Asia Pac. J. Math. 2025 **12**:31

FRACTIONAL BLACK-SCHOLES OPERATOR: A SOLUTION TO CERTAIN ANOMALIES OF THE CLASSICAL MODEL

M. ACHEHBOUNE

Department of Mathematics, Ibn Zohr University Agadir, Morocco

mustapha.achehboune@edu.uiz.ac.ma

Received Jan. 31, 2025

Abstract. This article aims to study the specific properties of the classical Black-Scholes operator in Banach and Hilbert spaces, with the objective of understanding the potential pathological behaviors of this operator, widely used in financial option pricing models. We suggest a remedy to overcome such anomalies by invoking the fractional Black-Scholes operator.

2020 Mathematics Subject Classification. 35B20; 35Q91.

Key words and phrases. Partial differential equations; Black-Scholes operator; Spectral Analysis; Stability; Perturbations; Fractional differential equations.

1. Introduction

The Black-Scholes equation is a partial differential equation (PDE) used to model the evolution of the price of a financial option (see [5] [19]). The basic equation is given by:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{1}$$

where:

— $V = V(S,t)$ represents the price of the option as a function of the price of the underlying asset S and time t.

— σ is the volatility of the asset, and r is the risk-free interest rate.

First, we will proceed to the reformulation of this operator by recalling certain known results of functional analysis, and applying them to the Black-Scholes operator (see [24] for further details).

The next section will be devoted to the transformation of this operator using the Feynman-Kac similarity transformation and to draw some useful results in finance [11].

The third section will focus on the analysis of the impact of perturbations on the coefficients of the

DOI: 10.28924/APJM/12-31

classical Black-Scholes operator and their effect on the resolution of the Black-Scholes equation (1) (see [16]). The section normally ends with an implementation of the results obtained.
In the final section, we will introduce the fractional Black-Scholes equation, outline a method for solving it, present an implementation of the results achieved, explore some applications of the fractional Black-Scholes operator in the field of finance, and conclude the discussion.

2. Reformulation of the Black-Scholes Equation into Operator

The Black-Scholes equation (1) can be reformulated in terms of a differential operator.
By setting $u(x,\tau) = V(S,t)$ with $x = \log(S)$ and $\tau = T - t$ (where T is the maturity of option), we obtain an equation of the following form:

$$\frac{\partial u}{\partial \tau} = \mathcal{L}u, \tag{2}$$

where $\mathcal{L}$ is an elliptic differential operator, which can be expressed as:

$$\mathcal{L}u = \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial u}{\partial x} - ru.$$

The continuity of an operator means that small variations in the input leads to small variations in the output, which can translate to the stability of solutions to partial differential equations (PDEs). Moreover, the compactness of an operator implies that bounded sequences of solutions have convergent subsequences. In finance, this could mean that if the volatility of the market σ increases slightly, the continuity of the operator $\mathcal{L}$ ensures that the change in the price of the option will induce a proportionally small change in the value of the option. This fact is crucial for reliable risk management strategies. Additionally, compactness ensures that the solutions of the perturbed equation remain within an appropriate functional space and do not diverge, even under varying market conditions.

2.1. Continuity and Compactness of the Black-Scholes operator:

The operator $\mathcal{L}$ can be analyzed on an appropriate spaces like the Sobolev space $H^2(\mathbb{R})$ (see [22]). Analysis spectral in this space allows us to study, among other things, the properties of continuity and compactness of $\mathcal{L}$.

2.1.1. *Study on the Sobolev Space $H^2(\mathbb{R})$:*

It is commonly acknowledged that:
$H^2(\mathbb{R}) := \left\{u \in L^2(\mathbb{R}) \mid u, u', u'' \in L^2(\mathbb{R})\right\}$ endowed with the standard norm

$$\|u\|_{H^2} := \left(\|u\|^2_{L^2} + \|u'\|^2_{L^2} + \|u''\|^2_{L^2}\right)^{1/2}$$

is a Sobolev space.

a) Continuity:

