Asia Pac. J. Math. 2025 **12**:30

AN EXPANSION OF EXTENDED REAL OPERATIONS FOR TWO QUADRATIC FUZZY NUMBERS TO 3-DIMENSIONAL SPACE

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Received Jan. 13, 2025

Abstract. We define extended real number operators for quadratic fuzzy numbers in three dimensions. We also prove that the calculated results extend the concepts from two dimensions to three dimensions and present them as graphs. When expanding from one to two dimensions, it was necessary to define a new operator due to the change in the alpha cut from a one-dimensional interval to a two-dimensional plane. Therefore, even in the case of three dimensions, operators between subsets (alpha cuts) of three dimensions must be defined differently from those in two dimensions. When the result of three dimensions is limited to two dimensions, graphically demonstrating its consistency with the result in two dimensions is very helpful for intuitive understanding and practical applications.

2020 Mathematics Subject Classification. 47S40; 03E72.

Key words and phrases. expansion; extended real operator; 3-dimensional quadratic fuzzy number.

1. Introduction

The quadratic fuzzy number, whose membership function is given by a quadratic equation, is one of the most studied and applied fuzzy numbers in fuzzy theory, along with the triangular fuzzy number. The quadratic fuzzy number is particularly widely used in fields such as fuzzy fractional programming, flow shop scheduling, and related optimization problems, and has been studied extensively, as evidenced by research papers [1, 2, 3]. We studied extended operators for general quadratic fuzzy numbers with a maximum value of $k < 1$ [4], and defined their extension to 2-dimensional spaces [5]. The expansion into 2-dimensional space takes the form of an elliptic parabola. The intersection with a vertical plane through the vertex of the elliptic parabola can be interpreted as a 1-dimensional quadratic fuzzy number in that plane. Likewise, when extended to three dimensions, the shape becomes an ellipsoid, and its largest horizontal section can be interpreted as a 2-dimensional quadratic fuzzy

DOI: 10.28924/APJM/12-30

number [6]. There have been many attempts to extend fuzzy numbers to two and three dimensions, but no studies have achieved consistency with existing results for one or two dimensions.

Several operators have been defined for fuzzy numbers, and significant research has been conducted on this topic. In particular, the extended real number operator defined by Zadeh serves as a fundamental operator between fuzzy numbers. The calculation of the extended real number operator in one dimension, between triangular and quadratic fuzzy numbers, is a well-documented topic in textbooks. We defined and calculated a new extended real number operator for quadratic fuzzy numbers in their 2-dimensional extensions [5]. In one dimension, the extended real number operator is defined through an alpha cut. In one dimension, an alpha cut is represented as an interval and is defined as an algebraic real number operation on the endpoints of the interval. However, in two dimensions, since the alpha cut appears as a subset in 2-dimensional space, it cannot be defined as an algebraic operation on real numbers as in the 1-dimensional case. Therefore, a new parametric operation was introduced to define and calculate an extended real number operator in two dimensions [7]. We also proved that limiting the results of the operation to one dimension agrees with the 1-dimensional results [8].

In this paper, we define extended real number operators for quadratic fuzzy numbers in three dimensions. We also prove that the calculated results extend the concepts from two dimensions to three dimensions and present them as graphs. When expanding from one to two dimensions, it was necessary to define a new operator due to the change in the alpha cut from a one-dimensional interval to a two-dimensional plane. Therefore, even in the case of three dimensions, operators between subsets (alpha cuts) of three dimensions must be defined differently from those in two dimensions. When the result of three dimensions is limited to two dimensions, graphically demonstrating its consistency with the result in two dimensions is very helpful for intuitive understanding and practical applications. A graph that was not presented in paper [5] has been added, and the result of division in paper [6] has been corrected.

2. Preliminaries

We define α-cut and α-set of the fuzzy set A on $\mathbb{R}$ with the membership function $\mu_A(x)$.

Definition 2.1. An *α-cut* of the fuzzy number A is defined by $A_\alpha = \{x \in \mathbb{R} \mid \mu_A(x) \geq \alpha\}$ if $\alpha \in (0, 1]$ and $A_0 = \mathrm{cl}\{x \in \mathbb{R} \mid \mu_A(x) > \alpha\}$. For $\alpha \in (0, 1)$, the set $A^\alpha = \{x \in X \mid \mu_A(x) = \alpha\}$ is said to be the *α-set* of the fuzzy set A, A^0 is the boundary of $\{x \in \mathbb{R} \mid \mu_A(x) > \alpha\}$ and $A^1 = A_1$.

Definition 2.2. [5] A fuzzy set A with a membership function

$$\mu_A(x,y) = \begin{cases} 1 - \left(\frac{(x-x_1)^2}{a^2} + \frac{(y-y_1)^2}{b^2}\right), & b^2(x-x_1)^2 + a^2(y-y_1)^2 \leq a^2b^2, \\ 0, & \text{otherwise}, \end{cases}$$

where $a, b > 0$ is called the *2-dimensional quadratic fuzzy number* and denoted by $[a,\ x_1,\ b,\ y_1]^2$.

The α-cut A_α of a 2-dimensional quadratic fuzzy number $A = [a, x_1, b, y_1]^2$ is an interior of ellipse in an xy-plane including the boundary

$$\begin{aligned} A_\alpha &= \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; b^2(x - x_1)^2 + a^2(y - y_1)^2 \leq a^2b^2(1 - \alpha)\right\} \\ &= \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; \frac{(x - x_1)^2}{a^2(1 - \alpha)} + \frac{(y - y_1)^2}{b^2(1 - \alpha)} \leq 1\right\}. \end{aligned}$$

Theorem 2.3. [7] *Let A be a continuous convex fuzzy number defined on $\mathbb{R}^2$ and $A^\alpha = \{(x, y) \in \mathbb{R}^2 | \mu_A(x, y) = \alpha\}$ be the α-set of A. Then for all $\alpha \in (0, 1)$, there exist continuous functions $f_1^\alpha(t)$ and $f_2^\alpha(t)$ defined on $[0, 2\pi]$ such that*

$$A^\alpha = \{(f_1^\alpha(t), f_2^\alpha(t)) \in \mathbb{R}^2 | 0 \leq t \leq 2\pi\}.$$

If A is a continuous convex fuzzy number defined on $\mathbb{R}^2$, then the α-cut A_α is a closed convex subset in $\mathbb{R}^2$.

