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NONLINEAR $\overrightarrow{p}(.)$-ANISOTROPIC PROBLEM INVOLVING NEUMANN BOUNDARY CONDITIONS AND L^1 DATA

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Abstract. This paper studies a class of multivalued anisotropic elliptic problems in variable exponent spaces, subject to Neumann boundary conditions, with data in L^∞ or L^1 data. Using approximation techniques, the theory of monotonicity in Banach spaces, and compactness arguments, we establish the existence of renormalized and entropy solutions. Additionally, by employing a comparison principle, we demonstrate the uniqueness of the entropy solution.

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1. Introduction

Over the last ten years, the study of partial differential equations and variational problems with the assumption of $p(.)$-growth has received significant attention within the mathematical community. The motivation for studying such equations stems from their ability to provide accurate mathematical models for describing the behavior of phenomena that can change state over time. In the literature, Chen et al. have demonstrated the importance of Sobolev spaces with variable exponents in the process of image restoration (see [9]). In a recent paper [14], Jean-Luc Henri et *al.* employed the $p(.)$-Laplacian operator to perform smoothing on digital images. They proved that its smoothing power plays a crucial role in the restoration process.

The variable exponent space appear also in the modeling of electro-rheological fluids (see [3], [11], [28]) thermorheological fluids.

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In general the so-called isotropic $p(.)$-Leray-Lions operator is used to model non-homogeneous materials in single direction. However, for materials that spread in several directions, the anisotropic $\overrightarrow{p}(.)$-Leray-Lions operator is used. For example, the anisotropic operator is used to describe the spread of an epidemic disease (see [4]).

In this paper, we study the existence and uniqueness of solutions for the following nonlinear multivalued elliptic anisotropic problem

$$(\mathcal{P})\begin{cases} -\displaystyle\sum_{j=1}^{N}\frac{\partial}{\partial x_j}a_j(x,\frac{\partial u}{\partial x_j})+\beta(u)\ \ni\ f \ \text{ in }\ \Omega \\ \\ \displaystyle\sum_{j=1}^{N}a_j(x,\frac{\partial u}{\partial x_j}).\eta_j=0 \qquad\qquad \text{on }\ \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open domain of $\mathbb{R}^N$ $(N\geq 3)$ with a smooth boundary $\partial\Omega$ and $\eta=(\eta_1,...,\eta_N)$ is the unit outward normal vector on $\partial\Omega$. For $j=1,...,N$, $a_j:\Omega\times\mathbb{R}\longrightarrow\mathbb{R}$ is a Carathéodory function satisfying classical hypotheses in the study of nonlinear problems.

The function $\beta:\mathbb{R}\to 2^{\mathbb{R}}$ is a maximal monotone mapping with $0\in\beta(0)$ and f belongs to $L^\infty(\Omega)$ or $L^1(\Omega)$.

We emphasize that for the Neumann boundary condition, we must seek the solution in Sobolev spaces with variable exponent, $W^{1,\overrightarrow{p}(.)}(\Omega)$. Unfortunately, in this space, we lose the Poincaré inequality, which is an important tool for obtaining coercivity in the Dirichlet case. Therefore, we must face with a non-coercive operator during the approximation process of the problem $(\mathcal{P})$. To overcome this difficulty, we add a strongly monotone perturbation term to the approximate problem $(\mathcal{P}_r)$, which allows us to obtain the coercivity of the associated operator. When the right-hand side data is an L^∞-function, we establish an L^∞-estimate on the approximation β_r of β, which ensures its weak convergence star in $L^\infty(\Omega)$. However, when the right-hand side data is an L^1-function, one can only establish an L^1-estimate on the sequence $(w_m)_{m\in\mathbb{N}}$ derived from the graph β. The convergence of this sequence in $L^1(\Omega)$ is a crucial step in the proof of the existence of a solution. But it is not easy to establish. To bypass this difficulty, we use a relatively compactness argument and obtain weak convergence of $(w_m)_{m\in\mathbb{N}}$ in $L^1(\Omega)$.

Note that there is a large literature on problem related to $(\mathcal{P})$. However, only particular cases of the data β have been considered.

The first attempts to tackle a problem like $(\mathcal{P})$ were made by Boureanu and Rădulescu in [7] (see also [25]), where they studied a non-homogeneous anisotropic Neumann problem with an obstacle. They established the existence and multiplicity of weak solutions and also identified conditions under which the uniqueness of the solution can be achieved. Given $\beta(u)=|u|^{p_M(x)-2}u$ (p_M to be define later), the authors in [6] used the techniques of minimization to obtain the existence of a weak solution to problem $(\mathcal{P})$. Furthermore, they also obtained the existence and uniqueness of an entropy solution using approximation methods.

In the case where β is a continuous and non-decreasing function such that $\beta(0) = 0$, Ouaro et *al.* [15,18] used the technique of maximal monotone operators in Banach spaces (see [29]) to prove the existence and uniqueness of entropy solutions of $(\mathcal{P})$ when the data f is an L^1-function or diffuse measure data. Furthermore, in collaboration with the second author (see [19]), they used the same techniques to prove the existence and uniqueness of a solution to the problem $(\mathcal{P})$ in the framework of maximal monotone graph β with bounded domain, i.e., $\overline{\text{Dom}(\beta)} = [a, b]$ with $a, b \in \mathbb{R}$ such that $a \leq 0 \leq b$.

We stress that the well-posedness of $(\mathcal{P})$ depends on how the domain of β is defined (for example, $\text{dom}(\beta) = \mathbb{R}$, $\text{dom}(\beta) \neq \mathbb{R}$ and closed, $\text{dom}(\beta) \neq \mathbb{R}$ and open).

A natural question that arising from our previous work [19] is: what happens when $\overline{\text{Dom}(\beta)} \neq [a, b]$? This paper aims to extend our main results in [19]. Roughly speaking, we establish the existence and uniqueness of a renormalized or entropy solution to the problem $(\mathcal{P})$ when the domain of β is the whole of $\mathbb{R}$ (i.e. $\text{Dom}(\beta) = \mathbb{R}$) instead of a bounded domain of $\mathbb{R}$. By this work, we also extend the work [10] from Dirichlet case to Neumann boundary condition.

In the framework of isotopic $p(.)$-Leray-Lions type operator, the authors in [27]) analyzed the existence and uniqueness of solution of the following

$$\begin{cases} -\nabla.a(x, \nabla u) + \beta(u) \ni \mu \text{ in } \Omega \\ \\ a(x, \nabla u).\eta = 0 \qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

It is important to mention that β was assumed to have a bounded domain. Recently, in [31], the problem (1.2) was reconsidered under the assumption that $\text{Dom}(\beta) = \mathbb{R}$. The problem $(\mathcal{P})$ can be viewed as the anisotropic version of the nonlinear isotropic problem (1.2). As far as problems like $(\mathcal{P})$ are concerned, we refer to [20–23].

Since this work is an extension of existing research, we will refer to [10, 19, 31] for certain proofs to avoid unnecessary repetition.

The rest of the paper is organized as follows. Section 2 presents some preliminaries on variable exponent spaces. In Section 3, we outline our key assumptions and introduce the concept of solutions. Section 4 establishes the existence of a renormalized solution to the problem $(\mathcal{P})$ for $f \in L^\infty(\Omega)$. In Section 5, we prove the existence of both a renormalized and entropy solution when f belongs to $L^1(\Omega)$. Finally, Section 6 demonstrates the uniqueness of the entropy solution.

2. Preliminary

In this section, we review some definitions and fundamental properties of anisotropic Lebesgue and Sobolev spaces.

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$, with smooth boundary $\partial\Omega$. We define the set

$$C_+(\overline{\Omega}) = \Big\{ p(.) : \overline{\Omega} \longrightarrow (1, \infty) \text{ continuous such that } 1 < p^- \leq p^+ < \infty \Big\},$$

where $p^- := \min_{x \in \overline{\Omega}} p(x)$ and $p^+ := \max_{x \in \overline{\Omega}} p(x)$.