To show that $\mathcal{L}$ is a continuous operator on $H^2(\mathbb{R})$, it is sufficient enough to verify that $\mathcal{L}$ is an bounded

operator.
Since $u \in H^2(\mathbb{R})$, the derivatives u' and u'' are in $L^2(\mathbb{R})$, we have:

$$\|\mathcal{L}u\|_{L^2} \leq C \left(\|u\|_{L^2} + \left\|u'\right\|_{L^2} + \left\|u''\right\|_{L^2}\right) \leq C'\|u\|_{H^2}$$

Where C and C' are strictly positive constants.

b)Compactness:

To prove that $\mathcal{L}$ is compact on $H^2(\mathbb{R})$, we must show that $\mathcal{L}$ sends bounded sets of $H^2(\mathbb{R})$ on relatively compact sets of $L^2(\mathbb{R})$.
A second-order differential operator with constant coefficients (like $\mathcal{L}$) is compact due to the compact inclusion of Sobolev spaces:

$$H^2(\mathbb{R}) \hookrightarrow L^2(\mathbb{R}).$$

This inclusion implies that any bounded set in $H^2(\mathbb{R})$ is relatively compact in $L^2(\mathbb{R})$.

2.2. **Similarity transformations:** In this section we will make the Black-Scholes operator $\mathcal{L}$ undergo similarity transformations to study its spectrum and demonstrate that certain transformations make this operator more easily analyzed.

A commonly used transformation in this context is the Feynman-Kac transformation (For more details see [10]), which links the Black-Scholes partial differential equation (2) to a more conventional diffusion equation.
Below is a detailed illustration of this approach:

The Black-Scholes operator in its transformed variables is given by:

$$\mathcal{L}u = \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial x^2} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial u}{\partial x} - ru$$

where $u = u(x,\tau), x = \log(S)$, et $\tau = T - t$
by considering a function $v(x,\tau)$ defined by:

$$u(x,\tau) = e^{\alpha x + \beta\tau}v(x,\tau)$$

where α and β are constants to be determined to simplify the operator $\mathcal{L}$. We will choose them in such a way as to eliminate the linear and constant terms. By substituting u into the Black-Scholes equation (2), we obtain:

$$\mathcal{L}\left(v\right) = \frac{1}{2}\sigma^2 v_{xx} + \left(\left(r - \frac{1}{2}\sigma^2\right) + \sigma^2\alpha\right)v_x + \left(\frac{1}{2}\sigma^2\alpha^2 - \beta + \left(r - \frac{1}{2}\sigma^2\right)\alpha - r\right)v. \tag{3}$$

To simplify this expression, we choose:

$$\alpha = -\frac{\left(r - \frac{1}{2}\sigma^2\right)}{\sigma^2} \quad \text{and} \quad \beta = -\frac{\left(r - \frac{1}{2}\sigma^2\right)^2}{2\sigma^2} - r \tag{4}$$

which gives:

$$u(x,\tau) = e^{-\frac{r-\frac{1}{2}\sigma^2}{\sigma^2}x-\left(\frac{\left(r-\frac{1}{2}\sigma^2\right)^2}{2\sigma^2}+r\right)\tau} v(x,\tau).$$

Substitute these values into the equation (3) we have:

$$\mathcal{L}v = \frac{1}{2}\sigma^2 v_{xx}$$

Thus, the operator $\mathcal{L}$ becomes:

$$\mathcal{L} = \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2},$$

which is simply the standard broadcast operator.

2.3. **Spectral Analysis.** The operator $\frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}$ is significantly easier to analyze. It is a well-documented elliptic differential operator (see [3])

-The spectrum of $\frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}$ on $L^2(\mathbb{R})$ is continuous and covers the interval $[0,\infty)$.

-The eigenfunctions of this operator are of the form e^{ikx} for $k \in \mathbb{R}$, with the associated eigenvalues $\lambda = \frac{1}{2}\sigma^2 k^2$.

It is worth mentioning that by using the Feynman-Kac transformation, we have simplified the Black-Scholes operator $\mathcal{L}$ into a standard diffusion operator. This simplification allows more direct and explicit spectral analysis and facilitates the understanding of the properties of $\mathcal{L}$. This approach demonstrates how similarity transformations can make operators more easily analyzable by converting them into simpler and better-understood forms.

2.4. **Spectral decomposition:** As detailed in [9], by employing the spectral theory of compact operators, we can decompose $\mathcal{L}$ in terms of its eigenvalues and eigenvectors. This decomposition permits us to better understand how the operator acts on different components of the Hilbert space.