Definition 2.4. [5] Let A and B be convex fuzzy numbers defined on $\mathbb{R}^2$ and

$$A^\alpha = \{(f_1^\alpha(t), f_2^\alpha(t)) \in \mathbb{R}^2 | 0 \leq t \leq 2\pi\},$$

$$B^\alpha = \{(g_1^\alpha(t), g_2^\alpha(t)) \in \mathbb{R}^2 | 0 \leq t \leq 2\pi\}$$

be the α-sets of A and B, respectively. For $\alpha \in (0, 1)$, we define that the parametric addition $A(+)_pB$, parametric subtraction $A(-)_pB$, parametric multiplication $A(\cdot)_pB$ and parametric division $A(/)_pB$ of two fuzzy numbers A and B are fuzzy numbers that have their α-sets as follows.

(1) $A(+)_pB : (A(+)_pB)^\alpha = \{(f_1^\alpha(t) + g_1^\alpha(t), f_2^\alpha(t) + g_2^\alpha(t)) \in \mathbb{R}^2 | 0 \leq t \leq 2\pi\}$

(2) $A(-)_pB : (A(-)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \in \mathbb{R}^2 | 0 \leq t \leq 2\pi\}$, where

$$x_\alpha(t) = \begin{cases} f_1^\alpha(t) - g_1^\alpha(t + \pi), & \text{if } 0 \leq t \leq \pi \\ f_1^\alpha(t) - g_1^\alpha(t - \pi), & \text{if } \pi \leq t \leq 2\pi \end{cases}$$

and

$$y_\alpha(t) = \begin{cases} f_2^\alpha(t) - g_2^\alpha(t + \pi), & \text{if } 0 \leq t \leq \pi \\ f_2^\alpha(t) - g_2^\alpha(t - \pi), & \text{if } \pi \leq t \leq 2\pi \end{cases}$$

(3) $A(\cdot)_pB : (A(\cdot)_pB)^\alpha = \{(f_1^\alpha(t) \cdot g_1^\alpha(t), f_2^\alpha(t) \cdot g_2^\alpha(t)) \in \mathbb{R}^2 | 0 \leq t \leq 2\pi\}$

(4) $A(/)_pB : (A(/)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \in \mathbb{R}^2 | 0 \leq t \leq 2\pi\}$, where

$$x_\alpha(t) = \frac{f_1^\alpha(t)}{g_1^\alpha(t + \pi)} \quad (0 \leq t \leq \pi), \quad x_\alpha(t) = \frac{f_1^\alpha(t)}{g_1^\alpha(t - \pi)} \quad (\pi \leq t \leq 2\pi)$$

and

$$y_\alpha(t) = \frac{f_2^\alpha(t)}{g_2^\alpha(t + \pi)} \quad (0 \leq t \leq \pi), \quad y_\alpha(t) = \frac{f_2^\alpha(t)}{g_2^\alpha(t - \pi)} \quad (\pi \leq t \leq 2\pi)$$

For $\alpha = 0$ and $\alpha = 1$, $(A(*)_pB)^0 = \lim_{\alpha\to 0^+}(A(*)_pB)^\alpha$ and $(A(*)_pB)^1 = \lim_{\alpha\to 1^-}(A(*)_pB)^\alpha$, where $* = +,\ -,\ \cdot,\ /$.

Theorem 2.5. [5] *Let $A = [a_1,\ x_1,\ b_1,\ y_1]^2$ and $B = [a_2,\ x_2,\ b_2,\ y_2]^2$ be two 2-dimensional quadratic fuzzy numbers. Then we have the following.*
(1) $A(+)_pB = \Big[a_1 + a_2,\ x_1 + x_2,\ b_1 + b_2,\ y_1 + y_2\Big]^2$
(2) $A(-)_pB = \Big[a_1 + a_2,\ x_1 - x_2,\ b_1 + b_2,\ y_1 - y_2\Big]^2$
(3) $(A(\cdot)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$, *where*

$$x_\alpha(t) = x_1x_2 + (x_1a_2 + x_2a_1)\sqrt{1-\alpha}\cos t + a_1a_2(1-\alpha)\cos^2 t$$

and

$$y_\alpha(t) = y_1y_2 + (y_1b_2 + y_2b_1)\sqrt{1-\alpha}\sin t + b_1b_2(1-\alpha)\sin^2 t.$$

(4) $(A(/)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$, *where*

$$x_\alpha(t) = \frac{x_1 + a_1\sqrt{1-\alpha}\cos t}{x_2 - a_2\sqrt{1-\alpha}\cos t} \quad \text{and} \quad y_\alpha(t) = \frac{y_1 + b_1\sqrt{1-\alpha}\sin t}{y_2 - b_2\sqrt{1-\alpha}\sin t}.$$

Thus $A(+)_pB$ and $A(-)_pB$ become 2-dimensional quadratic fuzzy numbers, but $A(\cdot)_pB$ and $A(/)_pB$ need not to be 2-dimensional quadratic fuzzy numbers.

Example 2.6. [5] Let $A = [6,\ 3,\ 8,\ 5]^2$ and $B = [4,\ 2,\ 5,\ 3]^2$. Then by Theorem 2.5, we have the following.
(1) $A(+)_pB = [10,\ 5,\ 13,\ 8]^2$
(2) $A(-)_pB = [10,\ 1,\ 13,\ 2]^2$
(3) $(A(\cdot)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$, where

$$x_\alpha(t) = 6 + 24\sqrt{1-\alpha}\cos t + 24(1-\alpha)\cos^2 t,$$
$$y_\alpha(t) = 15 + 49\sqrt{1-\alpha}\sin t + 40(1-\alpha)\sin^2 t.$$

(4) $(A(/)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$, where

$$x_\alpha(t) = \frac{3 + 6\sqrt{1-\alpha}\cos t}{2 - 4\sqrt{1-\alpha}\cos t} \quad \text{and} \quad y_\alpha(t) = \frac{5 + 8\sqrt{1-\alpha}\sin t}{3 - 5\sqrt{1-\alpha}\sin t}.$$

Thus $A(+)_pB$ and $A(-)_pB$ become 2-dimensional quadratic fuzzy numbers, but $A(\cdot)_pB$ and $A(/)_pB$ need not to be 2-dimensional quadratic fuzzy numbers.

