MONOTONE α-CONVEX GENERALIZED NONEXPANSIVE MAPPINGS AND FIXED POINT RESULTS

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Received Sep. 25, 2024

Abstract. This paper examines monotone α-convex generalized nonexpansive mappings and explores their relationships with other existing nonexpansive-type mappings. We establish new theorems on the existence and convergence of these mappings in the context of ordered Banach spaces. Moreover, we broaden the scope of results previously presented by M. R. Alfuraidan and M. A. Khamsi (Carpathian J. Math., 36(2): 199–204, 2020) and B. A. Bin Dehaish and M. A. Khamsi (Fixed Point Theory Appl., 2015, 2015:177, 7 pp).

2020 Mathematics Subject Classification. 47H10; 54H25; 47H09.
Key words and phrases. nonexpansive mapping; monotone mapping; uniformly convex space.

1. Introduction

In a Banach space $(\mathcal{E}, \|.\|)$ with $\mathcal{U}$ as a nonempty subset, a mapping $\Upsilon : \mathcal{U} \to \mathcal{U}$ is nonexpansive if $\|\Upsilon(\nu) - \Upsilon(\eta)\| \leq \|\nu - \eta\|$ for all $\nu, \eta \in \mathcal{U}$. A point $z \in \mathcal{U}$ is a fixed point of Υ if $\Upsilon(z) = z$ and $F(\Upsilon) := \{z \in \mathcal{U} : \Upsilon(z) = z\}$. A mapping Υ is called accretive if for each $\nu, \eta \in \mathcal{U}$ and $\lambda \geq 0$, the following condition holds:

$$\|\nu - \eta\| \leq \|\nu - \eta + \lambda(\Upsilon(\nu) - \Upsilon(\eta))\|.$$

The investigation of fixed points of nonexpansive mappings and zero points of accretive mappings extends the classical theory of successive approximations for strict contractions, which relies on the Banach contraction principle. It is well-established that if $\Upsilon : \mathcal{U} \to \mathcal{U}$ is a strict contraction, the Picard sequence $\nu_{n+1} = \Upsilon(\nu_n)$ converges strongly to the unique fixed point of Υ. In contrast, for nonexpansive mappings, the Picard sequence may fail to converge, and if a fixed point exists, it may not be unique. However, in sufficiently smooth spaces, such as Hilbert spaces, for any $\alpha \in (0, 1)$ and $\nu_1 \in \mathcal{U}$, the iterative process

$$\nu_{n+1} = (1 - \alpha)\nu_n + \alpha\Upsilon(\nu_n)$$

DOI: 10.28924/APJM/12-28

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converges weakly to a fixed point of the nonexpansive mapping $\Upsilon : \mathcal{U} \to \mathcal{U}$. This iterative process, along with the more general scheme $\nu_{n+1} = (1-\alpha_n)\nu_n + \alpha_n \Upsilon(\nu_n)$, is known as the Krasnosel'skiĭ-Mann (or Mann) iteration formula for finding fixed points of nonexpansive mappings. Despite nonexpansive mappings being regarded as one of the most significant subjects in what is known as metric fixed point theory, the literature contains a substantial body of research on broader classes of mappings beyond the nonexpansive type. Noteworthy contributions and advancements in the field can be found in works such as [8,14,15,19,23]. In 2008, Suzuki [23] introduced a class of mappings characterized by condition (C), which belong to the broader category of nonexpansive type mappings. Suzuki's investigation into these mappings significantly contributed to the development of important fixed point theorems.

Definition 1.1. [23]. Let $\mathcal{E}$ be a Banach space and $\mathcal{U}$ a subset of $\mathcal{E}$ such that $\mathcal{U} \neq \emptyset$. A mapping $\Upsilon : \mathcal{U} \to \mathcal{U}$ is said to satisfy condition (C) if

$$\frac{1}{2}\|\nu - \Upsilon(\nu)\| \leq \|\nu - \eta\| \text{ implies } \|\Upsilon(\nu) - \Upsilon(\eta)\| \leq \|\nu - \eta\| \ \forall\, \nu, \eta \in \mathcal{U}.$$

The author established a strong convergence theorem akin to Ishikawa's theorem [12], and a more nuanced weak convergence theorem aligned with the works of Edelstein and O'Brien [7].

García-Falset et al. [8] further generalized condition (C) into the following class of mappings.