For any $p \in C_+(\overline{\Omega})$, the variable exponent Lebesgue space is defined by

$$L^{p(.)}(\Omega) := \Big\{ u : u \text{ is a measurable real valued function such that } \int_\Omega |u|^{p(x)} dx < \infty \Big\},$$

endowed with the so-called Luxembourg norm

$$|u|_{p(.)} := \inf \Big\{ \lambda > 0 : \int_\Omega \Big| \frac{u(x)}{\lambda} \Big|^{p(x)} dx \leq 1 \Big\}.$$

Moreover, $(L^{p(.)}(\Omega), \|.\|_{p(.)})$ is a separable, reflexive and uniformly convex Banach space. Hence its dual space is isomorphic to $L^{p'(.)}(\Omega)$ where $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$ in Ω.

The $p(.)$-modular of the space $L^{p(.)}(\Omega)$ is the mapping $\rho_{p(.)} : L^{p(.)}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\rho_{p(.)}(u) := \int_\Omega |u|^{p(x)} dx.$$

For any $u \in L^{p(.)}(\Omega)$, the following inequalities (see [12], [13]) holds true.

$$\min \Big\{ |u|_{p(.)}^{p^-};\ |u|_{p(.)}^{p^+} \Big\} \leq \rho_{p(.)}(u) \leq \max \Big\{ |u|_{p(.)}^{p^-};\ |u|_{p(.)}^{p^+} \Big\}. \tag{2.1}$$

For any $u \in L^{p(.)}(\Omega)$ and $v \in L^{p'(.)}(\Omega)$, we have the Hölder type inequality

$$\Big| \int_\Omega uv dx \Big| \leq \Big(\frac{1}{p^-} + \frac{1}{(p')^-} \Big) |u|_{p(.)} |v|_{p'(.)}. \tag{2.2}$$

Let $p_1,\ p_2 \in C_+(\overline{\Omega})$ such that $p_1(x) \leq p_2(x)$ for any $x \in \Omega$, then the embedding $L^{p_2(.)}(\Omega) \hookrightarrow L^{p_1(.)}(\Omega)$ is continuous (see [24], Theorem 2.8).

Next, we introduce the variable exponent Sobolev space

$$W^{1,p(.)}(\Omega) := \Big\{ u \in L^{p(.)}(\Omega) :\ |\nabla u| \in L^{p(.)}(\Omega) \Big\},$$

with the norm

$$||u||_{1,p(.)} = \|u\|_{p(.)} + \|\nabla u\|_{p(.)}.$$

The space $(W^{1,p(.)}(\Omega), \|.\|_{1,p(.)})$ is a separable and reflexive Banach space.

We denote by $W_0^{1,p(.)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(.)}(\Omega)$, and its dual space will be denoted by $W^{-1,p'(.)}(\Omega)$.

Let $p_1(.), ..., p_N(.))$ be N variable exponents in $\mathcal{C}_+(\Omega)$, We denote by

$$p_M(x) := \max(p_1(x), ..., p_N(x)) \text{ and } p_m(x) := \min\left(p_1(x), ..., p_N(x)\right).$$

The anisotropic Sobolev space (see [25]) is defined by

$$W^{1,\overrightarrow{p}(.)}(\Omega) := \left\{u \in L^{p_M(.)}(\Omega) : \frac{\partial u}{\partial x_j} \in L^{p_j(.)}(\Omega),\ j = 1, ..., N\right\},$$

which are separable and reflexive Banach spaces under the norm

$$||u||_{\overrightarrow{p}(.)} = \sum_{j=1}^{N} \left|\frac{\partial u}{\partial x_j}\right|_{p_j(.)}. \tag{2.3}$$

We also defined by $W_0^{1,\overrightarrow{p}(.)}(\Omega)$, the closure of $\mathcal{C}_0^\infty(\Omega)$ with respect to the norm (2.3).

We introduce the numbers

$$q = \frac{N(\overline{p}-1)}{N-1};\ q^* = \frac{N(\overline{p}-1)}{N-\overline{p}} = \frac{Nq}{N-q}$$

and define $P_-^*,\ P_-^+,\ P_{-,\infty} \in \mathbb{R}^+$ by

$$P_-^* = \frac{N}{\displaystyle\sum_{j=1}^{N}\frac{1}{p_j^-} - 1},\ P_-^+ = \max\left\{p_1^-, ..., p_N^-\right\} \text{ and } P_{-,\infty} = \max\left\{P_-^+,\ P_-^*\right\},$$

where $\dfrac{N}{\overline{p}} = \displaystyle\sum_{j=1}^{N}\frac{1}{p_j^-}$.

Theorem 2.1. [24] *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ be a bounded open set and for all $i = 1, ..., N$, $p_j \in L^\infty(\Omega)$, $p_j(x) \geq 1$ a.e. in Ω. Then, for any $q \in L^\infty(\Omega)$ with $q(x) \geq 1$ a.e. in Ω such that*

$$ess \inf_{x\in\Omega}(p_M(x) - q(x)) > 0,$$

we have the compact embedding

$$W^{1,\overrightarrow{p}(.)}(\Omega) \hookrightarrow L^{q(.)}(\Omega).$$

Theorem 2.2. [30] *Let $p_1, ..., p_N \in [1, +\infty)$; $g \in W^{1,(p_1,...,p_N)}(\Omega)$ and*

$$q = \begin{cases} (\overline{p})^* & \text{if } (\overline{p})^* < N \\ \\ \in [1, +\infty) & \text{if } (\overline{p})^* \geq N. \end{cases}$$

Then, there exists a constant $C_4 > 0$ depending on $N, p_1, ..., p_N$ if $\overline{p} < N$ and also on q and $meas(\Omega)$ if $\overline{p} \geq N$ such that

$$||g||_{L^q(\Omega)} \leq C_4 \prod_{j=1}^{N} \left\|\frac{\partial g}{\partial x_j}\right\|_{L^{p_j}(\Omega)}^{\frac{1}{N}}. \tag{2.4}$$

We define the Marcinkiewicz space $\mathcal{M}^q(\Omega)(1 < q < +\infty)$ as the set of measurable function $g : \Omega \longrightarrow \mathbb{R}$ for which the distribution

$$\lambda_g(k) = meas(\{x \in \Omega : |g(x)| > k\}),\ k \geq 0 \tag{2.5}$$

satisfies an estimate of the form

$$\lambda_g(k) \leq Ck^{-q}, \text{ for some finite constant } C > 0. \tag{2.6}$$

We will use the following pseudo norm in $\mathcal{M}^q(\Omega)$

$$||g||_{\mathcal{M}^q(\Omega)} := \inf\{C > 0 : \lambda_g(k) \leq Ck^{-q}, \ \forall k > 0\}. \tag{2.7}$$

Throughout the paper, the truncation function T_k, of level $k > 0$ by

$$T_k(s) = \max\{-k, \min\{k; s\}\}. \tag{2.8}$$

It is obvious that $\lim_{k \to \infty} T_k(s) = s$ and $|T_k(s)| = \min\{|s|; k\}$.
Set $\mathcal{T}^{1,\overrightarrow{p}(.)}(\Omega)$ as the set of measurable functions $u : \Omega \longrightarrow \mathbb{R}$ such that $T_k(u) \in W^{1,\overrightarrow{p}(.)}(\Omega)$. We define the space $\mathcal{T}_{\mathcal{H}}^{1,\overrightarrow{p}(.)}(\Omega)$ as the set of functions $u \in \mathcal{T}^{1,\overrightarrow{p}(.)}(\Omega)$ such that there exists a sequence $(u_n)_{n \in \mathbb{N}} \subset W^{1,\overrightarrow{p}(.)}(\Omega)$ satisfying

$$u_n \longrightarrow u \text{ a.e. in } \Omega$$

and

$$\frac{\partial T_k(u_n)}{\partial x_j} \longrightarrow \frac{\partial T_k(u)}{\partial x_j} \text{ in } L^1(\Omega), \ \forall \, k > 0.$$

Proposition 1. [5] *Let $u \in \mathcal{T}^{1,p(.)}(\Omega)$ and $\gamma > 0$. For any $j = 1, ..., N$, there exists a unique measurable function $\vartheta_j : \Omega \longrightarrow \mathbb{R}^N$ such that*

$$\forall \, \gamma > 0 \ \frac{\partial T_\gamma(u)}{\partial x_j} = \vartheta_j \chi_{\{|u| < \gamma\}}, \text{ for a.e.} x \in \Omega,$$

where A denotes the characteristic function of a measurable set A.
The function ϑ_j are called the weak gradient of u and is still denoted by $\frac{\partial u}{\partial x_j}$. Moreover, if $u \in W^{1,p(.)}(\Omega)$, then $\vartheta \in (L^{p(.)}(\Omega))^N$ and $\vartheta_j = \frac{\partial u}{\partial x_j}$ in the usual sense.