Approximating $\mathcal{L}$ using finite rank operators enables us to control and simplify the analysis of its behavior, thereby mitigating potential pathological phenomena. To apply the spectral theory of compact operators to the Black-Scholes operator $\mathcal{L}$, we will undertake several key steps.

As seen before, using the Feynman-Kac transformation, the Black-Scholes operator becomes:

$$\mathcal{L} = \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2},$$

To apply spectral theory, we must restrict our domain to a finite interval, for example $[a,b]$, and impose border conditions.

Consider the Hilbert space $L^2([a,b])$ with the dot product:

$$\langle u, v\rangle_{L^2} = \int_a^b u(x)\overline{v(x)}dx$$

In this space, we consider the restriction of $\mathcal{L}$ with boundary conditions as done in [17], for example, Dirichlet conditions

$$u(a) = u(b) = 0. \tag{5}$$

It is well known that $\mathcal{L}$ is a self-adjoint differential operator with a discrete spectrum. The eigenfunctions $\phi_n(x)$ and the eigenvalues λ_n of $\mathcal{L}$ with the Dirichlet boundary conditions (5), are:

$$\phi_n(x) = \sqrt{\frac{2}{b-a}}\sin\left(\frac{n\pi(x-a)}{b-a}\right),$$

and

$$\lambda_n = \frac{1}{2}\sigma^2\left(\frac{n\pi}{b-a}\right)^2.$$

where $n \in \mathbb{N}$.

Thanks to this spectral decomposition, we can express a function $v \in L^2([a,b])$ as a series of its proper components:

$$v(x) = \sum_{n=1}^{\infty}\langle v, \phi_n\rangle_{L^2}\phi_n(x).$$

The action of the operator $\mathcal{L}$ on v is then:

$$\mathcal{L}v = \sum_{n=1}^{\infty}\lambda_n\langle v, \phi_n\rangle_{L^2}\phi_n(x)$$

- The eigenvalues λ_n determine the "strength" with which each eigencomponent is affected by the operator $\mathcal{L}$.

- The eigenfunctions $(\phi_n)_{n\in\mathbb{N}}$ form an orthonormal basis of the space $L^2([a,b])$. This allow us to represent any function $v \in L^2([a,b])$ in a decomposed manner.

- The operator $\mathcal{L}$ multiplies each eigencomponent by its corresponding eigenvalue, thus simplifying the analysis of its action.

When imposing Dirichlet boundary conditions: $v(a,\tau) = v(b,\tau) = 0$ we have:

$$\phi_n(x) = \sqrt{\frac{2}{b-a}}\sin\left(\frac{n\pi(x-a)}{b-a}\right), \text{ and } \lambda_n = \frac{1}{2}\sigma^2\left(\frac{n\pi}{b-a}\right)^2.$$

The solution $v(x,\tau)$ can be expressed as a series of these eigenfunctions:

$$v(x,\tau) = \sum_{n=1}^{\infty}c_n(\tau)\phi_n(x)$$

where $c_n(\tau)$ are the coefficients to be determined.

Let's substitute the series into the simplified equation:

$$\frac{\partial v}{\partial \tau} = \mathcal{L}v = \frac{1}{2}\sigma^2 \frac{\partial^2 v}{\partial x^2}.$$

This gives:

$$\sum_{n=1}^{\infty} \frac{\partial c_n(\tau)}{\partial \tau}\phi_n(x) = \sum_{n=1}^{\infty} \lambda_n c_n(\tau)\phi_n(x)$$

By equalizing the coefficients of the terms $\phi_n(x)$, we obtain an ordinary differential equation for each coefficient $c_n(\tau)$:

$$\frac{\partial c_n(\tau)}{\partial \tau} = \lambda_n c_n(\tau)$$

we then find:

$$c_n(\tau) = c_n(0)e^{\lambda_n \tau},$$

where $c_n(0)$ are determined by the initial conditions.