Definition 1.2. [8]. Let $\mathcal{E}$ be a Banach space and $\mathcal{U}$ a subset of $\mathcal{E}$ such that $\mathcal{U} \neq \emptyset$. A mapping $\Upsilon : \mathcal{U} \to \mathcal{U}$ is said to fulfill condition (E) if there exists $\mu \geq 1$ such that

$$\|\nu - \Upsilon(\eta)\| \leq \mu\|\nu - \Upsilon(\nu)\| + \|\nu - \eta\| \ \forall\, \nu, \eta \in \mathcal{U}.$$

The category of mappings satisfying condition (E) includes various significant classes of generalized nonexpansive mappings. Important results regarding nonexpansive mappings have been established within this framework, as discussed in [18].

Recently, Llorens-Fuster [14] introduced the class of partially nonexpansive mappings (PNE), which encompasses the Suzuki-nonexpansive mappings. Despite being defined on compact convex sets, PNE mappings may not necessarily have fixed points. The presence of fixed points for a mapping is not guaranteed by either the PNE condition or condition (E) alone. However, Llorens-Fuster demonstrated that by combining these properties—PNE and condition (E)—fixed points can indeed be ensured in Banach spaces possessing appropriate geometric properties in their norm structure.

Definition 1.3. Let $\Upsilon : \mathcal{U} \to \mathcal{U}$ be a mapping. A mapping Υ is called as partially nonexpansive, (in short, PNE), if

$$\left\|\Upsilon\left(\frac{1}{2}(\nu + \Upsilon(\nu))\right) - \Upsilon(\nu)\right\| \leq \frac{1}{2}\|\nu - \Upsilon(\nu)\|$$

for all $\nu \in \mathcal{U}$.

Recently, Shukla et al. [20] investigated a novel category of mappings known as α-convex generalized nonexpansive mappings (α-CGNE), and derived several theorems concerning their existence and convergence in Banach spaces possessing the Opial Property, characterized by uniform convexity in every direction (UCED).

Interest in monotone Lipschitzian mappings has grown significantly since Ran and Reurings [16] extended the Banach Contraction Principle to partially ordered metric spaces, a topic also explored by Turinici [24]. This advancement fueled further research into the development of metric fixed point theory for monotone Lipschitzian mappings, as investigated in studies such as [17,21,22]. In contrast, mappings that deviate from strict Lipschitzian conditions have attracted less attention. Notable contributions to the application of fixed point theory for monotone mappings are presented in Carl and Heikkilä comprehensive book [5]. For those seeking a deeper understanding of metric fixed point theory and the geometric structure of Banach spaces, Goebel and Kirk's foundational work [9] is highly recommended. In a related vein, Alfuraidan and Khamsi [1] expanded the study of condition (C) in ordered Banach spaces, deriving new existence and convergence theorems.

Building upon the insights from Shukla et al. [20] and others, we extend the class of α-convex generalized nonexpansive mappings in the context of ordered Banach spaces. This extension allows us to establish various results concerning the existence and convergence of partially nonexpansive mappings under specific assumptions. In doing so, we contribute to the broader framework presented in [1,3,20,23], offering new extensions, generalizations, and complementary perspectives.

2. Preliminaries

Consider $\mathcal{E}$ as a Banach space equipped with a partial order $\preceq$ that is compatible with its linear structure, meaning:

$$\nu \preceq \eta \text{ implies } \nu + z \preceq \eta + z,$$

$$\nu \preceq \eta \text{ implies } \lambda\nu \preceq \lambda\eta$$

for every $\nu, \eta \in \mathcal{E}$, $z \in \mathcal{E}$, and $\lambda \geq 0$. This ensures that all order intervals $[\nu, \rightarrow] = \{z \in \mathcal{E} : \nu \preceq z\}$ and $[\leftarrow, \eta] = \{z \in \mathcal{E} : z \preceq \eta\}$ are convex. Additionally, we assume that each $[\nu, \rightarrow]$ and $[\leftarrow, \eta]$ is closed. Thus, we define $(\mathcal{E}, \|\cdot\|, \preceq)$ as an ordered Banach space. If a monotone sequence $\{\nu_n\}$ possesses a subsequence that converges weakly to some z, then the entire sequence $\{\nu_n\}$ converges weakly to z. Furthermore, if $\{\nu_n\}$ is monotone increasing (resp. decreasing), then $\nu_n \preceq z$ (resp. $z \preceq \nu_n$).

