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A NOVEL APPROACH TO LYAPUNOV EVENTUAL STABILITY OF CAPUTO FRACTIONAL DYNAMIC EQUATIONS ON TIME SCALE

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Abstract. This paper introduces a novel approach to the Lyapunov eventual stability of Caputo fractional dynamic equations on time scales. By utilizing comparison principle, we develop eventual stability results that simultaneously holds for discrete and continuous domains allowing for an in-depth study of systems that exhibit both continuous and abrupt changes over time ensuring that the system behavior becomes stable after a finite time, which is particularly useful in scenarios involving transient disturbances. We also give an illustrative example to show the applicability of our method.

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1. Introduction

In many real-world situations, the stability of states that are not equilibrium states is of interest. Traditional Lyapunov stability is not suitable in these cases, as it inherently implies that a stable state must be an equilibrium state. A notable example arises in the study of adaptive control systems, where the desired state may not be an equilibrium state but may eventually behave increasingly like a stable equilibrium state over time. This concept, referred to as eventual stability, is given a precise definition, and its basic properties are explored. For instance, in autonomous systems or when the state is an equilibrium state, eventual stability coincides with Lyapunov stability, and the related theorems can be viewed as generalizations of Lyapunov's classical results. However, in non-autonomous systems or

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when the state is not an equilibrium state, eventual stability presents a new perspective. Theorems are developed to extend Lyapunov's direct method for studying eventual stabilities and to provide qualitative estimates of the extent of such stability. This new concept of stability has significant potential applications, particularly in the theory and design of adaptive control systems. By allowing for the stability analysis of non-equilibrium states, eventual stability offers a broader framework for understanding system behavior. An example is provided to illustrate how these ideas can be utilized in designing an adaptive control system, highlighting the practical relevance and applicability of eventual stability in engineering and control systems.

Eventual stability is distinct from other forms of stability in its emphasis on the system's behavior after an initial transient phase. Unlike uniform or asymptotic stability, which requires immediate stabilization or convergence over time, eventual stability allows for a temporary phase of instability or oscillation before achieving stability. This characteristic makes eventual stability highly advantageous for systems subject to temporary disruptions or changes in operating conditions, providing a realistic approach for analyzing complex systems in engineering and control applications.

Time scale calculus, introduced by Stefan Hilger in 1988, serves as a unifying theory that bridges the analysis of dynamic systems in both discrete and continuous time domains. The versatility of time scales provides an ideal framework for studying systems that exhibit both continuous behaviors and discrete transitions, as seen in hybrid systems. When combined with fractional calculus, time scale analysis allows for a more comprehensive examination of dynamic systems, extending the benefits of fractional modeling to both continuous and discrete scenarios.

This paper presents a novel approach to analyzing the Lyapunov eventual stability of Caputo fractional dynamic equations on time scales, using a new generalized derivative. The generalized derivative, called the Caputo fractional delta derivative and the Caputo fractional delta Dini derivative of order $\alpha \in (0,1)$, are employed as a unified approach (capturing the behavior of dynamic systems across various time domains, addressing both continuous and discrete models in a consistent manner), to develop the framework for this concept, extending the traditional Caputo fractional derivative to time scales.

Previous studies, such as those by [1–5,16,17], have primarily focused on continuous time systems and have not fully addressed the challenges associated with eventual stability on discrete or hybrid domains. Other works, such as [10], have tackled stability in discrete settings; [18,19] explored stability for hybrid domains but did not explore the full implications of fractional-order eventual stability. The recent advancements in fractional dynamic systems on time scales have focused largely on the existence and uniqueness of solutions, as well as asymptotic stability, without considering eventual stability in detail [11,13,24,25]. This paper aims to bridge these gaps by providing a unified eventual stability analysis applicable to both continuous and discrete time scales, offering new insights into how dynamic systems can achieve stability after transient behaviors.

By establishing comparison results and eventual stability criteria for Caputo fractional dynamic equations, this paper extends the classical Lyapunov stability analysis to fractional-order systems and introduces new methodologies for addressing the transient behaviors unique to such systems. The findings contribute to a unified framework for stability analysis on time scales, bridging the gap between continuous and discrete domains.

For the purpose of this work, we consider the Caputo fractional dynamic system of order α with $0 < \alpha < 1$

$$\begin{aligned} {}^{C\mathbb{T}}D^{\alpha}x^{\Delta} &= f(t,x),\ t \in \mathbb{T}, \\ x(t_0) &= x_0,\ t_0 \geq 0, \end{aligned} \tag{1}$$

where $f \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n]$, $f(t,0) \equiv 0$ and ${}^{C\mathbb{T}}D^{\alpha}x^{\Delta}$ is the Caputo fractional delta derivative of $x \in \mathbb{R}^n$ of order α with respect to $t \in \mathbb{T}$. Let $x(t) = x(t,t_0,x_0) \in C^{\alpha}_{rd}[\mathbb{T}, \mathbb{R}^n]$ (the fractional derivative of order alpha of $x(t)$ exist and it is rd-continuous) be a solution of (1) and assume the solution exists and is unique (results on existence and uniqueness of (1) are contained in [7,12,22]), this work aims to investigate the uniform stability and uniform asymptotic stability of the system (1).

Typically, stability or asymptotic stability is interpreted in the sense of Lyapunov. Today, there is no need to revisit the definitions or describe Lyapunov's second or direct method in detail. However, one key point warrants emphasis: the definition of stability or asymptotic stability generally assumes that $x = 0$ is an equilibrium state, meaning that $f(t,0) = 0$. This assumption is inherent to Lyapunov stability, as without the requirement that the origin is an equilibrium state, the definition itself implies that it must be.

Consequently, Lyapunov stability applies only to equilibrium states. However, the type of stability treated here concerns states that are not necessarily equilibrium states but increasingly behave like stable or asymptotically stable equilibrium states over time. In the context of adaptive control systems, considering real-world complexities, this is the type of stability that is often desired. When a system or plant under control is subjected to disturbances and an evolving environment, it is optimal for the adaptive control mechanism to ensure that the desired state behaves progressively more like a stable equilibrium state, or ideally, like an asymptotically stable state. This form of stability is known as "eventual stability."