The initial conditions are given by the initial value of the option, for example:

$$u(S,0) = g(S)$$

In terms of v, this gives:

$$v(x,0) = e^{-\alpha x} g\left(e^x\right)$$

We project $v(x,0)$ onto the eigenfunctions $\phi_n(x)$ to find $c_n(0)$:

$$c_n(0) = \int_a^b v(x,0)\phi_n(x)dx$$

So the final solution is:

$$v(x,\tau) = \sum_{n=1}^{\infty} c_n(0)e^{\lambda_n \tau}\phi_n(x)$$

and returning to the original variables:

$$u(S,t) = e^{\alpha x + \beta \tau} \sum_{n=1}^{\infty} c_n(0)e^{\lambda_n \tau}\phi_n(x),$$

with $x = \log(S)$, $\tau = T - t$, the values of α and β are given by formulas (4) above.

2.5. **Applications in Finance:** step by step:

Let's take a numerical example with fictitious values to illustrate:

- Strike price $K = 100$
- Volatility $\sigma = 0.2$
- Risk-free rate $r = 0.05$

- Maturity $T = 1$ year
- Interval $[a, b] = [\log(50), \log(150)]$

1. Calculation of Eigenfunctions:

$$\phi_n(x) = \sqrt{\frac{2}{b-a}} \sin\left(\frac{n\pi(x-a)}{b-a}\right)$$

2. Initial Conditions:

$$v(x, 0) = e^{-\alpha x} \max\left(e^x - 100, 0\right)$$

$$c_n(0) = \int_a^b v(x, 0)\phi_n(x)dx$$

3. Temporal evolution:

$$c_n(\tau) = c_n(0)e^{\lambda_n \tau}$$

4. Option Price:

$$u(S, t) = e^{\alpha x} \sum_{n=1}^{\infty} c_n(0)e^{\lambda_n \tau}\phi_n(x)$$

This method makes it possible to decompose and solve the Black-Scholes equation in an analytical manner, offering a detailed understanding of the impact of the different parameters on the price of the option (see [21]).

Remark: The approach described for solving the classical Black-Scholes equation using spectral decomposition is based on well-established concepts in applied mathematics, in particularly those related to partial differential equations (PDE), functional analysis, and the spectral theory of compact operators (for further investigations, see [8]).

3. Stability and Perturbations

In [1], we previously examined the impact of a perturbation of the Black-Scholes operator on the solution of the classical Black-Scholes equation. In the present work, we will focus on studying a perturbation applied specifically to the parameters σ (volatility) and r (risk-free interest rate), with the aim of thoroughly analyzing their influence on the solutions of this equation.

Studying the stability of $\mathcal{L}$ under perturbations helps us understand how small changes in parameters (such as σ or r) impact the solution (for reference, see [15]). This analysis is crucial for practical applications in finance where parameters may vary.

Returning to the Black-Scholes operator $\mathcal{L}$ for a European option which is given by:

$$\mathcal{L}u = \frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 u}{\partial S^2} + rS\frac{\partial u}{\partial S} - ru$$

and suppose that the parameters σ and r undergo small disturbances $\delta\sigma$ and δr. The new parameters become $\sigma' = \sigma + \delta\sigma$ and $r' = r + \delta r$.

The disturbed operator $\mathcal{L}'$ is then written:

$$\mathcal{L}'u = \frac{\partial u}{\partial t} + \frac{1}{2}(\sigma + \delta\sigma)^2 S^2 \frac{\partial^2 u}{\partial S^2} + (r + \delta r) S \frac{\partial u}{\partial S} - (r + \delta r) u$$

Let's develop $(\sigma + \delta\sigma)^2$:

$$(\sigma + \delta\sigma)^2 = \sigma^2 + 2\sigma\delta\sigma + (\delta\sigma)^2$$

Neglecting the higher order terms, we obtain:

$$\mathcal{L}'u \approx \mathcal{L}u + \delta\mathcal{L}u$$

where:

$$\delta\mathcal{L}u = \left(\sigma\delta\sigma S^2 \frac{\partial^2 u}{\partial S^2} + \delta r S \frac{\partial u}{\partial S} - \delta r u \right)$$