Definition 2.1. [9]. Let $\mathcal{U}$ be a nonempty subset of a Banach space $\mathcal{E}$. A sequence $\{\nu_n\}$ in $\mathcal{U}$ is said to be approximate fixed point sequence (in short, a.f.p.s.) for a mapping $\Upsilon : \mathcal{U} \to \mathcal{U}$ if $\lim_{n\to\infty} \|\nu_n - \Upsilon(\nu_n)\| = 0$.

Lemma 2.2. [9]. *Let $(\mathcal{E}, \|.\|)$ be a Banach space. Let $\{\nu_n\}$ and $\{\eta_n\}$ be two bounded sequences in $\mathcal{E}$ and $\lambda \in (0,1)$. Assume that $\nu_{n+1} = \lambda\eta_n + (1-\lambda)\nu_n$ and $\|\eta_{n+1} - \eta_n\| \leq \|\nu_{n+1} - \nu_n\|$, for any $n \in \mathbb{N}$. Then $\lim_{n\to\infty} \|\nu_n - \eta_n\| = 0$ holds.*

Definition 2.3. [2]. We say that $\mathcal{E}$ satisfies the monotone weak-Opial property if for any monotone sequence $\{\nu_n\}$ in $\mathcal{E}$ which converges weakly to ν, we have

$$\liminf_{n\to\infty} \|\nu_n - \nu\| < \liminf_{n\to\infty} \|\nu_n - y\|,$$

for any $y \neq \nu$ and y is greater or less than all the elements of the sequence $\{\nu_n\}$.

Definition 2.4. [2]. Let $\mathcal{E}$ be a Banach space and $\mathcal{U}$ a closed convex subset of $\mathcal{E}$ such that $\mathcal{U} \neq \emptyset$. Let $\Upsilon : \mathcal{U} \to \mathcal{U}$ be a mapping:

- The mapping Υ is said to be compact if $\Upsilon(\mathcal{U})$ has a compact closure.
- The mapping Υ is said to be weakly compact if $\Upsilon(\mathcal{U})$ has a weakly compact closure.

3. α-Convex Generalized Nonexpansive Mapping

Shukla et al. [20] considered the following category of mappings:

Definition 3.1. Consider $\mathcal{E}$ as a Banach space and $\mathcal{U}$ as a nonempty convex subset of $\mathcal{E}$. A mapping $\Upsilon : \mathcal{U} \to \mathcal{U}$ is termed α-convex generalized nonexpansive (α-CGNE) if there exists $\alpha \in (0,1)$ such that

$$\|\Upsilon((1-\alpha)\nu + \alpha\Upsilon(\nu)) - \Upsilon(\eta)\| \leq (1-\alpha)\|\nu - \eta\| + \alpha\|\Upsilon(\nu) - \eta\| \tag{3.1}$$

for all $\nu, \eta \in \mathcal{U}$.

Remark 3.2. For $\alpha = 0$, condition (3.1) reduces to that of a nonexpansive mapping. For $\alpha = 1$, condition (3.1) signifies a fundamentally nonexpansive mapping [11].

Example 3.3. [20]. Consider the set $\mathcal{U}$ as the interval $[0,4]$ within the real numbers, equipped with the standard norm. Define $\Upsilon : \mathcal{U} \to \mathcal{U}$ by

$$\Upsilon(\nu) = \begin{cases} 0, & \text{if } \nu \neq 4 \\ \frac{36}{19}, & \text{if } \nu = 4 \end{cases}$$

Initially, we demonstrate that Υ is an α-convex generalized mapping, where $\alpha = \frac{9}{10}$. On the other hand, Υ does not satisfy condition (C). Hence Υ is not nonexpansive mapping.

Proposition 3.4. [20]. *Let $\Upsilon : \mathcal{U} \to \mathcal{U}$ be an α-CGNE mapping with $F(\Upsilon) \neq \emptyset$. Then Υ is a quasinonexpansive.*

We prove the following important Lemma:

Lemma 3.5. *Suppose $\mathcal{E}$ is a Banach space and $\mathcal{U}$ represents a non-empty convex subset of $\mathcal{E}$. Let $\Upsilon : \mathcal{U} \to \mathcal{U}$ be an α-CGNE mapping. Then*

$$\|\nu - \Upsilon(\eta)\| \leq (1 + 2\alpha)\|\nu - \Upsilon(\nu)\| + \|\nu - \eta\|$$

holds for all $\nu, \eta \in \mathcal{U}$. That is, mapping Υ satisfies condition (E).

