d g(x)\,\Delta x.$$

□

Proposition 2.4 Let $f,g:[c,d]_S\to Y$ be $M\Delta$-integrable on $[c,d]_S$, and suppose that $f(x)\leq g(x)$ for all $x\in[c,d]_S$. Then

$$(M\Delta)\int_c^d f(x)\,\Delta x\leq(M\Delta)\int_c^d g(x)\,\Delta x.$$

Proof. Consider an integrable mapping $h:[c,d]_S\to Y$ such that $h(x)\geq 0$ for all $x\in[c,d]_S$. Then $\sigma(h,P)\geq 0$ for any partition P. If $n\in\mathbb{N}$ and Δ-gauge γ is the corresponding map, then

$$-(M\Delta)\int_c^d h(x)\,\Delta x\leq\sigma(h,P)-(M\Delta)\int_c^d h(x)\,\Delta x\leq|\sigma(h,P)-(M\Delta)\int_c^d h(x)\,\Delta x|\leq a_n.$$

For any γ-fine M-partition $P=\{([x_{i-1},x_i]_S,t_i),i=1,\ldots,n\}$ of $[c,d]_S$, we obtain

$$-(M\Delta)\int_c^d h(x)\,\Delta x\leq a_n.$$

Thus,

$$(M\Delta)\int_c^d h(x)\,\Delta x\geq 0.$$

Now, let $h(x)=g(x)-f(x)$. Then,

$$0\leq(M\Delta)\int_c^d h(x)\,\Delta x=(M\Delta)\int_c^d (g(x)-f(x))\,\Delta x=$$

$$(M\Delta)\int_c^d g(x)\,\Delta x-(M\Delta)\int_c^d f(x)\,\Delta x.$$

□

Theorem 2.5 (Bolzano-Cauchy Condition). A mapping $g : [c,d]_S \to Y$ is $M\Delta$-integrable if and only if the following condition is satisfied:

There exists an (o)-sequence $(b_n)_n$ in Y such that, for every n, there exists a Δ-gauge γ for $[c,d]_S$ such that

$$|\sigma(g, P_2) - \sigma(g, P_1)| < b_n,$$

for each γ-fine M-partition P_1, P_2 of $[c,d]_S$.

Proof. (*Necessity*) Clearly, integrability implies the Bolzano-Cauchy condition by the following inequality:

$$|\sigma(g, P_1) - \sigma(g, P_2)| \leq |\sigma(g, P_1) - (M\Delta)\int_c^d g(x)\,\Delta x| + |\sigma(g, P_2) - (M\Delta)\int_c^d g(x)\,\Delta x| < a_n + b_n.$$

(*Sufficiency*) For each n, there exists a Δ-gauge $\gamma_n(t)$ for $[c,d]_S$ such that

$$\gamma_{[c,d]_S} = \{\gamma(t) : \exists n, \gamma(t) = \gamma_n(t), t \in [c,d]_S\}.$$

Then, for $\gamma(t) \in \gamma_{[c,d]_S}$ and a γ-fine M-partition P of $[c,d]_S$, the set $\{\sigma(g,P)\}$ is bounded. Indeed, since Y is boundedly complete, there exist

$$m_n = \inf\{\sigma(g,P) : P \text{ is } \gamma_n\text{-fine}\}$$

and

$$M_n = \sup\{\sigma(g,P) : P \text{ is } \gamma_n\text{-fine}\}.$$

We have that $m_n \leq m_{n+1} \leq M_{n+1} \leq M_n$ for every n, and thus $M_n - m_n \leq b_n$. Given the fact that

$$|\sigma(g, P_2) - \sigma(g, P_1)| < b_n,$$

we conclude that

$$\sup_n m_n = \inf_n M_n = J.$$

Then, for every γ_n-fine M-partition $P = \{([x_{i-1}, x_i]_S, t_i), i = 1, \ldots, n\}$, we have

$$J - \sigma(g,P) \leq J - m_n \leq M_n - m_n \leq b_n,$$

and

$$\sigma(g,P) - J \leq M_n - J \leq M_n - m_n \leq b_n.$$

It follows that

$$|\sigma(g,P) - J| \leq b_n.$$

□

Proposition 2.5 Let $e \in (c,d)$, and let $g : [c,d]_S \to Y$ be $M\Delta$-integrable on both $[c,e]_S$ and $[e,d]_S$. Then g is $M\Delta$-integrable on $[c,d]_S$ as well, and

$$(M\Delta)\int_c^d g(x)\,\Delta x = (M\Delta)\int_c^e g(x)\,\Delta x + (M\Delta)\int_e^d g(x)\,\Delta x.$$