We seek a solution in the form:

$$u' = u + \delta u$$

where u is the solution of the original operator and δu is the correction due to the disturbance. Let's substitute u' into the perturbed operator:

$$\mathcal{L}'(u + \delta u) = 0$$

Neglecting the higher order terms, we obtain:

$$\mathcal{L}u + \mathcal{L}\delta u + \delta\mathcal{L}u = 0$$

As $\mathcal{L}u = 0$, it remains:

$$\mathcal{L}\delta u + \delta\mathcal{L}u = 0$$

3.1. **Numerical Solution.** To solve this numerically, we discretize the spatial variable S and the time variable t as follows:

$$S_{\min} = 0, \quad S_{\max} = 200, \quad T = 1$$

$$dS = 1, \quad dt = 0.01$$

This gives us:

$$N = \frac{S_{\max} - S_{\min}}{dS}, \quad M = \frac{T}{dt}$$

The discrete grids are:

$$S = \{S_{\min}, S_{\min} + dS, \ldots, S_{\max}\}, \quad t = \{0, dt, 2dt, \ldots, T\}$$

We initialize δu:

$$\delta u[:, 0] = 0$$

The finite difference scheme for δu is:

$$\begin{aligned}\delta u[i, j+1] =& \delta u[i, j] + dt\left(-\delta\mathcal{L} + 0.5\sigma^2 S[i]^2 \frac{\delta u[i+1, j] - 2\delta u[i, j] + \delta u[i-1, j]}{dS^2}\right) \\ &+ dt\left(rS[i]\frac{\delta u[i+1, j] - \delta u[i-1, j]}{2dS} - r\delta u[i, j]\right)\end{aligned}$$

Boundary conditions:

$$\delta u[0, :] = 0, \quad \delta u[N, :] = 0$$

The final perturbed solution is:

$$u' = u + \delta u$$

Graphical Representation By implementing the above, we obtain the graph below, titled: Figure 1.

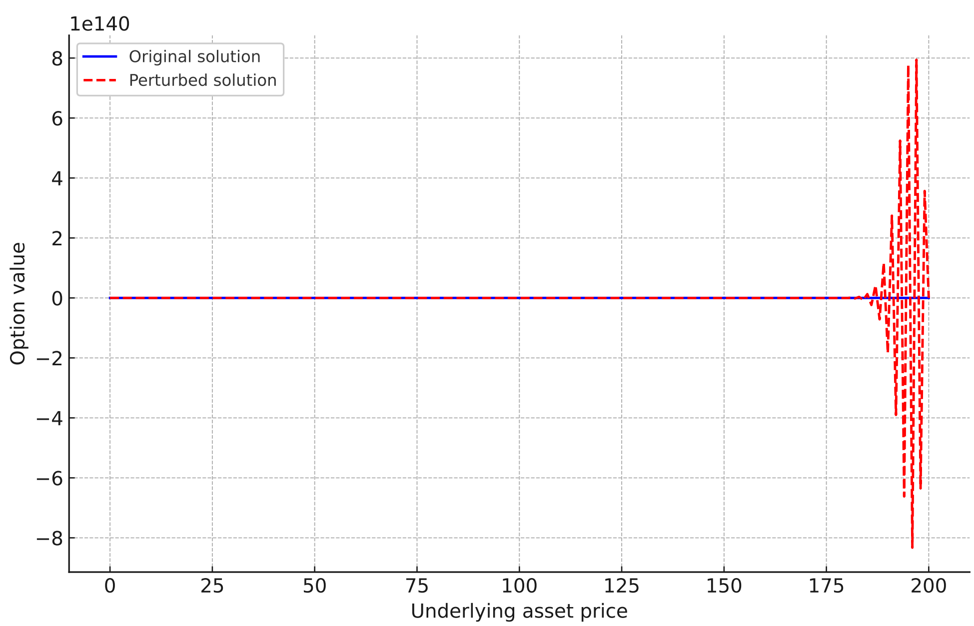

FIGURE 1. Impact of perturbations on the solution of the classical Black-Scholes equation.

3.2. **Interpretation of the graph:** The previous figure shows the impact of perturbations on the solution of the classical Black-Scholes equation.

Original solution (blue curve):

The blue curve shows the value of the option as a function of the underlying asset price without perturbations. This solution is obtained using the initial parameters σ and r.

Perturbed solution (red curve):

The red curve shows the value of the option after applying small perturbations to the parameters σ and r (these perturbations are denoted as $\delta\sigma$ and δr).

We can see that the red curve deviates slightly from the blue curve. This deviation indicates how the option value changes when the parameters σ and r are modified.