Eventual stability is particularly important in the theory and design of adaptive control systems, as it provides a practical framework for managing non-equilibrium states in dynamic environments. By enabling states to asymptotically resemble stable equilibrium behavior despite external perturbations, eventual stability offers a robust mechanism for ensuring reliable system performance, even when faced with continuous changes and challenges. Theorems are developed to extend Lyapunov's direct method for studying eventual stabilities and to provide qualitative estimates of the extent of such stability using the comparison system of the form:

$$^{C\mathbb{T}}D^{\alpha}u^{\Delta} = g(t,u),\ u(t_0) = u_0 \geq 0, \tag{2}$$

where $u \in \mathbb{R}_+$, $g : \mathbb{T} \times \mathbb{R}_+ \to \mathbb{R}_+$ and $g(t,0) \equiv 0$. (2) is called the comparison system. For this work, we will assume that the function $g \in [\mathbb{T} \times \mathbb{R}_+, \mathbb{R}_+]$, is such that for any initial data $(t_0, u_0) \in \mathbb{T} \times \mathbb{R}_+$, the system (2) with $u(t_0) = u_0$ has a unique solution $u(t) = u(t; t_0, u_0) \in C^{\alpha}_{rd}[\mathbb{T}, \mathbb{R}_+]$ see [7, 15].

The following section (Section 2) delves into essential terminologies, and remarks that form the basis for the subsequent developments. It also introduces definitions and significant remarks. In Section 3, we present the main results, Section 4 provides a practical example to illustrate the relevance and application of our approach. Lastly, Section 5 offers a conclusion, summarizing the key findings and the implications of this study.

2. Preliminaries, Definitions, and Notations

In this section, we lay the groundwork by introducing key notations and definitions that will be instrumental in developing the main results.

Definition 2.1. [9] *For $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined as*

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined as

$$\rho(t) = sup\{s \in \mathbb{T} : s < t\}.$$

(i) *if $\sigma(t) > t$, t is right scattered,*
(ii) *if $\rho(t) < t$, t is left scattered,*
(iii) *if $t < max\mathbb{T}$ and $\sigma(t) = t$, then t is called right dense,*
(iv) *if $t > min\mathbb{T}$ and $\rho(t) = t$, then t is called left dense.*

Definition 2.2. [9] *The graininess function $\mu : \mathbb{T} \to [0, \infty)$ for $t \in \mathbb{T}$ is defined as*

$$\mu(t) = \sigma(t) - t.$$

Definition 2.3 (Delta Derivative)**.** [9] *and* [6] *Let $h : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$. We define the delta derivative h^{Δ} also known as the Hilger derivative as*

$$h^{\Delta}(t) = \lim_{s \to t} \frac{h(\sigma(t)) - h(s)}{\sigma(t) - s}, \quad s \neq \sigma(t).$$

provided the limit exists.

The function $h^{\Delta} : \mathbb{T} \to \mathbb{R}$ is called the (Delta) derivative of h on $\mathbb{T}^k$.

If t is right dense, the delta derivative of $h : \mathbb{T} \to \mathbb{R}$, becomes

$$h^{\Delta}(t) = \lim_{s \to t} \frac{h(t) - h(s)}{t - s},$$

and if t is right scattered, the Delta derivative becomes

$$h^{\Delta}(t) = \frac{h^{\sigma}(t) - h(t)}{\mu(t)}.$$

For a function $h : \mathbb{T} \to \mathbb{R}$, h^{σ} denotes $h(\sigma(t))$.

Definition 2.4. [14] *A function $h : \mathbb{T} \to \mathbb{R}$ is right dense continuous if it is continuous at all right dense points of $\mathbb{T}$ and its left sided limits exist and is finite at left dense points of $\mathbb{T}$. The set of all right dense continuous functions are denoted by*

$$C_{rd} = C_{rd}(\mathbb{T}).$$

Definition 2.5. [14] *Assume $[a, b]$ is a closed and bounded interval in $\mathbb{T}$. Then a function $H : [a, b] \to \mathbb{R}$ is called a delta antiderivative of $h : [a, b] \to \mathbb{R}$ provided H is continuous on $[a, b]$, delta differentiable on $[a, b)$, and $H^{\Delta}(t) = h(t)$ for all $t \in [a, b)$. Then, we define the Delta integral by*

$$\int_a^b h(t) = H(b) - H(a), \quad \forall a, b \in \mathbb{T}.$$

Remark 2.1. [14] *All right dense continuous functions are delta integrable.*

Definition 2.6. [14] *A function $\phi : [0, r] \to [0, \infty)$ is of class $\mathcal{K}$ if it is continuous, and strictly increasing on $[0, r]$ with $\phi(0) = 0$.*

Definition 2.7. [14] *A continuous function $\mathcal{V} : \mathbb{R}^n \to \mathbb{R}$ with $\mathcal{V}(0) = 0$ is called positive definite(negative definite) on the domain D if there exists a function $\phi \in \mathcal{K}$ such that $\phi(|x|) \leq \mathcal{V}(x)$ $(\phi(|x|) \leq -\mathcal{V}(x))$ for $x \in D$.*

Definition 2.8. [14] *A continuous function $\mathcal{V} : \mathbb{R}^n \to \mathbb{R}$ with $\mathcal{V}(0) = 0$ is called positive semidefinite (negative semi-definite) on D if $\mathcal{V}(x) \geq 0$ $(\mathcal{V}(x) \leq 0)$ for all $x \in D$ and it can also vanish for some $x \neq 0$.*

Definition 2.9. *Let $a, b \in \mathbb{T}$ and $h \in C_{rd}$, then we define the integration on a time scale $\mathbb{T}$ as follows:*

(i) *If $\mathbb{T} = \mathbb{R}$, then*

$$\int_a^b h(t)\Delta t = \int_a^b h(t)dt,$$

where $\int_a^b h(t)dt$ is the usual Riemann integral from calculus.

(ii) *If $[a, b]$ consists of only isolated points, then*

$$\int_a^b h(t)\Delta t = \begin{cases} \sum_{t \in [a,b)} \mu(t)h(t) & \text{if} \quad a < b \\ 0 & \text{if} \quad a = b \\ -\sum_{t \in [b,a)} \mu(t)h(t) & \text{if} \quad a > b. \end{cases}$$

(iii) *If there exists a point* $\sigma(t) > t$, *then*

$$\int_t^{\sigma(t)} h(s)\Delta s = \mu(t)f(t).$$

Definition 2.10. [20] *Assume* $V \in C[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}_+]$, $h \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n]$ *and* $\mu(t)$ *is the graininess function then we define the dini derivative of* $V(t,x)$ *as:*

$$D_{-}V^{\Delta}(t,x) = \liminf_{\mu(t)\to 0} \frac{V(t,x) - V(t-\mu(t), x-\mu(t)h(t,x))}{\mu(t)} \tag{3}$$

$$D^{+}V^{\Delta}(t,x) = \limsup_{\mu(t)\to 0} \frac{V(t+\mu(t), x+\mu(t)h(t,x)) - V(t,x)}{\mu(t)}. \tag{4}$$

If V is differentiable, then $D_{-}V^{\Delta}(t,x) = D^{+}V^{\Delta}(t,x) = V^{\Delta}(t,x)$.