For any $r \in \mathbb{R}$ and any measurable function u on Ω, $[u = r]$, $[u \leq r]$ and $[u \geq r]$ denote, respectively the set
$\{x \in \Omega : u(x) = r\}$, $\{x \in \Omega : u(x) \leq r\}$, $\{x \in \Omega : u(x) \geq r\}$.
For any given $l, k > 0$, we define the function h_l by $h_l = \min\{(l + 1 - |r|)^+, 1\}$.
Let sign_0^+ be a function that assigns values as follows

$$\text{sign}_0^+(s) = \begin{cases} 1 \text{ if } s > 0, \\ 0 \text{ if } s \leq 0. \end{cases}$$

For $\delta > 0$, we define the function $H_\delta^+ : \mathbb{R} \to \mathbb{R}$ by

$$H_\delta^+(r) = \begin{cases} 1 \ \text{ if } r > \delta \\ \\ \dfrac{r}{\delta} \ \text{ if } 0 \leq r \leq \delta \\ \\ 0 \ \text{ if } r < 0. \end{cases}$$

It is evident that H_δ^+ is an approximation of sign_0^+.

Let β be a maximal monotone operator defined on $\mathbb{R}$, we denote by β_0 the main section of β; i.e.,

$$\beta^0(s) = \begin{cases} \text{minimal absolute value of } \ \beta(s) \ \text{ if } \beta(s) \ \neq \emptyset \\ +\infty & \text{if } [s, +\infty) \cap D(\beta) = \emptyset \\ -\infty & \text{if } (-\infty, s] \cap D(\beta) = \emptyset. \end{cases}$$

For a maximal monotone graph β in $\mathbb{R} \times \mathbb{R}$, for any $r \in (0, 1]$, the Yosida approximation β_r of β (see [1,2,8]) is given by $\beta_r = \dfrac{1}{r}(I - (I + r\beta)^{-1})$.

If $s \in Dom(\beta)$, $|\beta_r(s)| \leq |\beta^0(s)|$ and $\beta_r(s) \longrightarrow \beta^0(s)$, as $r \to 0$, and if $s \notin Dom(\beta)$, $|\beta_r(s)| \longrightarrow \infty$, as $r \to 0$.

Lemma 2.1. [26] *Let $(\beta_n)_{n \geq 1}$ be a sequence of maximal monotone graphs such that $\beta_n \to \beta$ in the sense of the graph (for $(x, y) \in \beta$, there exists $(x_n, y_n) \in \beta_n$ such that $x_n \to x$ and $y_n \to y$). We consider two sequences $(z_n)_{n \geq 1} \subset L^1(\Omega)$ and $(w_n)_{n \geq 1} \subset L^1(\Omega)$. We suppose that: $\forall n \geq 1, w_n \in \beta_n(z_n)$, $(w_n)_{n \geq 1}$ is bounded in $L^1(\Omega)$ and $z_n \to z$ in $L^1(\Omega)$. Then, $z \in dom(\beta)$.*

3. Assumptions and Notions of Solution

In this paper, we consider problem $(\mathcal{P})$ under the following assumptions on the data.

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$ and let $\overrightarrow{p}(.) = (p_1(.), ..., p_N(.))$ be a vector such that for any $j = 1, ..., N, \ p_j(.) \in C_+(\overline{\Omega})$.

For any $j = 1, ..., N$, let $a_j : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be a Carathéodory function satisfying :

• there exists a positive constant C_1 such that

$$|a_j(x, \xi)| \leq C_1 \Big(k_i(x) + |\xi|^{p_j(x)-1} \Big), \tag{3.1}$$

for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}$, where k_i is a non-negative function in $L^{p_i'(.)}(\Omega)$, with $\dfrac{1}{p_j(x)} + \dfrac{1}{p_i'(x)} = 1$;

• for $\xi,\ \eta\ \in \mathbb{R}$ with $\xi \neq \eta$ and for every $x \in \Omega$, there exists a positive constant C_2 such that

$$(a_j(x,\xi) - a_j(x,\eta))(\xi - \eta) \geq \begin{cases} C_2|\xi - \eta|^{p_j(x)} & \text{if } |\xi - \eta| \geq 1 \\ \\ C_2|\xi - \eta|^{p_j^-} & \text{if } |\xi - \eta| < 1 \end{cases} \tag{3.2}$$

and

• there exists a positive constant C_3 such that

$$a_j(x,\xi).\xi \geq C_3|\xi|^{p_j(x)}, \tag{3.3}$$

for every $\xi \in \mathbb{R}$ and almost every $x \in \Omega$.

Throughout this paper, we assume that

$$\frac{\overline{p}(N-1)}{N(\overline{p}-1)} < p_j^- < \frac{\overline{p}(N-1)}{N-\overline{p}}, \quad \frac{p_i^+ - p_j^- - 1}{p_j^-} < \frac{\overline{p} - N}{\overline{p}(N-1)} \tag{3.4}$$

and

$$\sum_{j=1}^{N} \frac{1}{p_j^-} > 1, \tag{3.5}$$

where $\dfrac{N}{\overline{p}} = \displaystyle\sum_{j=1}^{N} \frac{1}{p_j^-}$.

Definition 3.1. *A renormalized solution of problem* $(\mathcal{P})$ *is a couple of functions* $(u,w) \in \mathcal{T}_{\mathcal{H}}^{1,\overrightarrow{p}(.)}(\Omega) \times L^1(\Omega)$ *satisfying the following conditions.*

(i): $u : \Omega \longrightarrow \mathbb{R}$ *is measurable,* $u(x) \in dom(\beta(x))$, $w(x) \in\ \beta(u(x))$ *for a.e.* x *in* Ω,

(ii): *For all* $k > 0$, $S \in C_c^1(\mathbb{R})$ *and* $\varphi \in W^{1,\overrightarrow{p}(.)}(\Omega) \cap L^\infty(\Omega)$,

$$\sum_{j=1}^{N} \int_\Omega a_j\left(x, \frac{\partial u}{\partial x_j}\right) \frac{\partial}{\partial x_j}[S(u)\varphi]dx + \int_\Omega wS(u)\varphi dx = \int_\Omega fS(u)\varphi dx, \tag{3.6}$$

(iii):

$$\lim_{k\to\infty} \int_{\{l<|u|<l+1\}} a_j\left(x, \frac{\partial u}{\partial x_j}\right) \frac{\partial u}{\partial x_j} dx = 0. \tag{3.7}$$

We also introduce the notion of an entropy solution for the problem $(\mathcal{P})$, which will be useful in the proof of uniqueness.