In Example 2.6, the intersection of the graph of A with the vertical plane passing through the vertex $(3, 5)$ produces a one-dimensional quadratic fuzzy number A_1 in that plane. Likewise, intersecting the graph of B with the vertical plane passing through the vertex $(2, 3)$ yields a one-dimensional quadratic fuzzy number B_1. Let $A_1(*)B_1$ denote the result of applying a one-dimensional parametric operator to the quadratic fuzzy numbers A_1 and B_1. Example 2.6 shows the result of applying a two-dimensional parametric operator to the fuzzy numbers A and B in two dimensions. The intersection of $A(*)B$, obtained by applying the operator, with the vertical plane passing through the vertices is identical to $A_1(*)B_1$. That is, Theorem 2.5 extends to two dimensions and remains consistent with the one-dimensional result. Figures 1 to 4 represent the 2-dimensional quadratic fuzzy numbers $A, B, A(+)B$, and $A(-)B$, respectively. Similarly, Figures 5 and 6 depict A_1 and B_1, while Figures 7 and 8 illustrate the one-dimensional restrictions of $A(+)B$ and $A(-)B$.

Theorem 2.7. [8] *Parametric operations for two 2-dimensional quadratic fuzzy numbers in Definition 2.4 are the generalization of algebraic operation for two quadratic fuzzy numbers on* $\mathbb{R}$.

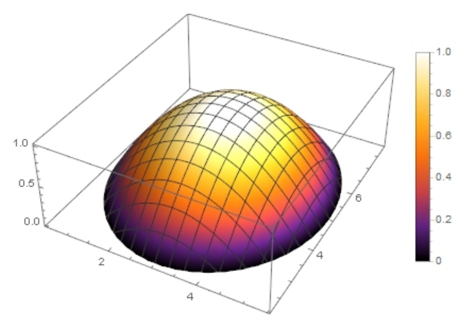


Figure 1. A

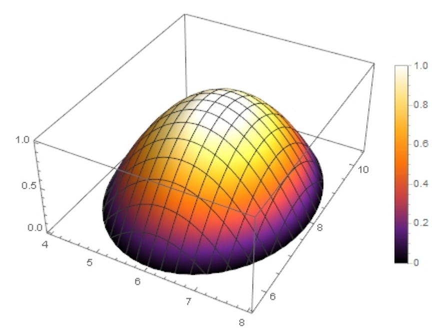


Figure 2. B

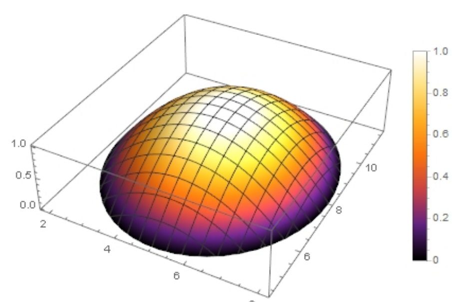


Figure 3. $A(+)B$

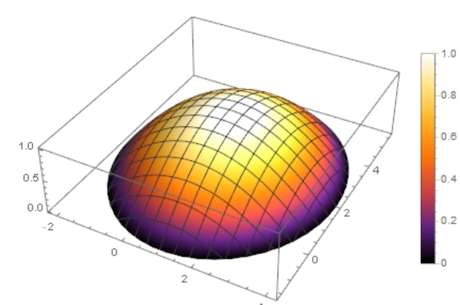


Figure 4. $A(-)B$

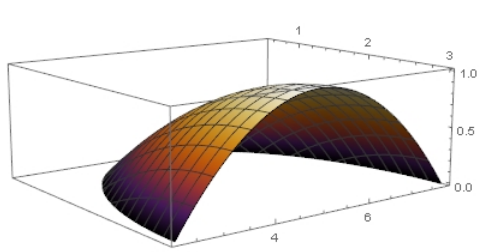


Figure 5. A_1

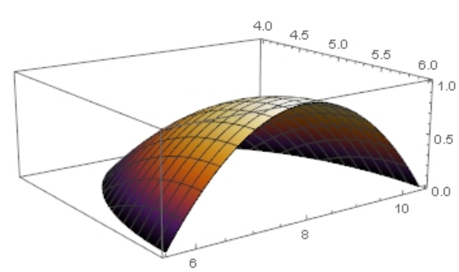


Figure 6. B_1

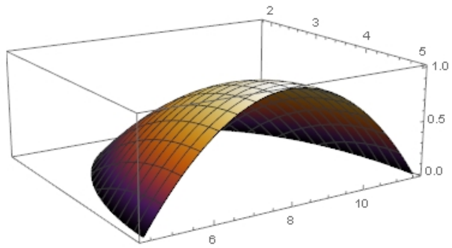


Figure 7. $A_1(+)B_1$

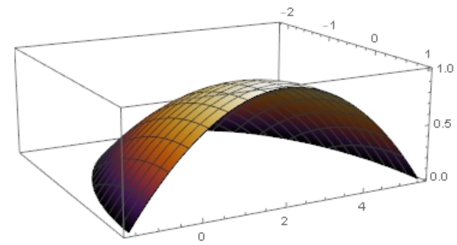


Figure 8. $A_1(-)B_1$

3. 3-DIMENSIONAL QUADRATIC FUZZY NUMBERS

In Chapter 3, we expand on the results of Chapter 2 by extending them to three dimensions. When a parametric operator is defined and calculated in three dimensions, the resulting projection should yield a consistent two-dimensional result. In other words, restricting the three-dimensional result to two dimensions must be consistent with the original two-dimensional outcome. In three dimensions, the horizontal plane (representing the largest circular cross-section) is analyzed instead of the vertical plane used in two dimensions. In the two-dimensional case, a one-dimensional fuzzy number appears in the vertical plane. By contrast, in three dimensions, a two-dimensional fuzzy number emerges, defined within an ellipse and its interior, in the horizontal plane. We show that the calculated results are consistent, both graphically and numerically.

Definition 3.1. [6] A fuzzy set A with a membership function

$$\mu_A(x,y,z) = \begin{cases} 1 - \left(\frac{(x-x_1)^2}{a^2} + \frac{(y-y_1)^2}{b^2} + \frac{(z-z_1)^2}{c^2}\right), & \text{if } \; b^2c^2(x-x_1)^2 + c^2a^2(y-y_1)^2 \\ & \qquad +a^2b^2(z-z_1)^2 \le a^2b^2c^2, \\ 0, \quad \text{otherwise}, & \end{cases}$$

where $a, b, c > 0$ is called the *3-dimensional quadratic fuzzy number* and denoted by $[a, x_1, b, y_1, c, z_1]^3$.