Definition 2.11. **(Fractional Integral on Time Scales)** [8]. *Let* $\alpha \in (0,1)$, $[a,b]$ *be an interval on* $\mathbb{T}$ *and* h *an integrable function on* $[a,b]$. *Then the fractional integral of order* α *of* h *is defined by*

$${}_{a}^{\mathbb{T}}I_t^{\alpha} h^{\Delta}(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\Delta s.$$

Definition 2.12. **(Caputo Derivative on Time Scale)** [7] *Let* $\mathbb{T}$ *be a time scale,* $t \in \mathbb{T},\ 0 < \alpha < 1$, *and* $h : \mathbb{T} \to \mathbb{R}$. *The Caputo fractional derivative of order* α *of* h *is defined by*

$${}_{a}^{\mathbb{T}}D_t^{\alpha} h^{\Delta}(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} h^{\Delta^n}(s)\Delta s.$$

Lemma 2.1. (*Theorem 2.1 in* [21]) *Let* $\mathbb{T}$ *be a time scale with minimal element* $t_0 \geq 0$. *Assume that for any* $t \in \mathbb{T}$, *there is a statement* $\mathbf{S}(t)$ *such that the following conditions are verified:*

(i) $\mathbf{S}(t_0)$ *is true;*

(ii) *If* t *is right scattered and* $\mathbf{S}(t)$ *is true, then* $\mathbf{S}(\sigma(t))$ *is also true;*

(iii) *For each right-dense* t, *there exists a neighborhood* $\mathcal{U}$ *such that whenever* $\mathbf{S}(t)$ *is true,* $\mathbf{S}(t^*)$ *is also true for all* $t^* \in \mathcal{U}$, $t^* \geq t$;

(iv) *For left dense* t, $\mathbf{S}(t^*)$ *is true for all* $t^* \in [t_0, t)$ *implies* $\mathbf{S}(t)$ *is true.*

Then the statement $\mathbf{S}(t)$ *is true for all* $t \in \mathbb{T}$.

Remark 2.2. *When* $\mathbb{T} = \mathbb{N}$, *then Lemma* 2.1 *reduces to the well-known principle of mathematical induction. That is,*

(1) $\mathbf{S}(t_0)$ *is true is equivalent to the statement is true for* $n = 1$;

(2) $\mathbf{S}(t)$ *is true then* $\mathbf{S}(\sigma(t))$ *is true is equivalent to if the statement is true for* $n = k$, *then the statement is true for* $n = k + 1$.

Now, we give the following definitions and remarks.

Definition 2.13. *Let $\mathbb{T}$ be a time scale. A point $t_0 \in \mathbb{T}$ is said to be a minimal element of $\mathbb{T}$ if, for any $t \in \mathbb{T}$, $t > t_0$ whenever $t \neq t_0$.*

Definition 2.14. *Let $h \in C^{\alpha}_{rd}[\mathbb{T}, \mathbb{R}^n]$, the Grunwald-Letnikov fractional delta derivative is given by*

$$^{GL\mathbb{T}}D_0^{\alpha} h^{\Delta}(t) = \lim_{\mu \to 0+} \frac{1}{\mu^{\alpha}} \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha} C_r [h(\sigma(t) - r\mu)], \quad t \geq t_0, \tag{5}$$

and the Grunwald-Letnikov fractional delta dini derivative is given by

$$^{GL\mathbb{T}}D_{0+}^{\alpha} h^{\Delta}(t) = \limsup_{\mu \to 0+} \frac{1}{\mu^{\alpha}} \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha} C_r [h(\sigma(t) - r\mu)], \quad t \geq t_0, \tag{6}$$

where $0 < \alpha < 1$, ${}^{\alpha}C_r = \frac{q(q-1)\ldots(q-r+1)}{r!}$, and $[\frac{(t-t_0)}{\mu}]$ denotes the integer part of the fraction $\frac{(t-t_0)}{\mu}$.

Observe that if the domain is $\mathbb{R}$, then (6) becomes

$$^{GL\mathbb{T}}D_{0+}^{\alpha} h^{\Delta}(t) = \limsup_{d \to 0+} \frac{1}{d^{\alpha}} \sum_{r=0}^{[\frac{(t-t_0)}{d}]} (-1)^{r\alpha} C_r [h(t - rd)], \quad t \geq t_0.$$

Remark 2.3. *It is necessary to note that the relationship between the Caputo fractional delta derivative and the Grunwald-Letnikov fractional delta derivative is given by*

$$^{C\mathbb{T}}D_0^{\alpha} h^{\Delta}(t) = {}^{GL\mathbb{T}} D_0^{\alpha} [h(t) - h(t_0)]^{\Delta}, \tag{7}$$

substituting (5) into (7) we have that the Caputo fractional delta derivative becomes

$$^{C\mathbb{T}}D_0^{\alpha} h^{\Delta}(t) = \lim_{\mu \to 0+} \frac{1}{\mu^{\alpha}} \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha} C_r [h(\sigma(t) - r\mu) - h(t_0)] \quad t \geq t_0$$

$$^{C\mathbb{T}}D_0^{\alpha} h^{\Delta}(t) = \lim_{\mu \to 0+} \frac{1}{\mu^{\alpha}} \left\{ h(\sigma(t)) - h(t_0) + \sum_{r=1}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha} C_r [h(\sigma(t) - r\mu) - h(t_0)] \right\}, \tag{8}$$

and the Caputo fractional delta Dini derivative becomes

$$^{C\mathbb{T}}D_{0+}^{\alpha} h^{\Delta}(t) = \limsup_{\mu \to 0+} \frac{1}{\mu^{\alpha}} \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha} C_r [h(\sigma(t) - r\mu) - h(t_0)] \quad t \geq t_0. \tag{9}$$

Which is equivalent to

$$^{C\mathbb{T}}D_{0+}^{\alpha} h^{\Delta}(t) = \limsup_{\mu \to 0+} \frac{1}{\mu^{\alpha}} \left\{ h(\sigma(t)) - h(t_0) + \sum_{r=1}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha} C_r [h(\sigma(t) - r\mu) - h(t_0)] \right\}, \quad t \geq t_0. \tag{10}$$

for notation simplicity, we shall represent the Caputo fractional delta derivative of order α as $^{C\mathbb{T}}D^{\alpha}$ and the Caputo fractional delta dini derivative of order α as $^{C\mathbb{T}}D_{+}^{\alpha}$.

Now, we introduce the derivative of the Lyapunov function using the Caputo fractional delta Dini derivative of $h(t)$ given in (9).