Definition 3.2. *An entropy solution of problem* $(\mathcal{P})$ *is a pair of functions* $(u,w) \in \mathcal{T}_{\mathcal{H}}^{1,\overrightarrow{p}(.)}(\Omega) \times L^1(\Omega)$ *such that* $u(x) \in dom(\beta(x))$, $w(x) \in\ \beta(u(x))$ *and for a.e* $x \in \Omega$,

$$\sum_{j=1}^{N} \int_\Omega a_j\left(x, \frac{\partial u}{\partial x_j}\right) \frac{\partial}{\partial x_j} T_k(u-\varphi)dx + \int_\Omega wT_k(u-\varphi)dx \leq \int_\Omega fT_k(u-\varphi)dx, \tag{3.8}$$

for all $\varphi \in W^{1,\overrightarrow{p}(.)}(\Omega) \cap L^\infty(\Omega)$ *such that* $\varphi(x) \in \beta(u(x))$ *for a.e* x *in* Ω.

4. Existence of Solution for L^∞ Data

Theorem 4.1. *Let $f \in L^\infty(\Omega)$. Then, there exists at least one renormalized solution (u, w) of the problem $(\mathcal{P})$.*

Proof. To prove this theorem we proceed by steps.

Step 1: Approximate problem

For every $r > 0$, we consider the Yosida regularization $\beta_r = \dfrac{1}{r}(I - (I + r\beta)^{-1})$ of β (see [8]).

Now, we consider the sequence of approximate problem

$$P(\beta_r, f)\begin{cases} -\displaystyle\sum_{j=1}^{N} \frac{\partial}{\partial x_j} a_j(x, \frac{\partial u_r}{\partial x_j}) + \beta_r(T_{\frac{1}{r}}(u_r)) + r|u_r|^{P_M(x)-2}u_r = f \text{ in } \Omega \\ \\ \displaystyle\sum_{j=1}^{N} a_j(x, \frac{\partial u_r}{\partial x_j}).\eta_j = 0 \qquad\qquad \text{on } \partial\Omega. \end{cases}$$

Proposition 2. (*see* [19] *Theorem 3.1*) *Let $f \in L^\infty(\Omega)$. Then, the problem $P(\beta_r, f)$ has at least one weak solution $u_r \in W^{1,\vec{p}(.)}(\Omega)$ in the sense that, $\beta_r(T_{\frac{1}{r}}(u_r)) \in L^1(\Omega)$ and for all $\varphi \in W^{1,\vec{p}(.)}(\Omega) \cap L^\infty(\Omega)$,*

$$\sum_{j=1}^{N} \int_\Omega a_j\left(x, \frac{\partial u_r}{\partial x_j}\right)\frac{\partial \varphi}{\partial x_j}dx + \int_\Omega \beta_r(T_{\frac{1}{r}}(u_r))\varphi dx + r\int_\Omega |u_r|^{P_M(x)-2}u_r\varphi dx = \int_\Omega f\varphi dx. \tag{4.1}$$

Step 2 : A priori estimates

Lemma 4.1. *Let $f \in L^\infty(\Omega)$ and $k > 0$. If u_r is a weak solution of problem $(\mathcal{P}_r)$, then,*

$$\sum_{j=1}^{N} \int_{\{|u_r| \le k\}} \left|\frac{\partial u_r}{\partial x_j}\right|^{p_j(x)} dx \le k(C_7 + 1), \tag{4.2}$$

$$||\beta_r(T_{\frac{1}{r}}(u_r))||_\infty \le \|f\|_\infty, \tag{4.3}$$

$$\sum_{j=1}^{N} \int_{\{l<|u_r|<l+k\}} a_j\left(x, \frac{\partial u_r}{\partial x_j}\right)\frac{\partial u_r}{\partial x_j}dx \le k\int_{\{|u_r|>l\}} |f|dx, \tag{4.4}$$

$$\int_\Omega |\nabla T_k(u_r)|^{p_m^-}dx \le C_6, \tag{4.5}$$

$$\sum_{j=1}^{N} \int_\Omega \left|\frac{\partial u}{\partial x_j}\right|^{p_j(x)} dx \ge C_6||\nabla u||^{p_m^-}_{L^{p_m^-}(\Omega)} - N meas(\Omega) \tag{4.6}$$

and

$$\sum_{j=1}^{N} \int_{\{l<|u_r|<l+k\}} a_j\left(x, \frac{\partial u_r}{\partial x_j}\right)\frac{\partial u_r}{\partial x_j}dx \le k||f||_\infty|\{|u_r| \ge l\}|. \tag{4.7}$$

Proof. By choosing $\varphi = T_k(u_r)$ as a test function in (4.1), one obtains

$$\sum_{j=1}^{N} \int_\Omega a_j\left(x, \frac{\partial u_r}{\partial x_j}\right) \frac{\partial T_k(u_r)}{\partial x_j} + \int_\Omega \beta_r(T_{\frac{1}{r}}(u_r))T_k(u_r)dx$$

$$+r\int_\Omega |u_r|^{P_M(x)-2}u_rT_k(u_r)dx = \int_\Omega fT_k(u_r)dx. \tag{4.8}$$

Due to the coercivity of a_j, the nondecreasing of $\beta_r \circ T_{\frac{1}{r}}$ and the fact that u_r and $T_k(u_r)$ have the same sing, one can conclude that all terms in right hand side of (4.8) are positive.

By neglecting some positive terms and taking into account (3.3), one obtains

$$C_5\sum_{j=1}^{N}\int_{\{|u_r|\leq k\}}\left|\frac{\partial u_r}{\partial x_j}\right|^{p_j(x)}dx \leq \int_\Omega fT_k(u_r)dx \leq \left|\int_\Omega fT_k(u_r)dx\right| \leq k||f||_\infty.$$

On the other hand, one has

$$\sum_{j=1}^{N}\int_{\{|u_r|\leq k\}}\left|\frac{\partial u_r}{\partial x_j}\right|^{p_j^-}dx = \sum_{j=1}^{N}\int_{\{|u_r|\leq k, |\frac{\partial u_r}{\partial x_j}|>1\}}\left|\frac{\partial u_r}{\partial x_j}\right|^{p_j^-}dx + \sum_{j=1}^{N}\int_{\{|u_r|\leq k, |\frac{\partial u_r}{\partial x_j}|\leq 1\}}\left|\frac{\partial u_r}{\partial x_j}\right|^{p_j^-}dx$$

$$\leq \sum_{j=1}^{N}\int_{\{|u_r|\leq k, |\frac{\partial u_r}{\partial x_j}|>1\}}\left|\frac{\partial u_r}{\partial x_j}\right|^{p_j(x)}dx + N\text{meas}(\Omega).$$

Combining the two last inequalities and setting $C_7 = \max\{N\text{meas}(\Omega), \dfrac{k\|f\|_\infty}{C_5}\}$, one obtains (4.2).

Now, we focus on the proof of (4.3). For that, one uses $\varphi_{r,\delta} = \frac{1}{\delta}[T_{k+\delta}(\beta_r(T_{\frac{1}{r}}(u_r))) - T_k(\beta_r(T_{\frac{1}{r}}(u_r)))]$ as a function test in (4.1) to get

$$\sum_{j=1}^{N}\int_\Omega a_j\left(x, \frac{\partial u_r}{\partial x_j}\right)\frac{\partial \varphi_{r,\delta}}{\partial x_j}dx + \int_\Omega \beta_r(T_{\frac{1}{r}}(u_r))\varphi_{r,\delta}dx + r\int_\Omega |u_r|^{P_M(x)-2}u_r\varphi_{r,\delta}dx = \int_\Omega f\varphi_{r,\delta}dx. \tag{4.9}$$

Having in mind that $\dfrac{\partial \varphi_{r,\delta}}{\partial x_j} = \begin{cases} \frac{1}{\delta}\beta_r'(T_{\frac{1}{r}}(u_r))\dfrac{\partial u_r}{\partial x_j} & \text{if } k \leq |\beta_r(T_{\frac{1}{r}}(u_r))| \leq k+\delta, \\ 0 & \text{elsewhere,} \end{cases}$

and the fact that β_r is nondecreasing, one deduce from (3.3) that the first term of (4.9) is positive. the third term is also positive due to the fact that u_r and $\varphi_{r,\delta}$ have the same sign.