Proof. By the triangle inequality, we have

$$\begin{aligned}
\|\nu - \Upsilon(\eta)\| &\leq \|\nu - \Upsilon((1-\alpha)\nu + \alpha\Upsilon(\nu))\| + \|\Upsilon((1-\alpha)\nu + \alpha\Upsilon(\nu)) - \Upsilon(\eta)\| \\
&\leq \|\nu - (1-\alpha)\nu + \alpha\Upsilon(\nu)\| + \|(1-\alpha)\nu + \alpha\Upsilon(\nu) - \Upsilon((1-\alpha)\nu + \alpha\Upsilon(\nu))\| \\
&\quad + \|\Upsilon((1-\alpha)\nu + \alpha\Upsilon(\nu)) - \Upsilon(\eta)\| \\
&= \alpha\|\nu - \Upsilon(\nu)\| + \|(1-\alpha)\nu + \alpha\Upsilon(\nu) - \Upsilon((1-\alpha)\nu + \alpha\Upsilon(\nu))\| \\
&\quad + \|\Upsilon((1-\alpha)\nu + \alpha\Upsilon(\nu)) - \Upsilon(\eta)\|.
\end{aligned}$$

By the condition on the mapping Υ, we have

$$\begin{aligned}
\|\nu - \Upsilon(\eta)\| &\leq \alpha\|\nu - \Upsilon(\nu)\| + \|(1-\alpha)\nu + \alpha\Upsilon(\nu) - \Upsilon((1-\alpha)\nu + \alpha\Upsilon(\nu))\| \\
&\quad + (1-\alpha)\|\nu - \eta\| + \alpha\|\Upsilon(\nu) - \eta\| \\
&\leq \alpha\|\nu - \Upsilon(\nu)\| + \|(1-\alpha)\nu + \alpha\Upsilon(\nu) - \Upsilon((1-\alpha)\nu + \alpha\Upsilon(\nu))\| \\
&\quad + (1-\alpha)\|\nu - \eta\| + \alpha\|\nu - \eta\| + \alpha\|\nu - \Upsilon(\nu)\| \\
&\leq 2\alpha\|\nu - \Upsilon(\nu)\| + \|(1-\alpha)\nu + \alpha\Upsilon(\nu) - \Upsilon((1-\alpha)\nu + \alpha\Upsilon(\nu))\| + \|\nu - \eta\| \\
&= 2\alpha\|\nu - \Upsilon(\nu)\| + \|\nu - \eta\| \\
&\quad + \|(1-\alpha)\nu + \alpha\Upsilon(\nu) + (1-\alpha)\Upsilon(\nu) - (1-\alpha)\Upsilon(\nu) - \Upsilon((1-\alpha)\nu + \alpha\Upsilon(\nu))\| \\
&\leq 2\alpha\|\nu - \Upsilon(\nu)\| + \|\nu - \eta\| + (1-\alpha)\|\nu - \Upsilon(\nu)\| + \|\Upsilon((1-\alpha)\nu + \alpha\Upsilon(\nu)) - \Upsilon(\nu)\|.
\end{aligned}$$

By the condition on the mapping Υ, we obtain

$$\begin{aligned}
\|\nu - \Upsilon(\eta)\| &\leq (1+\alpha)\|\nu - \Upsilon(\nu)\| + \|\nu - \eta\| + \alpha\|\nu - \Upsilon(\nu)\| \\
&\leq (1+2\alpha)\|\nu - \Upsilon(\nu)\| + \|\nu - \eta\|.
\end{aligned}$$

By taking $\mu = (1 + 2\alpha)$, we can conclude the proof. □

Example 3.6. Consider the mapping Υ defined on the interval $[0, 4]$, where

$$\Upsilon(\nu) = \begin{cases} 0 & \text{if } \nu \neq 4 \\ 3 & \text{if } \nu = 4. \end{cases}$$

Initially, we demonstrate that Υ fulfills condition (E). We examine three distinct nontrivial cases:

Case (1) $\nu \leq 3$ and $\eta = 4$. Then

$$\begin{aligned}\|\nu - T(\eta)\| = \|3-\nu\| &\leq \|\nu\| + \|4-\nu\| \\ &= \|\nu - T(\nu)\| + \|\eta - \nu\|.\end{aligned}$$