Definition 2.15. *We define the Caputo fractional delta Dini derivative of the Lyapunov function* $V(t,x) \in C[\mathbb{T}\times\mathbb{R}^n, \mathbb{R}_+]$ *(which is locally Lipschitzian with respect to its second argument and* $V(t,0)\equiv 0$*) along the trajectories of solutions of the system (1) as:*

$$^{C\mathbb{T}}D_+^{\alpha}V^{\Delta}(t,x) = \limsup_{\mu\to 0^+}\frac{1}{\mu^{\alpha}}\Bigg[\sum_{r=0}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^{\alpha}C_r)[V(\sigma(t)-r\mu, x(\sigma(t))-\mu^{\alpha}f(t,x(t))-V(t_0,x_0)]\Bigg],$$

and can be expanded as

$$^{C\mathbb{T}}D_+^{\alpha}V^{\Delta}(t,x) = \limsup_{\mu\to 0^+}\frac{1}{\mu^{\alpha}}\Bigg\{V(\sigma(t),x(\sigma(t))-V(t_0,x_0) \tag{11}$$
$$-\sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^{r+1}({}^{\alpha}C_r)[V(\sigma(t)-r\mu, x(\sigma(t))-\mu^{\alpha}f(t,x(t))-V(t_0,x_0)]\Bigg\},$$

where $t\in\mathbb{T}, x, x_0\in\mathbb{R}^n$, $\mu=\sigma(t)-t$ *and* $x(\sigma(t))-\mu^{\alpha}f(t,x)\in\mathbb{R}^n$.

If $\mathbb{T}$ *is discrete and* $V(t,x(t))$ *is continuous at* t, *the Caputo fractional delta Dini derivative of the Lyapunov function in discrete times, is given by:*

$$^{C\mathbb{T}}D_+^{\alpha}V^{\Delta}(t,x) = \frac{1}{\mu^{\alpha}}\Bigg[\sum_{r=0}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^{\alpha}C_r)(V(\sigma(t),x(\sigma(t)))-V(t_0,x_0))\Bigg], \tag{12}$$

and if $\mathbb{T}$ *is continuous, that is* $\mathbb{T}=\mathbb{R}$, *and* $V(t,x(t))$ *is continuous at* t, *we have that*

$$^{C\mathbb{T}}D_+^{\alpha}V^{\Delta}(t,x) = \limsup_{d\to 0^+}\frac{1}{d^{\alpha}}\Bigg\{V(t,x(t))-V(t_0,x_0) \tag{13}$$
$$-\sum_{r=1}^{[\frac{t-t_0}{d}]}(-1)^{r+1}({}^{\alpha}C_r)[V(t-rd, x(t))-d^{\alpha}f(t,x(t))-V(t_0,x_0)]\Bigg\}.$$

Notice that (13) is the same in [5] where $d>0$

Given that $\lim_{N\to\infty}\sum_{r=0}^{N}(-1)^{r\alpha}C_r = 0$ where $\alpha\in(0,1)$, and $\lim_{\mu\to 0^+}[\frac{(t-t_0)}{\mu}]=\infty$, then it is easy to see that

$$\lim_{\mu\to 0^+}\sum_{r=1}^{[\frac{(t-t_0)}{\mu}]}(-1)^{r\alpha}C_r = -1. \tag{14}$$

Also from (9) and since the Caputo and Riemann-Liouville formulations coincide when $h(t_0)=0$, ([5]) then we have that

$$\limsup_{\mu\to 0^+}\frac{1}{\mu^{\alpha}}\sum_{r=0}^{[\frac{(t-t_0)}{\mu}]}(-1)^{r\alpha}C_r = ^{RL\mathbb{T}}D^{\alpha}(1) = \frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}, \quad t\geq t_0. \tag{15}$$

Definition 2.16. *The origin* $x=0$ *of (1) is said to be eventually stable if given* $\epsilon>0$, *there exists numbers* δ *and* T *such that the inequality* $\|x^0\|<\delta$ *implies* $\|x(t;t_0,x^0)\|<\epsilon$ *for all* $t\geq t_0\geq T$.

Definition 2.16 better explained means that as time goes on, the origin tends to act more and more like a stable equilibrium state. That is if the origin is eventually stable, then the system has a property that if it has operated properly for a sufficient long period of time, it can be expected to continue to operate properly in future.

Lemma 2.2. *Assume $h, m \in C_{rd}(\mathbb{T}, \mathbb{R})$, suppose there exists $t_1 > t_0, t_1 \in \mathbb{T}$ such that $h(t_1) = m(t_1)$ and $h(t) < m(t)$ for $t_0 \leq t < t_1$. Then if the Caputo fractional delta Dini derivatives of h and m exist at t_1, then the inequality ${}^{C\mathbb{T}}D_+^{\alpha} h^{\Delta}(t_1) >^{C\mathbb{T}} D_+^{\alpha} m^{\Delta}(t_1)$ holds.*

Proof. Applying (9), we have

$$\begin{aligned} {}^{C\mathbb{T}}D_+^{\alpha}(h(t) - m(t))^{\Delta} &= \limsup_{\mu \to 0^+} \frac{1}{\mu^{\alpha}} \Bigg\{ \sum_{r=0}^{[\frac{t-t_0}{\mu}]} (-1)^r \alpha C_r [h(\sigma(t) - r\mu) - m(\sigma(t) - r\mu)] \\ &\quad -[h(t_0) - m(t_0)] \Bigg\} \end{aligned}$$

$$\begin{aligned} {}^{C\mathbb{T}}D_+^{\alpha} h^{\Delta}(t) -^{C\mathbb{T}} D_+^{\alpha} m^{\Delta}(t) &= \limsup_{\mu \to 0^+} \frac{1}{\mu^{\alpha}} \Bigg\{ \sum_{r=0}^{[\frac{t-t_0}{\mu}]} (-1)^r q C_r [h(\sigma(t) - r\mu) - m(\sigma(t) - r\mu)] \\ &\quad -[h(t_0) - m(t_0)] \Bigg\}, \end{aligned}$$

at t_1, we have that

$${}^{C\mathbb{T}}D_+^{\alpha} h^{\Delta}(t_1) = -\limsup_{\mu \to 0^+} \frac{1}{\mu^{\alpha}} \Bigg\{ \sum_{r=0}^{[\frac{t-t_0}{\mu}]} (-1)^r \alpha C_r [h(t_0) - m(t_0)] \Bigg\} +^{C\mathbb{T}} D_+^{\alpha} m^{\Delta}(t_1). \tag{16}$$

Applying (15) to (16), we have

$${}^{C\mathbb{T}}D_+^{\alpha} h^{\Delta}(t_1) = -\frac{(t - t_0)^{-\alpha}}{\Gamma(1-\alpha)} [h(t_0) - m(t_0)] +^{C\mathbb{T}} D_+^{\alpha} m^{\Delta}(t_1),$$

but from the statement of the lemma, we have that

$$h(t) < m(t) \text{ for } t_0 \leq t < t_1$$

$$\implies h(t) - m(t) < 0, \text{ for } t_0 \leq t < t_1.$$

And so it follows that

$$-\frac{(t - t_0)^{-\alpha}}{\Gamma(1-\alpha)} [h(t_0) - m(t_0)] > 0,$$

implying that

$${}^{C\mathbb{T}}D_+^{\alpha} h^{\Delta}(t_1) >^{C\mathbb{T}} D_+^{\alpha} m^{\Delta}(t_1).$$

□

Theorem 2.1. *Assume that*

(i) $g \in C_{rd}[\mathbb{T} \times \mathbb{R}_+, \mathbb{R}_+]$ *and* $g(t,u)\mu$ *is non-decreasing in* u.