Hence, neglecting some positive terms, one obtains

$$\int_\Omega \beta_r(T_{\frac{1}{r}}(u_r))\varphi_{r,\delta}dx \leq \int_\Omega f\varphi_{r,\delta}dx.$$

Then, arguing exactly as in [17], one obtains

$$\text{meas}\{k \leq |\beta_r(T_{\frac{1}{r}}(u_r))|\} = 0, \text{ for any } k > \|f\|_{L^\infty(\Omega)}.$$

So, we conclude that

$$\|\beta_r(T_{\frac{1}{r}}(u_r))\|_{L^\infty(\Omega)} \leq \|f\|_{L^\infty(\Omega)}.$$

For the proofs of (4.4)-(4.7) we refer to [21]. □

Proposition 3. [15,20,26,31]
Let $k>0$ large enough and suppose u_r a weak solution to problem $(\mathcal{P}_r)$. Then

$$meas\{|u_r|>k\} \leq \frac{\|f\|_\infty}{\min\{\beta_r(k), |\beta_r(-k)|\}}, \tag{4.10}$$

$$meas\left\{\left|\frac{\partial u_r}{\partial x_j}\right|>k\right\} \leq \frac{C_6}{k^{\frac{1}{(p_M)'}}} \tag{4.11}$$

and

$$meas\left\{|\nabla u_r|>k\right\} \leq \frac{C_7}{k^{p_m^-}} + \frac{\|f\|_\infty}{\min\{\beta_r(k), |\beta_r(-k)|\}}, \tag{4.12}$$

where C_6, C_7 are positive constants.

Lemma 4.2. [6,15] *For any $k>0$, there exists some constants $C_1,\ C_2>0$ such that:*

(i) $||u_r||_{\mathcal{M}^{q^*}(\Omega)} \leq C_1$;

(ii) $\|\frac{\partial u_r}{\partial x_j}\|_{\mathcal{M}^{p_j^- \frac{q}{\bar{p}}}(\Omega)} \leq C_2,\ \ \forall j=1,...,N.$

Proposition 4. [6,15,19] *Let $k>0$, and suppose u_r a weak solution to problem $(\mathcal{P}_r)$. Then, the sequence $(T_k(u_r))_{r>0}$ is bounded in $W^{1,p_m^-}(\Omega)$.*

Lemma 4.3. *Let suppose u_r a solution of the $(\mathcal{P}_r)$, then*

$$\int_\Omega (|\beta_r(T_{\frac{1}{r}}(u_r)|-k)^+ dx \leq \int_\Omega (|f|-k)^+ dx. \tag{4.13}$$

Proof. Taking $\varphi = H_\delta^+(\beta_r(T_{\frac{1}{r}}(u_r))-k)$ as a test function in (4.1), one obtains

$$\int_\Omega \beta_r(T_{\frac{1}{r}}(u_r))H_\delta^+(\beta_r(T_{\frac{1}{r}}(u_r))-k)dx + \sum_{j=1}^N \int_\Omega a_j\left(x, \frac{\partial u_r}{\partial x_j}\right)\frac{\partial H_\delta^+(\beta_r(T_{\frac{1}{r}}(u_r))-k)}{\partial x_j}dx$$
$$+r\int_\Omega |u_r|^{P_M(x)-2}u_r H_\delta^+(\beta_r(T_{\frac{1}{r}}(u_r))-k)dx = \int_\Omega f H_\delta^+(\beta_r(T_{\frac{1}{r}}(u_r))-k)dx.$$

Since u_r and $H_\delta^+(\beta_r(T_{\frac{1}{r}}(u_r))-k)$ have the same sign, the third term is positive.
Due to the nondecreasing of β_r, one has

$$\sum_{j=1}^N \int_\Omega a_j\left(x, \frac{\partial u_r}{\partial x_j}\right)\frac{\partial H_\delta^+(\beta_r(T_{\frac{1}{r}}(u_r))-k)}{\partial x_j}dx$$
$$= \sum_{j=1}^N \int_\Omega a_j\left(x, \frac{\partial u_r}{\partial x_j}\right)(H_\delta^+)'(\beta_r(T_{\frac{1}{r}}(u_r))-k)\beta_r'(T_{\frac{1}{r}}(u_r))\frac{\partial u_r}{\partial x_j}dx \geq 0.$$

Therefore,

$$\int_\Omega (\beta_r(T_{\frac{1}{r}}(u_r)-k)H_\delta^+(\beta_r(T_{\frac{1}{r}}(u_r))-k)dx \leq \int_\Omega (f-k)H_\delta^+(\beta_r(T_{\frac{1}{r}}(u_r))-k)dx.$$

Letting $\delta \to 0$ we obtain

$$\int_\Omega (\beta_r(T_{\frac{1}{r}}(u_r) - k)^+ dx \leq \int_\Omega (f-k)^+ dx. \tag{4.14}$$

In the same way, we have

$$\int_\Omega (\beta_r(T_{\frac{1}{r}}(u_r) + k)^- dx \leq \int_\Omega (f+k)^- dx. \tag{4.15}$$

By combining (4.14) and (4.15) follows (4.13). □

We establish a $L^\infty(\Omega)$-comparison principle of weak solutions.

Proposition 5. *Let $r > 0$ be fixed. If u_r^1 and u_r^2 are two solutions of $(\mathcal{P}_r)$ associated with $f_1, f_2 \in L^\infty(\Omega)$ respectively, then, the following inequality holds true,*

$$r\int_\Omega (|u_r|^{P_M(x)-2}u_r^1 - |u_r^2|^{P_M(x)-2}u_r^2)^+ dx \leq \int_\Omega (f_1 - f_2)\, sign_0^+(u_r^1 - u_r^2) dx. \tag{4.16}$$

Moreover, if $f_1 \leq f_2$ a.e. in Ω, then

$$u_r^1 \leq u_r^2 \text{ and } \beta_r(T_{\frac{1}{r}}(u_r^1)) \leq \beta_r(T_{\frac{1}{r}}(u_r^2)) \text{ a.e. in } \Omega. \tag{4.17}$$

Proof. By taking $\varphi = \dfrac{1}{k}T_k(u_r^1 - u_r^2)^+$ as a test function in (4.1) for f_1 and f_2 respectively. Then, subtracting the resulting equalities, one obtains

$$\sum_{j=1}^N \int_{A_r} \left(a_j(x, \frac{\partial u_r^1}{\partial x_j}) - a_j(x, \frac{\partial u_r^2}{\partial x_j}) \right) \frac{\partial}{\partial x_j}(u_r^1 - u_r^2) dx + \int_\Omega (\beta_r(T_{\frac{1}{r}}(u_r^1)) - \beta_r(T_{\frac{1}{r}}(u_r^2))\frac{1}{k}T_k(u_r^1 - u_r^2)^+ dx$$

$$+r\int_\Omega (|u_r^1|^{P_M(x)-2}u_r - |u_r^2|^{P_M(x)-2}u_r^2)\frac{1}{k}T_k(u_r^1 - u_r^2)^+ dx = \int_\Omega (f_1 - f_2)\frac{1}{k}T_k(u_r^1 - u_r^2)^+ dx \tag{4.18}$$

where $A_r = \{0 < u_r^1 - u_r^2) < k\}$.