Case (2) $\nu > 3$ and $\eta = 4$. Then

$$\begin{aligned}\|\nu - T(\eta)\| = \|\nu - 3\| &\leq 1 + \|4-\nu\| \\ &\leq \|\nu\| + \|4-\nu\| \\ &\leq \|\nu - T(\nu)\| + \|\eta - \nu\|.\end{aligned}$$

Case (3) If $\nu = 4, \eta \neq 4$, then

$$\begin{aligned}\|\nu - T(\eta)\| = \|4\| &\leq 4\|1\| + \|4-\eta\| \\ &\leq 4\|\nu - T(\nu)\| + \|\nu - \eta\|.\end{aligned}$$

Indeed at $\nu = 4$ and $\eta = 4$

$$\begin{aligned}\|\Upsilon\left((1-\alpha)\nu + \alpha\Upsilon(\nu)\right) - \Upsilon(\eta)\| &= \|\Upsilon\left((1-\alpha)4 + \alpha\Upsilon(4)\right) - \Upsilon(4)\| \\ &= \|\Upsilon\left(4-\alpha\right) - 3\| \\ &= \|0-3\| = 3 > \alpha = (1-\alpha)\|4-4\| + \alpha\|\Upsilon(4) - 4\| \\ &= (1-\alpha)\|\nu - \eta\| + \alpha\|\Upsilon(\nu) - \eta\|\end{aligned}$$

hold. Thus, the mapping Υ is not an α-CGNE.

4. Monotone α-CGNE

In this section we extend the class of α-CGNE in the setting of ordered Banach spaces.

Definition 4.1. Let $(\mathcal{E}, \|.\|, \preceq)$ be an ordered Banach space and $\mathcal{U}$ be a nonempty subset of $\mathcal{E}$. A mapping $\Upsilon : \mathcal{U} \to \mathcal{U}$ is said to be monotone if

$$\nu \preceq \eta \text{ implies } \Upsilon(\nu) \preceq \Upsilon(\eta)$$

for all $\nu, \eta \in \mathcal{U}$.

Definition 4.2. Let $(\mathcal{E}, \|.\|, \preceq)$ be an ordered Banach space and $\mathcal{U}$ be a nonempty convex subset of $\mathcal{E}$. A mapping $\Upsilon : \mathcal{U} \to \mathcal{U}$ is said to be monotone α-CGNE if Υ is monotone and there exists $\alpha \in (0,1)$ such that

$$\|\Upsilon\left((1-\alpha)\nu + \alpha\Upsilon(\nu)\right) - \Upsilon(\eta)\| \leq (1-\alpha)\|\nu - \eta\| + \alpha\|\Upsilon(\nu) - \eta\| \tag{4.1}$$

for all $\nu, \eta \in \mathcal{U}$ with $\nu \preceq \Upsilon(\nu) \preceq \eta$.

Lemma 4.3. *Suppose $\mathcal{E}$ is a Banach space and $\mathcal{U}$ represents a non-empty convex subset of $\mathcal{E}$. Let $\Upsilon : \mathcal{U} \to \mathcal{U}$ be a monotone α-CGNE mapping. Then*

$$\|\nu - \Upsilon(\eta)\| \leq (1+2\alpha)\|\nu - \Upsilon(\nu)\| + \|\nu - \eta\|$$

holds for all $\nu, \eta \in \mathcal{U}$ with $\nu \preceq \Upsilon(\nu) \preceq \eta$

The class of α-CGNE mappings enjoy an approximate fixed point property. We have a similar conclusion for monotone α-CGNE.

Lemma 4.4. *Consider an ordered Banach space $(\mathcal{E}, \|.\|, \preceq)$. Let $\mathcal{U}$ be a nonempty bounded convex subset of $\mathcal{E}$, and let $\Upsilon : \mathcal{U} \to \mathcal{U}$ be a monotone α-CGNE mapping. Suppose $\nu_0 \in \mathcal{U}$ such that ν_0 and $\Upsilon(\nu_0)$ are comparable. Define the sequence $\{\nu_n\}$ as follows:*

$$\nu_{n+1} = (1-\alpha)\nu_n + \alpha\Upsilon(\nu_n) \quad \forall\, n \in \mathbb{N} \cup \{0\}.$$

Then $\lim_{n\to\infty} \|\nu_n - \Upsilon(\nu_n)\| = 0$, i.e., $\{\nu_n\}$ is an a.f.p.s. of Υ.