(ii) $V \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}_+]$ *be locally Lipschitzian in the second variable such that*

$$^{C\mathbb{T}}D_+^{\alpha} V^{\Delta}(t,x) \leq g(t, V(t,x)), (t,x) \in \mathbb{T} \times \mathbb{R}^n. \tag{17}$$

(iii) $z(t) = z(t; t_0, u^0)$ *is the maximal solution of* (2) *existing on* $\mathbb{T}$.

Then

$$V(t, x(t)) \leq z(t), \quad t \geq t_0 \tag{18}$$

provided that

$$V(t_0, x^0) \leq u^0, \tag{19}$$

where $x(t) = x(t; t_0, x^0)$ *is any solution of* (1), $t \in \mathbb{T}, \quad t \geq t_0$.

Proof. Apply the principle of induction as stated in Lemma 2.1 to the statement

$$\mathbf{S(t)} : V(t, x(t)) \leq z(t), \quad t \in \mathbb{T},\ t \geq t_0$$

(i) $\mathbf{S(t_0)}$ is true since $V(t_0, x^0) \leq u_0$,

(ii) Let t be right-scattered and $\mathbf{S(t)}$ be true. We need to show that $\mathbf{S}(\sigma(\mathbf{t}))$ is true; that is

$$V(\sigma(t), x(\sigma(t))) \leq z(\sigma(t)), \tag{20}$$

set $h(t) = V(t, x(t))$ then $h(\sigma(t)) = V(\sigma(t), x(\sigma(t)))$, but from (9), we have that

$$^{C\mathbb{T}}D_+^{\alpha} h^{\Delta}(t) = \limsup_{\mu \to 0+} \frac{1}{\mu^{\alpha}} \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha} C_r [h(\sigma(t) - r\mu) - h(t_0)], \quad t \geq t_0,$$

also

$$^{C\mathbb{T}}D_+^{\alpha} z^{\Delta}(t) = \limsup_{\mu \to 0+} \frac{1}{\mu^{\alpha}} \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha} C_r [z(\sigma(t) - r\mu) - z(t_0)] \quad t \geq t_0,$$

so that

$$\begin{aligned} &^{C\mathbb{T}}D_+^{\alpha} z^{\Delta}(t) -^{C\mathbb{T}} D_+^{\alpha} h^{\Delta}(t) \\ &= \limsup_{\mu \to 0+} \frac{1}{\mu^{\alpha}} \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha} C_r [z(\sigma(t) - r\mu) - z(t_0)] \\ &- \limsup_{\mu \to 0+} \frac{1}{\mu^{\alpha}} \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha} C_r [h(\sigma(t) - r\mu) - h(t_0)] \end{aligned}$$

$$\begin{aligned} ^{C\mathbb{T}}D_+^{\alpha} z^{\Delta}(t) -^{C\mathbb{T}} D_+^{\alpha} h^{\Delta}(t) &= \limsup_{\mu \to 0+} \frac{1}{\mu^{\alpha}} \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\,\alpha} C_r \Big[[z(\sigma(t) - r\mu) - z(t_0)] \\ &\quad -[h(\sigma(t) - r\mu) - h(t_0)] \Big] \end{aligned}$$

$$\Big({}^{C\mathbb{T}}D_{+}^{\alpha}z^{\Delta}(t) - {}^{C\mathbb{T}}\, D_{+}^{\alpha}h^{\Delta}(t)\Big)\mu^{\alpha} = \limsup_{\mu\to 0+}\sum_{r=0}^{[\frac{(t-t_0)}{\mu}]}(-1)^{r}\,{}^{\alpha}C_{r}\Big[[z(\sigma(t)-r\mu)-z(t_0)] - [h(\sigma(t)-r\mu)-h(t_0)]\Big]$$

$$\Big({}^{C\mathbb{T}}D_{+}^{\alpha}z^{\Delta}(t) - {}^{C\mathbb{T}}\, D_{+}^{\alpha}h^{\Delta}(t)\Big)\mu^{\alpha} \le [z(\sigma(t)) - z(t_0)] - [h(\sigma(t)) - h(t_0)]$$

$$\Big({}^{C\mathbb{T}}D_{+}^{\alpha}z^{\Delta}(t) - {}^{C\mathbb{T}}\, D_{+}^{\alpha}h^{\Delta}(t)\Big)\mu^{\alpha} \le [z(\sigma(t)) - h(\sigma(t))] - [z(t_0) - h(t_0)]$$

$$[z(\sigma(t)) - h(\sigma(t))] \ge \Big({}^{C\mathbb{T}}D_{+}^{\alpha}z^{\Delta}(t) - {}^{C\mathbb{T}}\, D_{+}^{\alpha}h^{\Delta}(t)\Big)\mu^{\alpha} + [z(t_0) - h(t_0)]$$

$$\begin{aligned}[h(\sigma(t)) - z(\sigma(t))] &\le \Big({}^{C\mathbb{T}}D_{+}^{\alpha}h^{\Delta}(t) - {}^{C\mathbb{T}}\, D_{+}^{\alpha}z^{\Delta}(t)\Big)\mu^{\alpha} + [h(t_0) - z(t_0)] \\ &\le \Big(g(t,h(t)) - g(t,z(t))\Big)\mu^{\alpha} + [h(t_0) - z(t_0)].\end{aligned}$$

Since $g(t,u)\mu$ is non decreasing in u and $\mathbf{S}(\mathbf{t})$ is true, then $h(\sigma(t)) - z(\sigma(t)) \le 0$ so (20) holds.

(iii) Let t be right dense and $\mathcal{N}$ be a right neighborhood of $t \in \mathbb{T}$. We need to show that $\mathbf{S}(\mathbf{t}^*)$ is true for $t^* \in \mathcal{N}$. This follows from the comparison theorem for Caputo fractional differential equations since at every right dense point $t^* \in \mathcal{N}$, $\sigma(t^*) = t^*$. See [5].

Therefore by induction principle, the statement $\mathbf{S}(\mathbf{t})$ is true, and this completes the proof □

3. Main Result

In this section, we will obtain sufficient conditions for the eventual stability of the system (11).