Note that $\mu_A(x, y)$ is a elliptic paraboloid in $\mathbb{R}^2$, but we can not know the shape of $\mu_A(x, y, z)$ in $\mathbb{R}^3$. The α-cut A_α of a 3-dimensional quadratic fuzzy number $A = [a, x_1, b, y_1, c, z_1]^3$ is the following set

$$\begin{aligned} A_\alpha &= \left\{(x,y,z) \in \mathbb{R}^3 \;\middle|\; \frac{(x-x_1)^2}{a^2} + \frac{(y-y_1)^2}{b^2} + \frac{(z-z_1)^2}{c^2} \le 1-\alpha\right\} \\ &= \left\{(x,y,z) \in \mathbb{R}^3 \;\middle|\; \frac{(x-x_1)^2}{a^2(1-\alpha)} + \frac{(y-y_1)^2}{b^2(1-\alpha)} + \frac{(z-z_1)^2}{c^2(1-\alpha)} \le 1\right\}. \end{aligned}$$

Definition 3.2. [9] A 3-dimensional fuzzy number A defined on $\mathbb{R}^3$ is called *convex* fuzzy number if for all $\alpha \in (0, 1)$, the α-cuts

$$A_\alpha = \{(x,y,z) \in \mathbb{R}^3 | \mu_A(x,y,z) \ge \alpha\}$$

are convex subsets in $\mathbb{R}^3$.

Theorem 3.3. [9] *Let A be a continuous convex fuzzy number defined on $\mathbb{R}^3$ and $A^\alpha = \{(x,y,z) \in \mathbb{R}^3 | \mu_A(x,y,z) = \alpha\}$ be the α-set of A. Then for all $\alpha \in (0,1)$, there exist continuous functions $f_1^\alpha(s)$, $f_2^\alpha(s,t)$ and $f_3^\alpha(s,t)(0 \le s \le 2\pi, 0 \le t \le \frac{\pi}{2})$ such that*

$$A^\alpha = \{(f_1^\alpha(s), f_2^\alpha(s,t), f_3^\alpha(s,t)) \in \mathbb{R}^3 | 0 \le s \le 2\pi, 0 \le t \le \frac{\pi}{2}\}.$$

Definition 3.4. [9] Let A and B are two continuous convex fuzzy numbers defined on $\mathbb{R}^3$ and

$$\begin{aligned} A^\alpha &= \{(x,y,z) \in \mathbb{R}^3 | \mu_A(x,y,z) = \alpha\} \\ &= \{(f_1^\alpha(s), f_2^\alpha(s,t), f_3^\alpha(s,t)) \in \mathbb{R}^3 | 0 \le s \le 2\pi, 0 \le t \le \frac{\pi}{2}\}, \\ B^\alpha &= \{(x,y,z) \in \mathbb{R}^3 | \mu_B(x,y,z) = \alpha\} \\ &= \{(g_1^\alpha(s), g_2^\alpha(s,t), g_3^\alpha(s,t)) \in \mathbb{R}^3 | 0 \le s \le 2\pi, 0 \le t \le \frac{\pi}{2}\} \end{aligned}$$

be the α-sets of A and B, respectively. For $\alpha \in (0,1)$, we define that the parametric addition, parametric subtraction, parametric multiplication and parametric division of two fuzzy numbers A and B are fuzzy numbers that have their α-sets as follows.

(1) parametric addition $A(+)_pB$:

$$(A(+)_pB)^\alpha = \{(f_1^\alpha(s) + g_1^\alpha(s),\ f_2^\alpha(s,t) + g_2^\alpha(s,t), f_3^\alpha(s,t) + g_3^\alpha(s,t)) \in \mathbb{R}^3 | \\ 0 \le s \le 2\pi, 0 \le t \le \frac{\pi}{2}\}$$

(2) parametric subtraction $A(-)_pB$:

$$(A(-)_pB)^\alpha = \{(f_1^\alpha(s) - g_1^\alpha(s+\pi),\ f_2^\alpha(s,t) - g_2^\alpha(s+\pi,t),\ f_3^\alpha(s,t) - g_3^\alpha(s+\pi,t)) \in \mathbb{R}^3 | \\ 0 \le s \le \pi, 0 \le t \le \frac{\pi}{2}\},$$

$$(A(-)_pB)^\alpha = \{(f_1^\alpha(s) - g_1^\alpha(s-\pi),\ f_2^\alpha(s,t) - g_2^\alpha(s-\pi,t),\ f_3^\alpha(s,t) - g_3^\alpha(s-\pi,t)) \in \mathbb{R}^3 | \\ \pi \le s \le 2\pi, 0 \le t \le \frac{\pi}{2}\}$$

(3) parametric multiplication $A(\cdot)_pB$:

$$(A(\cdot)_pB)^\alpha = \{(f_1^\alpha(s) \cdot g_1^\alpha(s),\ f_2^\alpha(s,t) \cdot g_2^\alpha(s,t),\ f_3^\alpha(s,t) \cdot g_3^\alpha(s,t)) \in \mathbb{R}^3 | 0 \le s \le 2\pi, 0 \le t \le \frac{\pi}{2}\}.$$

(4) parametric division $A(/)_pB$:

$$(A(/)_pB)^\alpha = \{(\frac{f_1^\alpha(s)}{g_1^\alpha(s+\pi)}, \frac{f_2^\alpha(s,t)}{g_2^\alpha(s+\pi,t)}, \frac{f_3^\alpha(s,t)}{g_3^\alpha(s+\pi,t)}) \in \mathbb{R}^3 | 0 \le s \le \pi, 0 \le t \le \frac{\pi}{2}\},$$
$$(A(/)_pB)^\alpha = \{(\frac{f_1^\alpha(s)}{g_1^\alpha(s-\pi)}, \frac{f_2^\alpha(s,t)}{g_2^\alpha(s-\pi,t)}, \frac{f_3^\alpha(s,t)}{g_3^\alpha(s-\pi,t)}) \in \mathbb{R}^3 | \pi \le s \le 2\pi, 0 \le t \le \frac{\pi}{2}\}.$$

For $\alpha = 0$ and $\alpha = 1$, $(A(*)_pB)^0 = \lim_{\alpha \to 0^+}(A(*)_pB)^\alpha$ and $(A(*)_pB)^1 = \lim_{\alpha \to 1^-}(A(*)_pB)^\alpha$, where $* = +,\ -,\ \cdot,\ /$.