By the monotonicity of β_r and a_j the two first terms are positive. Then, neglecting some positive terms, one gets

$$r\int_\Omega (|u_r^1|^{P_M(x)-2}u_r - |u_r^2|^{P_M(x)-2}u_r^2)\frac{1}{k}T_k(u_r^1 - u_r^2)^+ dx = \int_\Omega (f_1 - f_2)\frac{1}{k}T_k(u_r^1 - u_r^2)^+ dx.$$

Therefore, passing to the limit as k tends to zero, one deduces that

$$r\int_\Omega (|u_r^1|^{P_M(x)-2}u_r^1 - |u_r^2|^{P_M(x)-2}u_r^2)^+ dx \leq \int_\Omega (f_1 - f_2)\, sign_0^+(u_r^1 - u_r^2) dx.$$

(4.17) is an immediate consequence of (4.18). □

Step 3 : Basic convergence results

We stress that it is well-known (see [6, 15, 19]) that Lemma 4.2 implies the following convergence results.

Lemma 4.4. *Let u_r be a solution of problem $(\mathcal{P}_r)$. Then, for $j = 1, ..., N$, as $n \to \infty$, one has*

$$a_j\left(x, \frac{\partial u_r}{\partial x_j}\right) \longrightarrow a_j\left(x, \frac{\partial u}{\partial x_j}\right) \text{ in } L^1(\Omega) \text{ a.e. } x \in \Omega. \tag{4.19}$$

$$a_j\left(x, \frac{\partial}{\partial x_j} T_k(u_r)\right) \longrightarrow a_j\left(x, \frac{\partial}{\partial x_j} T_k(u)\right) \text{ strongly in } L^1(\Omega) \text{ and weakly in } L^{p_i'(\cdot)}(\Omega). \tag{4.20}$$

Moreover (see [19]),

$$a_j\left(x, \frac{\partial u_r}{\partial x_j}\right)\frac{\partial u_r}{\partial x_j} \longrightarrow a_j\left(x, \frac{\partial u}{\partial x_j}\right)\frac{\partial u}{\partial x_j} \text{ in } L^1(\Omega) \text{ and a.e. in } \Omega. \tag{4.21}$$

Proposition 6. *Let $j = 1, ..., N$, and suppose u_r a weak solution to $(\mathcal{P}_r)$. Then,*

(i) *there exists $u \in W^{1,\overrightarrow{p}(\cdot)}(\Omega) \subset \mathcal{T}_{\mathcal{H}}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ such that $u \in dom(\beta)$ a.e. in Ω and*

$$u_r \longrightarrow u \text{ in measure and a.e. in } \Omega \text{ as } r \longrightarrow 0. \tag{4.22}$$

(ii) $\dfrac{\partial u_r}{\partial x_j}$ *converges in measure to the weak partial gradient of u.*

Lemma 4.5. [19] *Let $S \in C_c^1(\mathbb{R})$ and $\varphi \in W^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$.*

For any $j = 1, ..., N$, one has

$$\frac{\partial}{\partial x_j}(S(u_r)\varphi) \longrightarrow \frac{\partial}{\partial x_j}(S(u)\varphi) \text{ strongly in } L^1(\Omega) \text{ as } r \to 0$$

and

$$\lim_{r\to 0} r\int_\Omega |u_r|^{p_M(x)-2}u_r S(u_r)\varphi dx = 0. \tag{4.23}$$

Lemma 4.6. ([21,22])

$$\beta_r(T_{\frac{1}{r}}(u_r)) \rightharpoonup w \text{ weakly-* in } L^\infty(\Omega). \tag{4.24}$$

Step 3 : Passing to the limit

Let $S \in C_c^1(\mathbb{R})$ and $\varphi \in W^{\overrightarrow{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$ be arbitrary. Plugging $S_l(u_r)h(u)\varphi$ as a test function in (4.1), one obtains

$$\sum_{j=1}^N \int_\Omega a_j\left(x, \frac{\partial u_r}{\partial x_j}\right)\frac{\partial}{\partial x_j}[S_l(u_r)S(u)\varphi]dx + \int_\Omega \beta_r(T_{\frac{1}{r}}(u_r))S_l(u_r)S(u)\varphi dx$$
$$+r\int_\Omega |u_r|^{p_M(x)-2}u_r S_l(u_r)S(u)\varphi dx = \int_\Omega f S_l(u_r)S(u)\varphi dx.$$

Since $\dfrac{\partial}{\partial x_j}[S_l(u_r)S(u)\varphi] = S_l(u_r)\dfrac{\partial}{\partial x_j}[S(u)\varphi] + S(u)\varphi S_l'(u_r)\dfrac{\partial u_r}{\partial x_j}$, one obtain

$$\sum_{j=1}^N \int_\Omega S_l(u_r)a_j\left(x, \frac{\partial u_r}{\partial x_j}\right)\frac{\partial}{\partial x_j}[S(u)\varphi]dx + \sum_{j=1}^N \int_\Omega S(u)\varphi S_l'(u_r)a_j\left(x, \frac{\partial u_r}{\partial x_j}\right)\frac{\partial u_r}{\partial x_j}dx$$
$$+\int_\Omega \beta_r(T_{\frac{1}{r}}(u_r))S_l(u_r)S(u)\varphi dx + r\int_\Omega |u_r|^{p_M(x)-2}u_r S_l(u_r)S(u)\varphi dx$$

$$= \int_\Omega f S_l(u_r) S(u) \varphi dx. \tag{4.25}$$

By using the generalized Lebesgue dominated convergence theorem, one obtains

$$\lim_{r\to 0} r \int_\Omega |u_r|^{p_M(x)-2} u_r S_l(u_r) S(u) \varphi dx = 0. \tag{4.26}$$

According to [21] and (4.26), one pass to the limit as $r \downarrow 0$ in (4.25) to obtain.

$$\sum_{j=1}^N \int_\Omega S_l(u) a_j\left(x, \frac{\partial T_{l+1}(u)}{\partial x_j}\right) \frac{\partial}{\partial x_j}[S(u)\varphi] dx + \lim_{r\to 0} \sum_{j=1}^N \int_\Omega S(u)\varphi S_l'(u_r) a_j\left(x, \frac{\partial u_r}{\partial x_j}\right) \frac{\partial u_r}{\partial x_j} dx$$

$$\int_\Omega w S_l(u) S(u) \varphi dx = \int_\Omega f S_l(u) S(u) \varphi dx. \tag{4.27}$$

Since for any $k_0 > 0$ such that supp$S \subset [-k_0, k_0]$, one can replace u by $T_{k_0}(u)$ in the equality above. This also implies that, $S_l'(u) = S_l'(T_{k_0}(u)) = 0$ if $l + 1 > k_0$ and $S_l(u) = S_l(T_{k_0}(u)) = 1$ if $l > k_0$.

Remark 4.1. *According to [21], one has*

$$\begin{aligned}
\Big|\sum_{j=1}^N \int_\Omega S(u)\varphi S_l'(u_r) a_j\left(x, \frac{\partial u_r}{\partial x_j}\right) \frac{\partial u_r}{\partial x_j} dx\Big| &= \Big|\sum_{j=1}^N \int_{\{l<|u_r|<l+k\}} S(u)\varphi a_j\left(x, \frac{\partial u_r}{\partial x_j}\right) \frac{\partial u_r}{\partial x_j} dx\Big| \\
&\le const(S, \|\varphi\|_\infty) \sum_{j=1}^N \int_\Omega a_j\left(x, \frac{\partial u_r}{\partial x_j}\right) \frac{\partial u_r}{\partial x_j} dx \\
&\le const(k, S, \|\varphi\|_\infty) \sum_{j=1}^N \int_{\{|u_r|>l\}} f dx \\
&\le const(||f\|_\infty, k, S, \|\varphi\|_\infty) \sum_{j=1}^N meas\{|u_r| > l\} \\
&\le const(N, ||f\|_\infty, k, S, \|\varphi\|_\infty) meas\{|u_r| > l\}.
\end{aligned}$$

According to (4.10) *in Proposition* 3, *meas*$\{|u_r| > l\} \longrightarrow 0$ *as* $l \to \infty$. *Then, we deduce from above inequality that*

$$\lim_{l\to\infty} \lim_{r\to 0} \sum_{j=1}^N \int_\Omega S(u)\varphi S_l'(u_r) a_j\left(x, \frac{\partial u_r}{\partial x_j}\right) \frac{\partial u_r}{\partial x_j} dx = 0.$$

Therefore, passing to the limit as $l \to \infty$, one obtains

$$\int_\Omega \sum_{j=1}^N a_j\left(x, \frac{\partial u}{\partial x_j}\right) \frac{\partial}{\partial x_j}[S(u)\varphi] dx + \int_\Omega w S(u)\varphi dx = \int_\Omega f S(u) \varphi dx.$$

Now, we focus on the proof (3.7).