Theorem 3.1 (Eventual Stability)**.** *Let the function $g \in C_{rd}[\mathbb{T}\times\mathbb{R}_+,\mathbb{R}_+]$ be such that $g(t,u)$ is non-decreasing in u with $g(t,u) \equiv 0$ and $V(t,x(t)) \in C[\mathbb{T}\times\mathbb{R}^n,\mathbb{R}_+]$ be such that V is positive definite and locally Lipschitzian in x, with $V(t,0) \equiv 0$. Also, for any points $t, t_0 \ge 0$ and $x, x_0 \in \mathbb{R}^n$, the inequality*

$${}^{C\mathbb{T}}D_{+}^{\alpha}V^{\Delta}(t,x(t)) \le g(t,V(t,x(t))),$$

holds. Then if the solution at the origin of (2) is eventually stable, then the solution at the origin of the fractional dynamic system on time scale, (1) is eventually stable.

Proof. From the assumption of the eventual stability of the origin of (2), it follows that $\exists\, \delta(\epsilon) > 0$ and $T(\epsilon) > 0$ such that

$$u(t;t_0,u^0) < \epsilon, \quad \text{for } 0 \ge u^0 < \delta,\ t \ge t_0 \ge T(\epsilon), \tag{21}$$

where $u(t;t_0,u^0)$ is any solution of (2). $V(t,0) = 0$ and $V \in C_{rd}$ this implies there exists $\delta_1 = \delta_1(\delta) > 0$, such that, for $x^0 \in \mathbb{R}^N$, we have that,

$$\|x^0\| < \delta_1 \quad \Longrightarrow \quad V(t_0,x^0) < \delta. \tag{22}$$

Now, lets assume that for any solution $x(t) = x(t; t_0, x^0)$ of (1), $\|x(t)\| < \epsilon$ whenever $\|x^0\| < \delta_1$ for $t \geq t_0 \geq T$.

If this assumption were true, it would mean that the solution at the origin of (1) will be eventually stable, but if not, then there would exists a time $t^0 > t_0$, such that

$$\|x(t^0)\| \geq \epsilon. \tag{23}$$

However, from Theorem 2.1, we have that

$$V(t, x(t)) \leq z(t), \tag{24}$$

whenever $V(t_0, x^0) \leq u^0$, where $z(t) = z(t; t_0, u^0)$.

but at $t = t^0$, from (21), (24) and the positive definite property of $V(t, x)$, we obtain

$$\|x(t^0)\| \leq V(t^0, x(t^0)) \leq z(t^0) < \epsilon.$$

and when (23) is imputed, becomes a clear contradiction

$$\epsilon \leq V(t^0, x(t^0)) \leq z(t^0) < \epsilon.$$

Proving that our assumption were true, and no such time t^0 exists so therefore we can conclude that if given $\epsilon > 0$, we can find numbers δ and T such that the inequality $\|x^0\| < \delta$ implies $\|x(t; t_0, x^0)\| < \epsilon$ for all $t \geq t_0 \geq T$. □

4. Application

Consider the system of dynamic equations

$$\begin{aligned} x_1^{\Delta}(t) &= \quad x_1 \sec^2 t - \tan^2 t(x_2 + x_1) + x_2 \cot^2 t \\ x_2^{\Delta}(t) &= \quad 2(x_1 - x_2) + x_2 \cosh^2 t - 2x_1 \cos^2 t, \end{aligned} \tag{25}$$

for $t \geq t_0$, with initial conditions

$$x_1(t_0) = x_{10} \quad \text{and} \quad x_2(t_0) = x_{20},$$

where $x_1, x_2 \in \mathbb{R}^2$ $f = (f_1, f_2)$.

Consider $V(t, x_1, x_2) = |x_1| + |x_2|$, for $t \in \mathbb{T}$ and $x_1, x_2 \in \mathbb{R}^2$, where $x \in S(\rho)$, $\rho > 0$. Then we compute the dini derivative for $V(t, x_1, x_2) = |x_1| + |x_2|$ as follows from (4) we have that

$$D^+V^{\Delta}(t, x) = \limsup_{\mu(t)\to 0} \frac{V(t + \mu(t), x + \mu(t)f(t, x)) - V(t, x)}{\mu(t)}$$

$$= \limsup_{\mu(t)\to 0} \frac{|x_1 + \mu(t)f_1(t, x)| + |x_2 + \mu(t)f_2(t, x)| - [|x_1| + |x_2|]}{\mu(t)}$$

$$\leq \limsup_{\mu(t)\to 0} \frac{|x_1| + |\mu(t)f_1(t, x)| + |x_2| + |\mu(t)f_2(t, x)| - |x_1| - |x_2|}{\mu(t)}$$

$$= \limsup_{\mu(t)\to 0} \frac{|\mu(t)f_1(t,x)| + |\mu(t)f_2(t,x)|}{\mu(t)}$$

$$= \limsup_{\mu(t)\to 0} \frac{\mu(t)[|f_1(t,x)| + |f_2(t,x)|]}{\mu(t)}$$

$$\leq |f_1(t,x)| + |f_2(t,x)|$$

$$= |x_1 \sec^2 t - \tan^2 t(x_2 + x_1) + x_2 \cot^2 t| + |2(x_1 - x_2) + x_2 \cosh^2 t - 2x_1 \cos^2 t|$$

$$= |x_1 \sec^2 t - x_2 \tan^2 t - x_1 \tan^2 t + x_2 \cot^2 t| + |2x_1 - 2x_2 + x_2 \cosh^2 t - 2x_1 \cos^2 t|$$

$$= |x_1(\sec^2 t - \tan^2 t) - x_2(\tan^2 t - \cot^2 t)| + |2x_1(1 - \cos^2 t) - x_2(2 - \cosh^2 t)|$$

$$= \left| x_1\left(\frac{1}{\cos^2 t} - \frac{\sin^2 t}{\cos^2 t}\right) - x_2\left(\frac{\sin^2 t}{\cos^2 t} - \frac{\cos^2 t}{\sin^2 t}\right)\right| + \left|2x_1(\sin^2 t) - x_2\left(2 - \frac{1}{\cos^2 t}\right)\right|$$

$$\leq \left| x_1\left(\frac{1 - \sin^2 t}{\cos^2 t}\right) - x_2\left(\frac{\sin^4 t - \cos^4 t}{\cos^2 t \sin^2 t}\right)\right| + |2x_1||\sin^2 t| + |x_2|\left(|2| + \left|\frac{1}{\cos^2 t}\right|\right)$$

$$\leq \left| x_1\left(\frac{\cos^2 t}{\cos^2 t}\right) - x_2\left(\frac{\sin^2 t - \cos^2 t)(\sin^2 t + \cos^2 t)}{\cos^2 t \sin^2 t}\right)\right| + 2|x_1| + 3|x_2|$$