Proof. Since g is $M\Delta$-integrable on $[c,e]_S$ and on $[e,d]_S$, for every two (o)-sequences $(a_n)_n$ and $(b_n)_n$ in Y, there exist corresponding Δ-gauges $\gamma_1 = (\gamma_l^1(t), \gamma_r^1(t))$ and $\gamma_2 = (\gamma_l^2(t), \gamma_r^2(t))$ for $[c,d]_S$, such that for every n and for a γ_1-fine M-partition $P_1 = \{([x_{k-1}, x_k]_S, t_k), k = 1, \ldots, n\}$ of $[c,e]_S$ and a γ_2-fine M-partition $P_2 = \{([x_{j-1}, x_j]_S, t_j), j = 1, \ldots, n\}$ of $[e,d]_S$, the following inequalities hold:

$$|\sigma(g, P_1) - (M\Delta)\int_c^e g(x)\,\Delta x| < a_n,$$

$$|\sigma(g, P_2) - (M\Delta)\int_e^d g(x)\,\Delta x| < b_n.$$

We define a Δ-gauge $\gamma(t) = (\gamma_l(t), \gamma_r(t))$ on $[c,d]_S$ in the following way:

$$\gamma_l(t) = \begin{cases} \gamma_l^1(t) & \text{if } t < e, \\ \gamma_l^1(t) & \text{if } t = e = \omega(e), \\ \min\{\gamma_l^1(t), \mu(e)/2\} & \text{if } t = e > \omega(e), \\ \min\{\gamma_l^2(t), t - e\} & \text{if } t > e. \end{cases}$$

Similarly, define $\gamma_r(t)$ as follows:

$$\gamma_r(t) = \begin{cases} \min\{\gamma_r^1(t), \max\{\theta(t), e - t\}\} & \text{if } t < e, \\ \gamma_r^2(t) & \text{if } t \geq e. \end{cases}$$

Then, for any γ-fine M-partition $P = \{([x_{i-1}, x_i]_S, t_i), i = 1, \ldots, n\}$ of $[c,d]_S$, there are γ_1-fine M-partition $P_1 = \{([x_{k-1}, x_k]_S, t_k), k = 1, \ldots, n\}$ of $[c,e]_S$ and γ_2-fine M-partition $P_2 = \{([x_{j-1}, x_j]_S, t_j), j = 1, \ldots, n\}$ of $[e,d]_S$, such that

$$\sigma(g, P) = \sigma(g, P_1) + \sigma(g, P_2).$$

Therefore,

$$|\sigma(g, P) - (M\Delta)\int_c^e g(x)\,\Delta x - (M\Delta)\int_e^d g(x)\,\Delta x| \leq$$

$$|\sigma(g, P_1) - (M\Delta)\int_c^e g(x)\,\Delta x| + |\sigma(g, P_2) - (M\Delta)\int_e^d g(x)\,\Delta x| \leq a_n + b_n.$$

Thus, we have shown that g is $M\Delta$-integrable on $[c,d]_S$, and

$$(M\Delta)\int_c^d g(x)\,\Delta x = (M\Delta)\int_c^e g(x)\,\Delta x + (M\Delta)\int_e^d g(x)\,\Delta x.$$

□

Theorem 3.3. Let $G = \{g_k : S \to Y; k \in \mathbb{N}\}$ be uniformly $M\Delta$-integrable on $[c,d]_S$ and assume that $g_k \to g$ converges with a common (o)-sequence. Then g is $M\Delta$-integrable on $[c,d]_S$ and

$$(o)\text{-}\lim_{n\to\infty}(M\Delta)\int_c^d g_n(x)\,\Delta x = (M\Delta)\int_c^d g(x)\,\Delta x.$$

Proof. By assumption, there exists an (o)-sequence $(b_n)_n$ of elements of Y such that for every n, there exist two Δ-gauges, $\gamma_1 = (\gamma_l^1(t), \gamma_r^1(t))$ and $\gamma_2 = (\gamma_l^2(t), \gamma_r^2(t))$, for $[c,d]_S$, such that for every n and a γ_1-fine M-partition $P_1 = \{([x_{k-1}, x_k]_S, t_k), k = 1, \ldots, n\}$ of $[c,d]_S$ and a γ_2-fine M-partition $P_2 = \{([x_{j-1}, x_j]_S, t_j), j = 1, \ldots, n\}$ of $[c,d]_S$, the following inequalities hold:

$$|\sigma(g_n, P_1) - (M\Delta)\int_c^d g_n(x)\,\Delta x| < b_n,$$

$$|\sigma(g_n, P_2) - (M\Delta)\int_c^d g_n(x)\,\Delta x| < b_n.$$

Let $\gamma = \min\{\gamma_1, \gamma_2\}$. By the convergence with a common (o)-sequence, we have

$$|\sigma(g_n, P_1) - \sigma(g, P_1)| \le \sum_{i=1}^{n} |g_n(t_i) - g(t_i)|(x_i - x_{i-1}) < (d-c)a_n,$$

for each γ-fine M-partition P_1 of $[c,d]_S$ and $n \ge \omega_1$. Similarly, we have

$$|\sigma(g_n, P_2) - \sigma(g, P_2)| \le \sum_{j=1}^{n} |g_n(t_j) - g(t_j)|(x_j - x_{j-1}) < (d-c)a_n,$$

for each γ-fine M-partition P_2 of $[c,d]_S$ and $n \ge \omega_2$.

Let $n > \max\{\omega_1, \omega_2\}$. Then,

$$|\sigma(g, P_1) - \sigma(g, P_2)| = |\sigma(g, P_1) - \sigma(g_n, P_1)| + |\sigma(g_n, P_1) - (M\Delta)\int_c^d g_n(x)\,\Delta x|$$

$$+|(M\Delta)\int_c^d g_n(x)\,\Delta x - \sigma(g_n, P_2)| + |\sigma(g_n, P_2) - \sigma(g, P_2)| < 2(b_n + (d-c)a_n),$$

for each γ_1-fine M-partition $P_1 = \{([x_{k-1}, x_k]_S, t_k), k = 1, \ldots, n\}$ of $[c,d]_S$ and γ_2-fine M-partition $P_2 = \{([x_{j-1}, x_j]_S, t_j), j = 1, \ldots, n\}$ of $[c,d]_S$. Therefore, by the Cauchy-Bolzano condition, g is $M\Delta$-integrable.

Since g is $M\Delta$-integrable, there exists an (o)-sequence $(b_n)_n$ of elements of Y such that for every n, there exists a Δ-gauge $\gamma' = (\gamma_l'(t), \gamma_r'(t))$ for $[c,d]_S$, such that for each γ'-fine M-partition $P' = \{([x_{k-1}, x_k]_S, t_k), k = 1, \ldots, r\}$ of $[c,d]_S$, the inequalities hold:

$$|\sigma(g, P') - (M\Delta)\int_c^d g(x)\,\Delta x| < c_n.$$

By uniform $M\Delta$-integrability, there exists an (o)-sequence $(b_n)_n$ of elements of Y for every n such that

$$|\sigma(g_n, P') - (M\Delta)\int_c^d g_n(x)\,\Delta x| < b_n,$$

for each γ'-fine M-partition $P' = \{([x_{k-1}, x_k]_S, t_k), k = 1, \ldots, r\}$ of $[c, d]_S$. By the convergence with a common (o)-sequence, we have

$$|\sigma(g_n, P') - \sigma(g, P')| \le \sum_{k=1}^{r} |g_n(t_k) - g(t_k)|(x_k - x_{k-1}) < (d - c)a_n,$$

for each γ'-fine M-partition $P' = \{([x_{k-1}, x_k]_S, t_k), k = 1, \ldots, r\}$ of $[c, d]_S$ and $n \ge \omega$. Then,

$$|(M\Delta)\int_c^d g(x)\,\Delta x - (M\Delta)\int_c^d g_n(x)\,\Delta x| =$$

$$|(M\Delta)\int_c^d g(x)\,\Delta x - \sigma(g, P')| + |\sigma(g, P') - \sigma(g_n, P')| + |\sigma(g_n, P') - (M\Delta)\int_c^d g_n(x)\,\Delta x| < (d-c)a_n + b_n + c_n,$$

for each γ'-fine M-partition $P' = \{([x_{k-1}, x_k]_S, t_k), k = 1, \ldots, r\}$ of $[c, d]_S$. It follows that

$$(o)\text{-}\lim_{n\to\infty}(M\Delta)\int_c^d g_n(x)\,\Delta x = (M\Delta)\int_c^d g(x)\,\Delta x,$$

□

4. Conclusion

In this paper, we prove some basic properties and a uniform convergence theorem for the McShane delta ($M\Delta$-integral) of Riesz space-valued functions defined on time scales. We demonstrate that under appropriate conditions, the McShane delta integral leads to the same result in Riesz spaces as the Henstock integral for Riesz-space-valued functions on time scales. This establishes the equivalence between the two integration theories in this context. Our results extend the applicability of the $M\Delta$-integral to Riesz space-valued functions on time scales, offering a unified framework for integration in both discrete and continuous settings.

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