From (4.10), one has (see [16,19])

$$\text{meas}(\{|u_r| > l\}) \to 0 \text{ uniformly as } l \to \infty.$$

Hence, passing to the limit in (4.4) as $l \to \infty$ and using (3.3), one obtains

$$\lim_{l\to\infty} \int_{\{l<|u|<l+1\}} a_j\left(x, \frac{\partial u}{\partial x_j}\right)\frac{\partial u}{\partial x_j}dx = 0.$$

We end the proof of Theorem 4.1 by using the same argument as in [10] to prove the subdifferential argument : $u(x) \in dom(\beta(x))$, $w(x) \in \beta(u(x))$ for a.e. x in Ω. □

5. Existence of Solution for L^1 Data

Theorem 5.1. *Let $f \in L^1(\Omega)$ and assume that (3.1)-(3.5) hold true. Then, the problem $(\mathcal{P})$ has at least one renormalized solution.*

Step 1: Approximate problem and a priori estimates

Let us consider, for any $m > 0$, the approximated problem

$$(\mathcal{P}_m)\begin{cases} -\displaystyle\sum_{j=1}^{N} \frac{\partial}{\partial x_j} a_j(x, \frac{\partial u_m}{\partial x_j}) + \beta_{\frac{1}{m}}(u_m) = f_m \text{ in } \Omega \\ \\ \displaystyle\sum_{j=1}^{N} a_j(x, \frac{\partial u_m}{\partial x_j}).\eta_j = 0 \qquad \text{on } \partial\Omega, \end{cases}$$

where $f_m = T_m(f)$. Note that (f_m) is a sequence of L^∞-functions which converges strongly to f in $L^1(\Omega)$ and verify $|f_m| \le |f|$.

$\beta_{\frac{1}{m}}(.) : \mathbb{R} \to \mathbb{R}$ is the Yosida approximation of $\beta(.)$ such that: for any $u \in W^{1,p(.)}(\Omega)$

$$\langle \beta_{\frac{1}{m}}(u), u\rangle \ge 0,\ |\beta_{\frac{1}{m}}(u)| \le m|u| \text{ and } \lim_{m\to\infty} \beta_{\frac{1}{m}}(u) = \beta(u).$$

By Theorem 4.1, the problem $(\mathcal{P}_m)$ has at least one solution $(u_m, w_m) \in W^{1,\vec{p}(.)}(\Omega) \times L^\infty(\Omega)$. Namely, $u_m \in \mathrm{Dom}(\beta_{\frac{1}{m}})$, $w_m \in \beta_{\frac{1}{m}}(u_m)$ a.e. in Ω and

$$\int_\Omega \sum_{j=1}^{N} a_j\left(x, \frac{\partial u_m}{\partial x_j}\right)\frac{\partial}{\partial x_j}[S(u_m)\varphi]dx + \int_\Omega w_m S(u_m)\varphi dx = \int_\Omega f_m S(u_m)\varphi dx, \tag{5.1}$$

for every $S \in C_c^1(\mathbb{R})$ and $\varphi \in W^{1,\vec{p}(.)}(\Omega) \cap L^\infty(\Omega)$.

Remark 5.1. *According to [10] (see the proof of Theorem 4.2), the sequences $(u_m)_{m>1}$ and $(w_m)_{m>1}$ are increasing.*

Remark 5.2. *Setting $S = S_l$, $\varphi = T_k(u_m)$ in the above equality, then, arguing analogously as in [22] (see Lemma 6) one obtains*

$$\sum_{j=1}^{N} \int_\Omega \left|\frac{\partial u_m}{\partial x_j}\right|^{p_j(x)} dx \le \frac{k\|f\|_1}{C_5}, \tag{5.2}$$

$$\|w_m\|_1 \le \|f\|_1. \tag{5.3}$$

Let us also stress that it follows from (5.2) that

$$\sum_{j=1}^{N}\int_{\Omega}\left|\frac{\partial u_m}{\partial x_j}\right|^{p_j}dx \leq const(k, f, C_5, \Omega, N). \tag{5.4}$$

As consequences to inequality (5.4), the estimates (i) and (ii) in Lemma 4.2 are satisfied by the sequence $(u_m)_{m>0}$. Therefore, as in the previous section, the convergence results in Lemmas 4.4 and 4.5, and Proposition 6 hold true for the the sequence $(u_m)_{m>0}$.

Lemma 5.1. *The sequence $(w_m)_{m>0}$ satisfies*

$$w_m \rightharpoonup w \text{ in } L^1(\Omega), \text{ as } m \to \infty. \tag{5.5}$$

Proof. Let (u_n^r, b_n^r) be a solution of the following problem

$$\begin{cases} \beta_r(T_{\frac{1}{r}}(u_m^r)) - \displaystyle\sum_{j=1}^{N}\frac{\partial}{\partial x_j}a_j(x, \frac{\partial u_m^r}{\partial x_j}) = f_m & \text{in } \Omega \\ \\ \displaystyle\sum_{j=1}^{N} a_j(x, \frac{\partial u_m^r}{\partial x_j}).\eta_j = 0 & \text{on } \partial\Omega. \end{cases}$$

According to Lemma 4.3, the following estimate holds true.

$$\int_{\Omega}(|\beta_r(T_{\frac{1}{r}}(u_m^r)| - k)^+dx \leq \int_{\Omega}(|f_m| - k)^+dx. \tag{5.6}$$

Since $\beta_r(T_{\frac{1}{r}}(u_m^r) \rightharpoonup w_m$ in $L^\infty(\Omega)$ as r goes to 0, we get

$$\int_{\Omega}(|w_m| - k)^+dx \leq \int_{\Omega}(|f_m| - k)^+dx. \tag{5.7}$$

From the inequality (5.7), one deduces that the sequence $(w_m)_{m\in\mathbb{N}}$ is relatively weakly compact in $L^1(\Omega)$ (see [17]). Therefore, up to a subsequence, one obtains

$$w_m \rightharpoonup w \text{ weakly in } L^1(\Omega) \text{ as } m \to \infty.$$

Since $S(u_m)\varphi \overset{*}{\rightharpoonup} S(u)\varphi$ in $L^\infty(\Omega)$ as $n \to \infty$, and $f_m \to f$, one has

$$\int_{\Omega} f_n S(u_n)v dx \to \int_{\Omega} f S(u) v dx. \tag{5.8}$$

From Lemma 5.1, we deduce that

$$\int_{\Omega} w_n S(u_n)v dx \to \int_{\Omega} w S(u)\varphi dx. \tag{5.9}$$

According to [19], one has

$$\lim_{m\to\infty}\int_{\Omega}\sum_{j=1}^{N}a_j\left(x, \frac{\partial u_m}{\partial x_j}\right)\frac{\partial}{\partial x_j}[S(u)\varphi]dx = \int_{\Omega}\sum_{j=1}^{N}a_j\left(x, \frac{\partial u}{\partial x_j}\right)\frac{\partial}{\partial x_j}[S(u)\varphi]dx. \tag{5.10}$$

Combining (5.8), (5.9) and (5.10), we pass to the limit in (5.1), as , $m \to \infty$ and obtain (3.6). □