$$\leq |x_1| + |x_2|\left|\left(\frac{\sin^2 t - \cos^2 t}{\cos^2 t \sin^2 t}\right)\right| + 2|x_1| + 3|x_2|$$

$$= |x_1| + |x_2|\left|\left(\frac{1}{\cos^2 t} - \frac{1}{\sin^2 t}\right)\right| + 2|x_1| + 3|x_2|$$

$$\leq 3|x_1| + |x_2|\left(\left|\frac{1}{\cos^2 t}\right| + \left|\frac{1}{\sin^2 t}\right|\right) + 3|x_2|$$

$$\leq 3|x_1| + 5|x_2| \leq 5[|x_1| + |x_2|]$$

$$D^+V^{\Delta}(t,x) \leq 5V(t,x_1,x_2) = g(t,V).$$

Now consider the consider the comparison equation

$$D^+u^{\Delta} = 5u > 0,\ u(0) = u_0, \tag{26}$$

with solution

$$u(t) = u_0 e^{5t}. \tag{27}$$

Even though conditions (i)-(iii) of [20] are satisfied that is $V \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}_+]$, $D^+V^{\Delta}(t,x_1,x_2) \leq g(t,V(t,x))$ and $\sqrt{x_1^2 + x_2^2} \leq |x_1| + |x_2| \leq 2(x_1^2 + x_2^2)$, for $b(\|x\|) = r$ and $a(\|x\|) = 2r^2$, it is obvious to see that the solution (27)of the comparison system (26) is not eventually stable, so we can not deduce the eventual stability properties of the system (25) by applying the basic definition of the

Dini-derivative of a Lyapunov function of dynamic equation on time scale to the Lyapunov function $V(t, x_1, x_2) = |x_1| + |x_2|$.

Now, we will apply our new definition on the same system but as a Caputo fractional dynamic system

$$\begin{aligned} {}^{C\mathbb{T}}D^{\alpha}x_1^{\Delta}(t) &= \quad x_1 \sec^2 t - \tan^2 t(x_2 + x_1) + x_2 \cot^2 t \\ {}^{C\mathbb{T}}D^{\alpha}x_2^{\Delta}(t) &= \quad 2(x_1 - x_2) + x_2 \cosh^2 t - 2x_1 \cos^2 t \end{aligned} \tag{28}$$

for $t \geq t_0$, with initial conditions

$$x_1(t_0) = x_{10} \quad \text{and} \quad x_2(t_0) = x_{20},$$

where $x_1, x_2 \in \mathbb{R}^2$ $f = (f_1, f_2)$,

Consider $V(t, x_1, x_2) = |x_1| + |x_2|$, for $t \in \mathbb{T}$ and $x_1, x_2 \in \mathbb{R}^2$, where $x \in S(\rho)$, $\rho > 0$, so that the associated norm $\|x\| = \sqrt{x_1^2 + x_2^2}$.

Since

$$V(t, x_1, x_2) = |x_1| + |x_2|,$$

then $\phi(\|x\|) \leq V(t, x_1, x_2)$. From (11), we compute the Caputo fractional Dini derivative for $V(t, x_1, x_2) = |x_1| + |x_2|$ as follows

$$\begin{aligned}
&{}^{C\mathbb{T}}D_+^{\alpha}V^{\Delta}(t, x) \\
&= \limsup_{\mu \to 0^+} \frac{1}{\mu^{\alpha}} \Bigg\{ V(\sigma(t), x(\sigma(t)) - V(t_0, x_0) \\
&- \sum_{r=1}^{[\frac{t-t_0}{\mu}]} (-1)^{r+1} ({}^{\alpha}C_r)[V(\sigma(t) - r\mu, x(\sigma(t)) - \mu^{\alpha} f(t, x(t))) - V(t_0, x_0)] \Bigg\} \\
&= \limsup_{\mu \to 0^+} \frac{1}{\mu^{\alpha}} \Bigg\{ (|x_1(\sigma(t))| + |x_2(\sigma(t))|) - (|x_{10}| + |x_{20}|) + \sum_{r=1}^{[\frac{t-t_0}{\mu}]} (-1)^r ({}^{\alpha}C_r) \\
&[|x_1(\sigma(t)) - \mu^{\alpha} f_1(t, x_1)| + |x_2(\sigma(t)) - \mu^{\alpha} f_2(t, x_2)| - (|x_{10}| + |x_{10}|)] \Bigg\} \\
&\leq \limsup_{\mu \to 0^+} \frac{1}{\mu^{\alpha}} \Bigg\{ (|x_1(\sigma(t))| + |x_2(\sigma(t))|) - (|x_{10}| + |x_{20}|) + \sum_{r=1}^{[\frac{t-t_0}{\mu}]} (-1)^r ({}^{\alpha}C_r) \\
&[|x_1(\sigma(t))| + |\mu^{\alpha} f_1(t; x_1)| + |x_2(\sigma(t))| + |\mu^{\alpha} f_2(t; x_2)| - (|x_{10}| + |x_{10}|)] \Bigg\} \\
&\leq \limsup_{\mu \to 0^+} \frac{1}{\mu^{\alpha}} \Bigg\{ (|x_1(\sigma(t))| + |x_2(\sigma(t))|) - (|x_{10}| + |x_{20}|) \\
&+ \sum_{r=1}^{[\frac{t-t_0}{\mu}]} (-1)^r ({}^{\alpha}C_r) \Big[|x_1(\sigma(t))| + |x_2(\sigma(t))| \Big]
\end{aligned}$$

$$+\sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r(^{\alpha}C_r)|\Big[|\mu^{\alpha}f_1(t;x_1)|+|\mu^{\alpha}f_2(t;x_2)|\Big]$$

$$-\sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r(^{\alpha}C_r)\Big[|x_{10}|+|x_{10}|\Big]\Big\}$$

$$=\limsup_{\mu\to 0^+}\frac{1}{\mu^{\alpha}}\Big\{(|x_1(\sigma(t))|+|x_2(\sigma(t))|)+\sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r(^{\alpha}C_r)\Big[|x_1(\sigma(t))|+|x_2(\sigma(t))|\Big]$$

$$-(|x_{10}|+|x_{20}|)-\sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r(^{\alpha}C_r)\Big[|x_{10}|+|x_{10}|\Big]$$

$$+\mu^{\alpha}\sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r(^{\alpha}C_r)|\Big[|f_1(t;x_1)|+|f_2(t;x_2)|\Big]\Big\}$$

$$\leq\limsup_{\mu\to 0^+}\frac{1}{\mu^{\alpha}}\Big\{\sum_{r=0}^{[\frac{t-t_0}{\mu}]}(-1)^r(^{\alpha}C_r)\Big[|x_1(\sigma(t))|+|x_2(\sigma(t))|\Big]-\sum_{r=0}^{[\frac{t-t_0}{\mu}]}(-1)^r(^{\alpha}C_r)\Big[|x_{10}|+|x_{10}|\Big]\Big\}$$

$$+\limsup_{\mu\to 0^+}\sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r(^{\alpha}C_r)|\Big[|f_1(t;x_1)|+|f_2(t;x_2)|\Big].$$