Theorem 3.5. [6] *Let* $A = [a_1, x_1, b_1, y_1, c_1, z_1]^3$ *and* $B = [a_2, x_2, b_2, y_2, c_2, z_2]^3$ *be two 3-dimensional quadratic fuzzy numbers. Then we have the followings.*

(1) $A(+)_p B = \Big[a_1 + a_2, x_1 + x_2, b_1 + b_2, y_1 + y_2, c_1 + c_2, z_1 + z_2\Big]^3$

(2) $A(-)_p B = \Big[a_1 + a_2, x_1 - x_2, b_1 + b_2, y_1 - y_2, c_1 + c_2, z_1 - z_2\Big]^3$

(3) $(A(\cdot)_p B)^\alpha = \{(x_\alpha(s), y_\alpha(s,t), z_\alpha(s,t)) \in \mathbb{R}^3 \mid 0 \le s \le 2\pi, 0 \le t \le \frac{\pi}{2}\}$, *where*

$$x_\alpha(s) = x_1x_2 + (x_1a_2 + x_2a_1)\sqrt{1-\alpha}\cos s + a_1a_2(1-\alpha)\cos^2 s,$$

$$y_\alpha(s,t) = y_1y_2 + (y_1b_2 + y_2b_1)\sqrt{1-\alpha}\sin s\cos t + b_1b_2(1-\alpha)\sin^2 s\cos^2 t$$

and

$$z_\alpha(s,t) = z_1z_2 + (z_1c_2 + z_2c_1)\sqrt{1-\alpha}\sin s\sin t + c_1c_2(1-\alpha)\sin^2 s\sin^2 t.$$

(4) $(A(/)_p B)^\alpha = \{(x_\alpha(s), y_\alpha(s,t), z_\alpha(s,t)) \in \mathbb{R}^3 \mid 0 \le s \le 2\pi, 0 \le t \le \frac{\pi}{2}\}$, *where*

$$x_\alpha(s) = \frac{x_1 + a_1\sqrt{1-\alpha}\cos s}{x_2 - a_2\sqrt{1-\alpha}\cos s}, \qquad y_\alpha(s,t) = \frac{y_1 + b_1\sqrt{1-\alpha}\sin s\cos t}{y_2 - b_2\sqrt{1-\alpha}\sin s\cos t},$$

and

$$z_\alpha(s,t) = \frac{z_1 + c_1\sqrt{1-\alpha}\sin s\sin t}{z_2 - c_2\sqrt{1-\alpha}\sin s\sin t}.$$

Thus $A(+)_pB$ *and* $A(-)_pB$ *become 3-dimensional quadratic fuzzy numbers, but* $A(\cdot)_pB$ *and* $A(/)_pB$ *need not to be 3-dimensional quadratic fuzzy numbers.*

Theorem 3.6. *Parametric operations for two 3-dimensional quadratic fuzzy numbers in Definition 3.4 are the generalization of algebraic operation for two quadratic fuzzy numbers on* $\mathbb{R}^2$ *in Definition 2.4.*

The result of the operation on two two-dimensional quadratic fuzzy numbers is proven in Theorem 2.5. Therefore, it is sufficient to show that restricting the three-dimensional result of Theorem 3.6 to two dimensions is consistent with Theorem 2.5.

Proof. The proof of Theorem 3.6 is established by showing that when the result of Theorem 3.5 is restricted to two dimensions, it aligns with Theorem 2.5. Consider the cases where $z_1 = 0$ in the three-dimensional quadratic fuzzy number A and $z_2 = 0$ in B.

Consider two 3-dimensional quadratic fuzzy numbers $A = [a_1,\ x_1,\ b_1,\ y_1,\ c_1,\ 0]^3$ and $B = [a_2,\ x_2,\ b_2,\ y_2,\ c_2,\ 0]^3$. By Theorem 3.5,

(1) $A(+)_pB = [a_1 + a_2,\ x_1 + x_2,\ b_1 + b_2,\ y_1 + y_2,\ c_1 + c_2,\ 0]^3$

(2) $A(-)_pB = [a_1 + a_2,\ x_1 - x_2,\ b_1 + b_2,\ y_1 - y_2,\ c_1 + c_2,\ 0]^3$

(3) $(A(\cdot)_pB)^\alpha = \{(x_\alpha(s), y_\alpha(s,t), z_\alpha(s,t)) \in \mathbb{R}^3 \mid 0 \le s \le 2\pi, 0 \le t \le \frac{\pi}{2}\}$, where

$$x_\alpha(s) = x_1x_2 + (x_1a_2 + x_2a_1)\sqrt{1-\alpha}\cos s + a_1a_2(1-\alpha)\cos^2 s,$$

$$y_\alpha(s,t) = y_1y_2 + (y_1b_2 + y_2b_1)\sqrt{1-\alpha}\sin s\cos t + b_1b_2(1-\alpha)\sin^2 s\cos^2 t$$

and

$$z_\alpha(s,t) = c_1c_2(1-\alpha)\sin^2 s\sin^2 t.$$

(4) $(A(/)_pB)^\alpha = \{(x_\alpha(s), y_\alpha(s,t), z_\alpha(s,t)) \in \mathbb{R}^3 \mid 0 \leq s \leq 2\pi, 0 \leq t \leq \frac{\pi}{2}\}$, where

$$x_\alpha(s) = \frac{x_1 + a_1\sqrt{1-\alpha}\cos s}{x_2 - a_2\sqrt{1-\alpha}\cos s}, \quad y_\alpha(s,t) = \frac{y_1 + b_1\sqrt{1-\alpha}\sin s\cos t}{y_2 - b_2\sqrt{1-\alpha}\sin s\cos t},$$

and

$$z_\alpha(s,t) = \frac{c_1\sqrt{1-\alpha}\sin s\sin t}{c_2\sqrt{1-\alpha}\sin s\sin t} = \frac{c_1}{c_2}.$$

The three-dimensional case is defined as the region encompassing the ellipsoid and its interior. Therefore, the intersection with the $z = 0$ plane becomes a two-dimensional quadratic fuzzy number defined within the ellipse and its interior. The result of the operation for two-dimensional quadratic fuzzy numbers was established in Theorem 2.5. Thus, it suffices to show that restricting the result of Theorem 3.6 to two dimensions is consistent with Theorem 2.5.

The intersections of these three-dimensional quadratic fuzzy numbers with the plane $z = 0$ are described as follows.