6. Existence of Entropy Solution

Theorem 6.1. *Let $f \in L^1(\Omega)$. Then, the problem $(\mathcal{P})$ has at least one entropy solution.*

Proof. Let (u_m, w_m) a sequence of solutions to the problem $(\mathcal{P}_m)$ and $t > 0$. Then, for any $v \in W_0^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$, taking $T_t(u_n - v)$ as test function in (5.1) and setting $L = t + \|v\|_\infty$, we obtain

$$\int_\Omega w_m T_t(u_m - v)dx + \int_\Omega \sum_{j=1}^N a_j\left(x, \frac{\partial u_m}{\partial x_j}\right)\frac{\partial}{\partial x_j}T_t(u_m - v)dx$$
$$= \int_\Omega f_m T_t(u_m - v)dx. \tag{6.1}$$

Notice that if $|u_m| \geq L$, then $|u_m - v| \geq |u_m| - \|v\|_\infty > t$. Therefore, $\{|u_m - v| \leq t\} \subseteq \{|u_m| \leq L\}$, which gives

$$\begin{cases} \displaystyle\int_\Omega \sum_{j=1}^N a_j\left(x, \frac{\partial u_m}{\partial x_j}\right)\frac{\partial}{\partial x_j}T_t(u_m - v)dx \\ \displaystyle= \int_\Omega \sum_{j=1}^N a_j\left(x, \frac{\partial T_L(u_m)}{\partial x_j}\right)\left(\frac{\partial T_L(u_m)}{\partial x_j} - \frac{\partial v}{\partial x_j}\right)\chi_{\{u_m - v|\leq t\}}dx \\ \displaystyle\int_\Omega \sum_{j=1}^N \left[a_j\left(x, \frac{\partial T_L(u_m)}{\partial x_j}\right) - a_j\left(x, \frac{\partial v}{\partial x_j}\right)\right]\left(\frac{\partial T_L(u_m)}{\partial x_j} - \frac{\partial v}{\partial x_j}\right)\chi_{\{u_m - v|\leq t\}}dx \\ \displaystyle+ \int_\Omega \sum_{j=1}^N a_j\left(x, \frac{\partial v}{\partial x_j}\right)\left(\frac{\partial T_L(u_m)}{\partial x_j} - \frac{\partial v}{\partial x_j}\right)\chi_{\{u_m - v|\leq t\}}dx. \end{cases}$$

Using Fatou's Lemma, we obtain

$$\begin{cases} \displaystyle\liminf_{m\to\infty} \int_\Omega \sum_{j=1}^N a_j\left(x, \frac{\partial u_m}{\partial x_j}\right)\frac{\partial}{\partial x_j}T_t(u_m - v)dx \\ \displaystyle\geq \int_\Omega \sum_{j=1}^N a_j\left(x, \frac{\partial T_L(u_m)}{\partial x_j}\right)\left(\frac{\partial T_L(u_m)}{\partial x_j} - \frac{\partial v}{\partial x_j}\right)\chi_{\{u_m - v|\leq t\}}dx \\ \displaystyle+ \lim_{m\to\infty} \int_\Omega \sum_{j=1}^N a_j\left(x, \frac{\partial v}{\partial x_j}\right)\left(\frac{\partial T_L(u_m)}{\partial x_j} - \frac{\partial v}{\partial x_j}\right)\chi_{\{u_m - v|\leq t\}}dx. \end{cases} \tag{6.2}$$

Since

$$\begin{cases} \displaystyle\lim_{m\to\infty} \int_\Omega \sum_{j=1}^N a_j\left(x, \frac{\partial v}{\partial x_j}\right)\left(\frac{\partial T_L(u_m)}{\partial x_j} - \frac{\partial v}{\partial x_j}\right)\chi_{\{u_m - v|\leq t\}}dx \\ \displaystyle= \int_\Omega \sum_{j=1}^N a_j\left(x, \frac{\partial v}{\partial x_j}\right)\left(\frac{\partial T_L(u)}{\partial x_j} - \frac{\partial v}{\partial x_j}\right)\chi_{\{u - v|\leq t\}}dx, \end{cases}$$

we deduce from (6.2), that

$$\begin{cases} \liminf_{n\to\infty} \int_\Omega \sum_{j=1}^N a_j\left(x, \frac{\partial u_m}{\partial x_j}\right) \frac{\partial}{\partial x_j} T_t(u_m - v)dx \\ \\ \geq \int_\Omega \sum_{j=1}^N a_j\left(x, \frac{\partial v}{\partial x_j}\right)\left(\frac{\partial T_L(u)}{\partial x_j} - \frac{\partial v}{\partial x_j}\right)\chi_{\{u-v|\leq t\}}dx \\ \\ = \int_\Omega a(x, T_L(u), \nabla v)(\nabla T_L(u) - \nabla v)\chi_{\{u-v|\leq t\}}dx \\ \\ = \int_\Omega \sum_{j=1}^N a_j\left(x, \frac{\partial v}{\partial x_j}\right)\frac{\partial}{\partial x_j}T_t(u-v)dx. \end{cases}$$

Since $T_t(u_m - v) \overset{*}{\rightharpoonup} T_t(u-v)$ in $L^\infty(\Omega)$ and $f_m \to f$ in $L^1(\Omega)$ as $m \to \infty$, one deduces that

$$\int_\Omega f_m T_t(u_m - v)dx \to \int_\Omega f T_t(u-v)dx. \tag{6.3}$$

Since $w_m \rightharpoonup w$ weakly in $L^1(\Omega)$ and $T_t(u_m - v) \overset{*}{\rightharpoonup} T_t(u-v)$ in $L^\infty(\Omega)$ as $m \to \infty$, we obtain

$$\int_\Omega w_m T_t(u_m - v)dx \to \int_\Omega w T_t(u-v)dx. \tag{6.4}$$

Passing to the limit in (6.1), we obtain the entropy inequality (3.8).

□

7. Uniqueness of Solution

Here we analyze the uniqueness of the solution of the problem $(\mathcal{P})$.

Theorem 7.1. *Let (u, w) and (v, d) be two entropy solutions of the problem $(\mathcal{P})$ and let f be in $L^1(\Omega)$. Then, we have*

$$u - v = c \text{ and } w = d \text{ a.e. in } \Omega,$$

where c is a constant.

Proof. Let f be in $L^1(\Omega)$. By writing the entropy inequality corresponding to (u, w) with test function v and (v, d) with test function u. Adding up the both results, we obtain

$$\int_\Omega w T_k(u-v)dx + \sum_{j=1}^N \int_\Omega \left(a_j(x, \frac{\partial u}{\partial x_j}) - a_j(x, \frac{\partial v}{\partial x_j})\right)\frac{\partial}{\partial x_j}T_k(u-v)dx \leq \int_\Omega f T_k(u-v)dx \tag{7.1}$$

Reasonning as in [19] (see also [31]), we obtain

$$u - v = c \text{ and } w = d \text{ a.e. in } \Omega.$$

□

Corollary 1. *Let $f \in L^1(\Omega)$ and let (u, w) and (v, d) be two entropy solutions of the problem $(\mathcal{P})$. If the graph $\beta(.)$ is a strictly increasing and continuous function, then, we have*

$$u = v \ \textit{and} \ w = d \ \textit{a.e. in } \Omega.$$

proof. It follows the same as in [10]. □

8. Conclusion

In this paper, we have addressed the nonlinear multivalued $\overrightarrow{p}(.)$-anisotropic problem under Neumann boundary conditions with $L^\infty(\Omega)$ or $L^1(\Omega)$ data. By leveraging the approximation techniques and the theory of maximal monotone operators in Banach spaces, we established the existence of a renormalized and an entropy solution of the problem. Additionally, by the comparison principle, we prove the uniqueness of the entropy solution. The main results broaden the understanding of numerous recent works in the literature [10,18,19,31].

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