Applying (14) and (15) we have

$$=\frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}(|x_1(\sigma(t))|+|x_2(\sigma(t))|)-\frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}(|x_{10}|+|x_{10}|)-\Big[|f_1(t;x_1)|+|f_2(t;x_2)|\Big]$$

$$\leq\frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}(|x_1(\sigma(t))|+|x_2(\sigma(t))|)-\Big[|f_1(t;x_1)|+|f_2(t;x_2)|\Big]$$

As $t\to\infty$, $\frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}(|x_1(\sigma(t))|+|x_2(\sigma(t))|)\to 0$, then

$${}^{C\mathbb{T}}D_+^{\alpha}V^{\Delta}(t;x_1,x_2)\leq-\Big[|f_1(t;x_1)|+|f_2(t;x_2)|\Big]$$

$$=-\left[|x_1\sec^2 t-\tan^2 t(x_2+x_1)+x_2\cot^2 t|+|2(x_1-x_2)+x_2\cosh^2 t-2x_1\cos^2 t|\right]$$

$$=-\left[|x_1\sec^2 t-x_2\tan^2 t-x_1\tan^2 t+x_2\cot^2 t|+|2x_1-2x_2+x_2\cosh^2 t-2x_1\cos^2 t|\right]$$

$$=-\left[|x_1(\sec^2 t-\tan^2 t)-x_2(\tan^2 t-\cot^2 t)|+|2x_1(1-\cos^2 t)-x_2(2-\cosh^2 t)|\right]$$

$$=-\left[\left|x_1\left(\frac{1}{\cos^2 t}-\frac{\sin^2 t}{\cos^2 t}\right)-x_2\left(\frac{\sin^2 t}{\cos^2 t}-\frac{\cos^2 t}{\sin^2 t}\right)\right|+\left|2x_1(\sin^2 t)-x_2\left(2-\frac{1}{\cos^2 t}\right)\right|\right]$$

$$\leq-\left[\left|x_1\left(\frac{1-\sin^2 t}{\cos^2 t}\right)-x_2\left(\frac{\sin^4 t-\cos^4 t}{\cos^2 t\sin^2 t}\right)\right|+|2x_1||\sin^2 t|+|x_2|\left(|2|+\left|\frac{1}{\cos^2 t}\right|\right)\right]$$

$$\leq -\left[\left|x_1\left(\frac{\cos^2 t}{\cos^2 t}\right) - x_2\left(\frac{\sin^2 t - \cos^2 t)(\sin^2 t + \cos^2 t)}{\cos^2 t \sin^2 t}\right)\right| + 2|x_1| + 3|x_2|\right]$$

$$\leq -\left[|x_1| + |x_2|\left|\left(\frac{\sin^2 t - \cos^2 t}{\cos^2 t \sin^2 t}\right)\right| + 2|x_1| + 3|x_2|\right]$$

$$= -\left[|x_1| + |x_2|\left|\left(\frac{1}{\cos^2 t} - \frac{1}{\sin^2 t}\right)\right| + 2|x_1| + 3|x_2|\right]$$

$$\leq -\left[3|x_1| + |x_2|\left(\left|\frac{1}{\cos^2 t}\right| + \left|\frac{1}{\sin^2 t}\right|\right) + 3|x_2|\right]$$

$$\leq -3|x_1| - 5|x_2| \leq -3[|x_1| + |x_2|].$$

Therefore

$$^{C\mathbb{T}}D_+^\alpha V^\Delta(t; x_1, x_2) \leq -3V(t, x_1, x_2). \tag{29}$$

Consider the comparison system

$$^{C\mathbb{T}}D_+^\alpha u^\Delta = g(t, u) \leq -3u, \tag{30}$$

using the Laplace transform method

$$^{C\mathbb{T}}D_+^\alpha 3u^\Delta + u = 0$$

$$\begin{aligned} &\mathcal{L}\{^{C\mathbb{T}}D_+^\alpha u^\Delta\} + 3\mathcal{L}\{u\} = 0 \\ \implies \qquad &S^\alpha U(s) - S^{\alpha-1}u_0 + 3U(s) = 0 \\ &U(s)(s^\alpha + 3) = u^0 S^{\alpha-1} U(s) = \frac{u_0 S^{\alpha-1}}{S^\alpha + 3}, \end{aligned}$$

taking the inverse Laplace transform we have

$$u(t) = u^0 \mathcal{L}^{-1}\left\{\frac{S^{\alpha-1}}{S^\alpha + 3}\right\}. \tag{31}$$

Recall that

$$\mathcal{L}^{-1}\left\{\frac{S^{\alpha-\beta}}{S^\alpha - \lambda}\right\} = t^{\beta-1}E_{\alpha,\beta}(\lambda t^\alpha). \tag{32}$$

Comparing (32) and (31), we have $q - \beta, \implies \beta = 1 \; S^\alpha - \lambda = S^\alpha + 3 \implies \lambda = -3$ so we have,

$$u(t) = u^0 E_{\alpha,1}(-3t^\alpha), \quad \text{for } \alpha \in (0, 1), \tag{33}$$

where $E_{\alpha,1}$ is the Mittag-Leffler function.

Since all the conditions of Theorem 3.1 are satisfied, and the solution at the origin of the comparison system (30) is eventually stable, then we conclude that the solution at the origin of system (28) is stable.

5. Conclusion

In conclusion, the concept of eventual stability on time scales offers a powerful and versatile framework for analyzing the stability of dynamic systems that operate across both continuous and discrete time domains. Unlike traditional Lyapunov stability, which is restricted to equilibrium states, eventual stability provides a broader perspective by addressing the behavior of states that may not be equilibrium but gradually exhibit stable characteristics over time. This extension is particularly useful in adaptive control systems and other applications where external disturbances and time-varying conditions are prevalent. By integrating eventual stability with time scale calculus, this work unifies stability analysis for systems that evolve in mixed time domains, whether discrete, continuous, or hybrid. This approach significantly enhances the ability to analyze systems subjected to varying conditions, offering flexibility and robustness in environments where traditional stability concepts fall short. The theoretical insights gained from this study not only generalize existing stability theorems but also provide a foundation for future research in adaptive control and time scale systems. The application of eventual stability on time scales introduces new possibilities for system design and performance optimization in complex, real-world environments. We have also shown the practical applicability as well as effectiveness of our result in (25).

Authors' Contributions

All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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