(1) $A(+)_pB$; Note that

$$\mu_{A(+)_pB}(x,y,z) = 1 - \Big((\frac{x-x_1-x_2}{a_1+a_2})^2 + (\frac{y-y_1-y_2}{b_1+b_2})^2 + (\frac{z}{c_1+c_2})^2 \Big).$$

If $z = 0$ and $\mu_{A(+)_pB}(x,y,z) = 0$,

$$(\frac{x-x_1-x_2}{a_1+a_2})^2 + (\frac{y-y_1-y_2}{b_1+b_2})^2 = 1.$$

Thus the intersection is the 2-dimensional quadratic fuzzy number $C = [a_1+a_2,\ x_1+x_2,\ b_1+b_2,\ y_1+y_2]^2$.

(2) $A(-)_pB$; Note that

$$\mu_{A(-)_pB}(x,y) = 1 - \Big((\frac{x-x_1+x_2}{a_1+a_2})^2 + (\frac{y-y_1+y_2}{b_1+b_2})^2 + (\frac{z}{c_1+c_2})^2 \Big).$$

If $z = 0$ and $\mu_{A(-)_pB}(x,y,z) = 0$,

$$(\frac{x-x_1+x_2}{a_1+a_2})^2 + (\frac{y-y_1+y_2}{b_1+b_2})^2 = 1.$$

Thus the intersection is the 2-dimensional quadratic fuzzy number $D = [a_1+a_2,\ x_1-x_2,\ b_1+b_2,\ y_1-y_2]^2$.

(3) $A(\cdot)_pB$; If $z=0, c_1c_2(1-\alpha)\sin^2 s\sin^2 t=0$. This means $0\leq s\leq 2\pi$ and $t=0$. Thus the intersection is a fuzzy number E on $\mathbb{R}^2$ with α-cut $(A(\cdot)_pB)_\alpha=\{(x_\alpha(s),y_\alpha(s))\in\mathbb{R}^2\mid 0\leq s\leq 2\pi\}$, where

$$x_\alpha(s)=x_1x_2+(x_1a_2+x_2a_1)\sqrt{1-\alpha}\cos s+a_1a_2(1-\alpha)\cos^2 s$$

and

$$y_\alpha(s)=y_1y_2+(y_1b_2+y_2b_1)\sqrt{1-\alpha}\sin s+b_1b_2(1-\alpha)\sin^2 s.$$

(4) $A(/)_pB$; If $z=0, \frac{c_1}{c_2}=0$. Since this case cannot occur, it cannot be specifically stated that three dimensions are an extension of two dimensions. Ideally, the result would be $t=0$, as in case (3). However, in the case of division, there are instances where the result is infinite, such as $1/0$, depending on the value of s, which makes it difficult to calculate or graph. Nevertheless, when we restrict the case from three dimensions to two dimensions and draw the graph (Figure 11), keeping the variable t constant, we observe that it closely resembles the two-dimensional graph (Figure 12).
The proof is complete for (1), (2), and (3). □

Example 3.7. [6] Consider $A=[6,\ 3,\ 8,\ 5,\ 4,\ 7]^3$ and $B=[4,\ 2,\ 5,\ 3,\ 6,\ 4]^3$. Subsequently, the following observations hold:

(1) $A(+)_pB=[10,\ 5,\ 13,\ 8,\ 10,\ 11]^3$

(2) $A(-)_pB=[10,\ 1,\ 13,\ 2,\ 10,\ 3]^3$

(3) $(A(\cdot)_pB)^\alpha=\{(x_\alpha(s),y_\alpha(s,t),z_\alpha(s,t))\mid 0\leq s\leq 2\pi,-\frac{\pi}{2}\leq t\leq\frac{\pi}{2}\}$, where

$$x_\alpha(s)=6+24\sqrt{1-\alpha}\cos s+24(1-\alpha)\cos^2 s,$$
$$y_\alpha(s,t)=15+49\sqrt{1-\alpha}\sin s\cos t+40(1-\alpha)\sin^2 s\cos^2 t,$$

and $$z_\alpha(s,t)=28+58\sqrt{1-\alpha}\sin s\sin t+24(1-\alpha)\sin^2 s\sin^2 t.$$

(4) $(A(/)_pB)^\alpha=\{(x_\alpha(s),y_\alpha(s,t),z_\alpha(s,t))\mid 0\leq s\leq 2\pi,-\frac{\pi}{2}\leq t\leq\frac{\pi}{2}\}$, where

$$x_\alpha(s)=\frac{3+6\sqrt{1-\alpha}\cos s}{2-4\sqrt{1-\alpha}\cos s}, y_\alpha(s,t)=\frac{5+8\sqrt{1-\alpha}\sin s\cos t}{3-5\sqrt{1-\alpha}\sin s\cos t}, z_\alpha(s,t)=\frac{7+4\sqrt{1-\alpha}\sin s\sin t}{4-6\sqrt{1-\alpha}\sin s\sin t}.$$

Thus $A(+)_pB$ and $A(-)_pB$ become 3-dimensional quadratic fuzzy numbers, but $A(\cdot)_pB$ and $A(/)_pB$ are not 3-dimensional quadratic fuzzy numbers.

4. Overall flow chart through example graph

In Chapter 2, since it involves a two-dimensional fuzzy set, the graph is represented in three dimensions. In Chapter 3, however, since it involves a three-dimensional fuzzy set, it is more complex to represent the graph in three dimensions. Therefore, the fuzzy set is depicted as a graph in three dimensions, with the membership function value corresponding to each element of the domain represented by the color intensity at that point. Since the membership function values range from 0 to 1, a bar graph illustrating the function values through color intensity is provided.

We demonstrate that the result of Example 3.7 is a dimensional extension of the result from Example 2.6. We verify the proof in Theorem 3.6 with a graph. The intersection of the result of the operator $A(+)B$ in three dimensions with the plane $z = 11$ forms a two-dimensional fuzzy set. Since the intersection is two-dimensional, the graph can be represented in three dimensions. This graph matches the one presented in Chapter 2.