Proof. Since order intervals are convex, if $\nu_0 \preceq \Upsilon(\nu_0)$ (or equivalently, $\Upsilon(\nu_0) \preceq \nu_0$), the sequence $\{\nu_n\}$ exhibits monotonic behavior: it is monotone increasing if $\nu_0 \preceq \Upsilon(\nu_0)$, and monotone decreasing if $\Upsilon(\nu_0) \preceq \nu_0$. Furthermore, it can be straightforwardly demonstrated by induction that under the assumption $\nu_0 \preceq \Upsilon(\nu_0)$, for every $n \in \mathbb{N}$, the following inequalities hold:

$$\nu_n \preceq \nu_{n+1} \preceq \Upsilon(\nu_n) \preceq \Upsilon(\nu_{n+1}).$$

Again,

$$\alpha\left(\nu_n - \Upsilon\left(\nu_n\right)\right) = (\nu_n - \nu_{n+1}), \tag{4.2}$$

and ν_n is comparable to $\Upsilon(\nu_n)$. Using the fact that, Υ is monotone α-CGNE mapping implies that

$$\begin{aligned} \|\Upsilon\left(\nu_{n+1}\right) - \Upsilon\left(\nu_n\right)\| &= \|\Upsilon\left((1-\alpha)\nu_n + \alpha\Upsilon(\nu_n)\right) - \Upsilon\left(\nu_n\right)\| \\ &\leq (1-\alpha)\|\nu_n - \nu_n\| + \alpha\left\|\nu_n - \Upsilon\left(\nu_n\right)\right\| \\ &= \alpha\left\|\nu_n - \Upsilon\left(\nu_n\right)\right\|. \end{aligned}$$

From (4.2)

$$\|\Upsilon\left(\nu_{n+1}\right) - \Upsilon\left(\nu_n\right)\| \leq \|\nu_{n+1} - \nu_n\|.$$

for any $n \in \mathbb{N}$. Using Lemma 2.2, we conclude that $\lim_{n\to\infty} \|\nu_n - \Upsilon\left(\nu_n\right)\| = 0$. □

By incorporating compactness into the assumptions of Lemma 4.4, we achieve the initial convergence outcome for an iteration linked to an α-CGNE mapping.

Theorem 4.5. *Consider an ordered Banach space $(\mathcal{E}, \|\cdot\|, \preceq)$. Let $\mathcal{U}$ be a nonempty bounded closed convex subset of $\mathcal{E}$. Assume that $\Upsilon : \mathcal{U} \to \mathcal{U}$ is a compact monotone α-CGNE mapping. Suppose $\nu_0 \in \mathcal{U}$ such that ν_0 and $\Upsilon(\nu_0)$ are comparable. Define the sequence $\{\nu_n\}$ as follows:*

$$\nu_{n+1} = (1-\alpha)\nu_n + \alpha\Upsilon(\nu_n) \quad \forall\, n \in \mathbb{N} \cup \{0\}.$$

Then $\{\nu_n\}$ converges to a point $z \in F(\Upsilon)$.

Proof. Without loss of generality, let us assume $\nu_0 \preceq \Upsilon(\nu_0)$. According to Lemma 4.4, we know that $\lim_{n\to\infty} \|\nu_n - \Upsilon(\nu_n)\| = 0$, given that Υ is compact. Consequently, there exists a point $z \in \mathcal{U}$ and a subsequence $\{\Upsilon(\nu_{\phi(n)})\}$ converging to z. Note that $\mathcal{U}$ being closed implies $z \in \mathcal{U}$. Clearly, the subsequence $\{\nu_\phi(n)\}$ also converges to z. Since

$$\nu_n \preceq \nu_{n+1} \preceq \Upsilon\left(\nu_n\right) \preceq \Upsilon\left(\nu_{n+1}\right),$$

and order intervals are closed, we conclude that $\nu_n \preceq \Upsilon\left(\nu_n\right) \preceq z$, for any $n \in \mathbb{N}$. Since Υ is monotone α-CGNE mapping, from Lemma 4.3 we have

$$\left\|\nu_{\phi(n)} - \Upsilon(z)\right\| \leq (1+2\alpha)\left\|\Upsilon\left(\nu_{\phi(n)}\right) - \nu_{\phi(n)}\right\| + \left\|\nu_{\phi(n)} - z\right\|$$