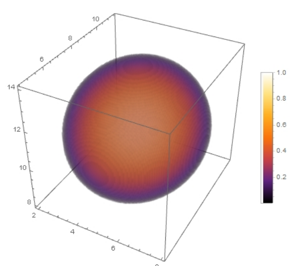


Figure 9. $(A(+)B)^3$

$\cap$ plane $z = 11 \Rightarrow$

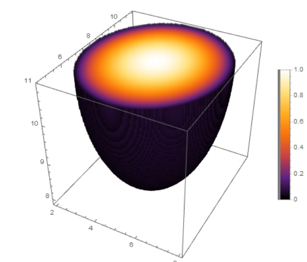


Figure 10. $(A(+)B)^3/2$

3 dim'al graph(domain: plane $z = 11$) $\Rightarrow$

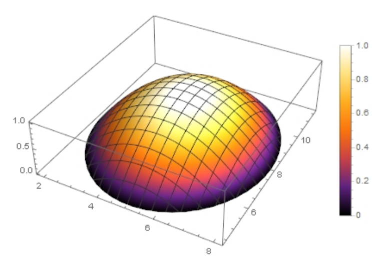
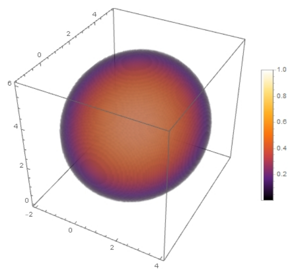


Figure 11. 3-dim'al graph

The intersection of the result of the operator $A(-)B$ in three dimensions with the plane $z = 3$ forms a two-dimensional fuzzy set. Since the intersection is two-dimensional, the graph can be represented in three dimensions. This graph matches the one in Chapter 2.



$\cap$ plane $z = 3 \Rightarrow$

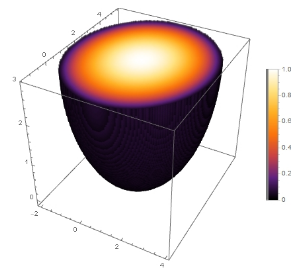


Figure 12. $(A(-)B)^3$

Figure 13. $(A(-)B)^3/2$

3 dim'al graph(domain: plane $z = 3$) $\Rightarrow$

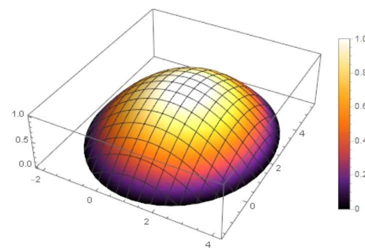
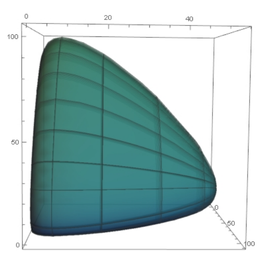


Figure 14. 3-dim'al graph



when $\alpha = 0.01 \Rightarrow$

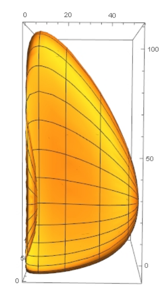


Figure 15. $(A(\cdot)B)^3$

Figure 16. $(A(\cdot)B)^3,\ \alpha = 0.01$

2 dim'al graph($t = 0$) $\Rightarrow$

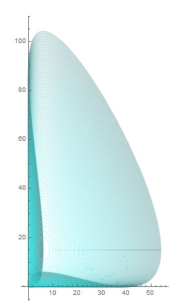


Figure 17. 2-dim'al graph

The graph of the operator $A(\cdot)B$ in three dimensions, along with the graph for $\alpha = 0.01$, is presented. The graph for $\alpha = 0.01$ represents the three-dimensional set where $\alpha = 0.01$. When $t = 0$, it becomes a two-dimensional graph, and the two-dimensional expansion can be inferred by comparing the two graphs.

Remark 4.1. $A(+)B$ and $A(-)B$ are sliced at $z = 11$ and $z = 3$ to aid understanding, but the same result is obtained even when cut by any plane that has a non-empty intersection. Let us consider a two-dimensional fuzzy set whose domain is the non-empty intersection set, as described earlier. This fuzzy set is identical to the result of applying the operator to the two-dimensional fuzzy set discussed in Chapter 2.

(Figure 1)

```
Plot3D[1 - ((x - 3)^2/6 + (y - 5)^2/8), {x, y} \[Element] Ellipsoid[{3, 5},
{Sqrt[6], Sqrt[8]}], PlotPoints -> 50, ColorFunction -> "SunsetColors",
BoxRatios -> {Sqrt[6], Sqrt[8], 1}, PlotLegends -> Automatic]
```

(Figure 3)

```
Plot3D[1 - ((x - 5)^2/10 + (y - 8)^2/13), {x, y} \[Element] Ellipsoid[{5, 8},
{Sqrt[10], Sqrt[13]}], PlotPoints -> 50, ColorFunction -> "SunsetColors",
BoxRatios -> {Sqrt[10], Sqrt[13], 1},PlotLegends -> Automatic]
```

(Figure 5)

```
reg1 = ImplicitRegion[0 <= (x - 3)^2/6 + (y - 5)^2/8 <= 1 && x <= 3, {x, y}];
Plot3D[1 - ((x - 3)^2/6 + (y - 5)^2/8), {x, y}\[Element] reg1,PlotPoints -> 50,
ColorFunction -> "SunsetColors",BoxRatios -> {Sqrt[6], Sqrt[8], 1},
PlotLegends -> Automatic]
```

(Figure 7)

```
reg1 = ImplicitRegion[0 <= (x - 5)^2/10 + (y - 8)^2/13 <= 1 && x <= 5, {x, y}];
Plot3D[1 - ((x - 5)^2/10 + (y - 8)^2/13), {x, y} \[Element] reg1, PlotPoints
-> 50, ColorFunction -> "SunsetColors", BoxRatios -> {Sqrt[6], Sqrt[8], 1},
PlotLegends -> Automatic]
```

(Figure 9)

```
DensityPlot3D[1 -((x - 5)^2/10 + (y - 8)^2/13 + (z - 11)^2/10), {x, y, z}
\[Element]Ellipsoid[{5, 8, 11}, {Sqrt[10], Sqrt[13], Sqrt[10]}], PlotPoints ->
100, ColorFunction -> "SunsetColors",  OpacityFunction -> 0.05, BoxRatios ->
{Sqrt[10],Sqrt[13],Sqrt[10]},PlotLegends -> Automatic]
```

(Figure 10)

```
reg1 = ImplicitRegion[0 <= ((x - 5)^2/10 + (y - 8)^2/13 + (z - 11)^2/10) <= 1
&& z <= 11, {x, y, z}];
```

```
DensityPlot3D[1 -((x - 5)^2/10 + (y - 8)^2/13 + (z - 11)^2/10), {x, y, z}
\[Element] reg1,PlotPoints -> 100, ColorFunction ->"SunsetColors", Opacity
Function -> 1, BoxRatios -> {Sqrt[10],Sqrt[13], Sqrt[10]}, PlotLegends ->
Automatic]
```

(Figure 12)

```
DensityPlot3D[1 -((x - 1)^2/10 + (y - 2)^2/13 + (z - 3)^2/10),{x, y, z}
\[Element] Ellipsoid[{1, 2, 3}, {Sqrt[10], Sqrt[13], Sqrt[10]}],PlotPoints ->
100, ColorFunction -> "SunsetColors", OpacityFunction -> 0.05, BoxRatios ->
{Sqrt[10], Sqrt[13], Sqrt[10]}, PlotLegends -> Automatic]
```

(Figure 13)

```
reg1 = ImplicitRegion[0 <= ((x - 1)^2/10 + (y - 2)^2/13 + (z - 3)^2/10) <= 1
&& z <= 3, {x, y, z}];
DensityPlot3D[1 -((x - 1)^2/10 + (y - 2)^2/13 + (z - 3)^2/10),{x, y, z}
\[Element] reg1, PlotPoints -> 100, ColorFunction -> "SunsetColors",
OpacityFunction -> 1, BoxRatios -> {Sqrt[10], Sqrt[13], Sqrt[10]},
PlotLegends -> Automatic]
```

(Figure 15)

```
g[a_] := ParametricPlot3D[{6 + 24 Sqrt[1 - a] Cos[s] + 24 (1 - a)(Cos[s])^2,
15 + 49 Sqrt[1 - a]  Sin[s] Cos[t] + 40 (1 - a)(Sin[s])^2 (Cos[t])^2, 28 +
58 Sqrt[1 - a]  Sin[s] Sin[t] + 24(1 - a) (Sin[s])^2 (Sin[t])^2}, {s, 0, 2
Pi}, {t, -Pi/2, Pi/2}, PlotStyle -> Directive[RGBColor[0.2, 0.5 + a/2,
0.5 + a/2], Opacity[0.3]], BoxRatios -> {1, 1, 1}];
tg = Table[g[i], {i, 0, 1.0, 0.01}]; Show[tg]
```

(Figure 16)

```
ParametricPlot3D[{6 + 24 Sqrt[1 - 0.01] Cos[s] + 24 (1 - 0.01)(Cos[s])^2,
15 + 49 Sqrt[1 - 0.01]  Sin[s] Cos[t] + 40 (1 - 0.01) (Sin[s])^2
(Cos[t])^2, 28 + 58 Sqrt[1 - 0.01] Sin[s] Sin[t] + 24 (1 - 0.01)
(Sin[s])^2 (Sin[t])^2}, {s, 0, 2 Pi}, {t,-Pi/2, Pi/2}, Axes -> True]
```

(Figure 17)

```
g[a_] := ParametricPlot[{6 + 24 Sqrt[1 - a] Cos[s] + 24(1 - a)(Cos[s])^2,
15 + 49 Sqrt[1 - a]  Sin[s] + 40 (1 - a) (Sin[s])^2}, {s, 0, 2 Pi},
```

```
PlotStyle -> Directive[RGBColor[0.2,0.5 + a/2, 0.5 + a/2], Opacity[0.3]]];
tg =Table[g[i], {i, 0, 1.0, 0.01}]; Show[tg]
```

5. Conclusion

In Chapter 2, we generalized the quadratic fuzzy number to two dimensions and presented the results of a study on extended real operations [5]. We also included graphs illustrating the example results in this paper. In one-dimensional space, the operator was defined using the real operation of the alpha cut, but in two-dimensional space, the alpha cut was defined as a parametric operator. Nevertheless, the results proved that it is an extended concept of one-dimensional space [8]. The graphs of the examples demonstrate that $A(+)B$ and $A(-)B$ preserve the form of the quadratic fuzzy number well, while the other operations exhibit a more complex structure. Representing these operations as graphs makes them significantly easier to understand and apply.

In Chapter 3, we generalized the quadratic fuzzy number to three-dimen-sional space and presented the results for the extended real number operator [6]. The graph of an example result was presented in a previous paper. In the case of three dimensions, since alpha cuts are subsets of three-dimensional space, operators must be defined differently from those in two-dimensional space. It was proven that restricting the results in three dimensions to two dimensions aligns with the results in two dimensions. Since the membership function is defined on three dimensions, a four-dimensional space is required to fully represent it as a graph. However, for each point in the three-dimensional domain, the membership function value can be visualized in three-dimensional space by representing it as color intensity. This is feasible because the membership function values lie between 0 and 1. Nevertheless, in the actual graph, only the surface is visible, making it impossible to discern the values at internal points. The function values for internal points can be determined by slicing the graph at specific planes.

In Chapter 4, we presented a graph sliced to reveal the internal values clearly. Additionally, it was explained in the graph that this was a two-dimensional generalization. Figure 9 shows the result of $A(+)B$ in three-dimensional space. The intersection with the plane $z = 11$ reduces it to two dimensions, as depicted in Figure 10. Since a graph defined in two-dimensional space can represent function values on the z-axis in a more intuitive manner, Figure 11 illustrates these function values using color intensity. We proved that Figure 11 coincides with the two-dimensional result, demonstrating that the three-dimensional result is ultimately a generalization of the two-dimensional case. While the three-dimensional graph of the result may be difficult to interpret directly, its representation as a color-coded graph makes it easier to understand and apply. Even in three dimensions, $A(+)B$ and $A(-)B$ retain the structure of a three-dimensional quadratic fuzzy number, while the results of other operations exhibit more complex forms.

This dimensional expansion is expected to advance research in fuzzy ranking, quadratic fuzzy regression, optimality conditions, and quadratic programming in the future [10, 11]. Thanks to the well-structured and unique nature of $A(+)B$ and $A(-)B$, they can be applied across various fields without modification. Furthermore, modifications to the forms of $A(\cdot)B$ and $A(/)B$ enable their use in diverse applications. This paper contributes to the development of applications involving quadratic fuzzy numbers by expanding their dimensions. These applications include solving fuzzy multi-objective optimization problems, quadratic fuzzy equations, and least squares algorithms [12, 13, 14], with additional potential applications expected across various fields. Above all, this work lays the foundation for future research extending into four dimensions.

Acknowledgments. This research was supported by the 2025 scientific promotion program funded by Jeju National University (Jin Won Park).

Authors' Contributions. All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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