for any $n \in \mathbb{N}$. Therefore $\left\{\nu_{\phi(n)}\right\}$ also converges to $\Upsilon(z)$. Hence $\Upsilon(z) = z$. The quasi-nonexpansiveness of Υ and $\nu_n \preceq z$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned} \|\nu_{n+1} - z\| &\leq (1-\alpha)\left\|\Upsilon\left(\nu_n\right) - z\right\| + \alpha\left\|\nu_n - z\right\| \\ &\leq (1-\alpha)\left\|\nu_n - z\right\| + \alpha\left\|\nu_n - z\right\| \\ &= \left\|\nu_n - z\right\|, \end{aligned}$$

for any $n \in \mathbb{N}$. The sequence $\{\|\nu_n - z\|\}$ is a decreasing sequence of positive numbers of which a subsequence goes to 0. Thus, $\lim_{n\to\infty} \|\nu_n - z\| = 0$ and $\{\nu_n\}$ converges to z. □

Theorem 4.6. *Consider an ordered Banach space $(\mathcal{E}, \|.\|, \preceq)$ that satisfies the monotone weak-Opial property. Let $\mathcal{U}$ be a nonempty bounded closed convex subset of $\mathcal{E}$, and let $\Upsilon : \mathcal{U} \to \mathcal{U}$ be a weakly compact monotone α-CGNE mapping. Suppose $\nu_0 \in \mathcal{U}$ such that ν_0 and $\Upsilon(\nu_0)$ are comparable. Define the sequence $\{\nu_n\}$ as follows:*

$$\nu_{n+1} = (1-\alpha)\nu_n + \alpha\Upsilon(\nu_n)\ \forall\, n \in \mathbb{N} \cup \{0\}.$$

Then $\{\nu_n\}$ converges weakly to a point $z \in F(\Upsilon)$.

Proof. We suppose $\nu_0 \preceq \Upsilon(\nu_0)$. From Lemma 4.4, it follows that $\lim_{n\to\infty} \|\nu_n - \Upsilon(\nu_n)\| = 0$. Since Υ is weakly compact, there exists a point z and a subsequence $\{\Upsilon(\nu_{\phi(n)})\}$ that converges weakly to z. Given that $\mathcal{U}$ is closed and convex, we have $z \in \mathcal{U}$. Additionally, the sequence $\{\nu_{\phi(n)}\}$ also weakly converges to z. Because $\{\nu_n\}$ is monotone increasing, $\{\nu_n\}$ weakly converges to z, as does $\{\Upsilon(\nu_n)\}$. Since

$$\nu_n \preceq \Upsilon(\nu_n) \preceq z$$

the monotonicity of Υ implies

$$\nu_n \preceq \Upsilon(z) \text{ for any } n \in \mathbb{N}.$$

Assuming $\Upsilon(z) \neq z$, the monotone weak-Opial property yields:

$$\liminf_{n\to\infty} \|\nu_n - z\| < \liminf_{n\to\infty} \|\nu_n - \Upsilon(z)\|.$$

Given that Υ is an α-CGNE mapping, we have:

$$\|\nu_n - \Upsilon(z)\| \leq (1+2\alpha)\|\Upsilon(\nu_n) - \nu_n\| + \|\nu_n - z\|$$

for any $n \in \mathbb{N}$. Consequently,

$$\liminf_{n\to\infty} \|\nu_n - \Upsilon(z)\| \leq \liminf_{n\to\infty} \|\nu_n - z\|.$$

This contradiction forces $\Upsilon(z) = z$ and $z \in F(\Upsilon)$. □

Building primarily on the foundations established in the two preceding theorems, we get an existence fixed point theorem for monotone α-CGNE mapping.

Theorem 4.7. *Consider an ordered Banach space $(\mathcal{E}, \|.\|, \preceq)$. Let $\mathcal{U}$ be a nonempty convex subset of $\mathcal{E}$, and let $\Upsilon : \mathcal{U} \to \mathcal{U}$ be a monotone α-CGNE mapping. Suppose $\nu_0 \in \mathcal{U}$ such that ν_0 and $\Upsilon(\nu_0)$ are comparable. Assume one of the following conditions holds:*

(a) *$\mathcal{U}$ is compact;*

(b) *$\mathcal{U}$ is weakly compact and $\mathcal{E}$ satisfies the monotone weak-Opial property.*

Then Υ has a fixed point.

Authors' Contributions. All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Funding information. This work was supported by Directorate of Research and Innovation, Walter Sisulu University, South Africa.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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