The differences between the two curves can be more or less pronounced depending on the magnitude of the perturbations. A small difference indicates that the solution is relatively stable to small perturbations, while a large difference indicates increased sensitivity.

3.2.1. *Applications in finance:* In this subsection, we will provide some applications of the aforementioned concepts in the field of finance.

Risk management:

Understanding how small variations in parameters influence option values allows traders and risk managers to better assess uncertainty and make more informed decisions. For example, if an option is very sensitive to changes in σ (volatility), it may be necessary to implement hedging strategies to protect against unexpected market movements.

Option pricing:

Option pricing models rely on parameters such as volatility σ and the interest rate r. Knowing how these parameters affect option values helps to adjust prices more accurately and predict potential variations.

Investors can use this information to adjust their portfolios and optimize their investment strategies by considering the sensitivity of options to perturbations.

4. Fractional Black-Scholes Model

Although the Black-Scholes model is a fundamental tool in finance for pricing options, it has several notable drawbacks (see [12], [14], [6], [20] [2]):

The model assumes that the volatility of the underlying asset is constant over the life of the option. In reality, volatility can vary significantly depending on market conditions, which can make valuations inaccurate.

The model is limited to European options, which can only be exercised at maturity, which restricts its application to American options.

Black and Scholes postulate that returns follow a normal distribution, neglecting extreme behavior and asymmetries often observed in financial markets.
The model makes the assumption that the risk-free interest rate is constant, when in reality rates can fluctuate, thereby affecting the valuation of options.
The model does not take into account transaction costs, such as brokerage commissions, which can influence the profitability of options strategies.
The model does not take into account irrational investor behavior and market effects, such as speculative bubbles, which can impact asset prices.

The fractional model emerges as a response to the limitations of the classic Black-Scholes model in finance (see [26]), offering a more flexible and realistic approach to options pricing and financial market analysis. For advanced fractional calculus, we refer to the recent papers [7] in a general and depp framework and to [23] for newest results ans statistical treatment of Black-Sholes operator.
In sum, the fractional model represents a significant advance in the understanding and evaluation of options, providing a robust alternative to the limitations of the Black-Scholes model. This approach allows researchers and practitioners to better understand market realities and improve the precision of financial assessments.

4.1. **Fractional Black-Scholes equation.** A typical form of the fractional Black-Scholes equation is:

$$\frac{\partial^\alpha V}{\partial t^\alpha} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \ (\alpha \in \mathbb{R}) \tag{6}$$

where $\frac{\partial^\alpha}{\partial t^\alpha}$ represents the Caputo fractional derivative of order α with respect to the given time by:

$$\frac{\partial^\alpha V}{\partial t^\alpha} = {}^C D_t^\alpha V(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{V^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds$$

Or:

- n is an integer such that $n-1 < \alpha < n$
- Γ is the Gamma function
- $V^{(n)}$ is the n-th ordinary derivative of V

If α holds $0 < \alpha < 1$, Caputo's fractional derivative simplifies to:

$${}^C D_t^\alpha V(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{V'(s)}{(t-s)^\alpha} ds$$

When α is a positive integer, the Caputo fractional derivative of order α of $V(t)$ becomes the ordinary derivative of order α.

We apply time series analysis methods to estimate the order of the fractional derivative α. Techniques such as Hurst exponent or fractional regression methods can be used for this estimation. We use optimization techniques to adjust α in order to minimize the error between the theoretical option prices obtained by the fractional model and the market prices.

In summary, α is chosen to model specific market characteristics that cannot be captured by an ordinary first-order derivative. Empirical analysis and model fits play a key role in determining the appropriate value of α (see [13] [4] [25])

Remark

- Using Caputo's fractional derivative of order α in the Black-Scholes equation helps capture complex features of financial markets, such as long memory effects.
- Applying Caputo's fractional derivative of order α with respect to time in the Black-Scholes equation is often preferred over the Riemann-Liouville fractional derivative for several practical and theoretical reasons [25].

4.2. **Existence and uniqueness of the solution to the fractional equation of B-S.** To demonstrate the existence of the solution to the fractional Black-Scholes equation, we will use the variational method, and to apply it, we need to rewrite the fractional Black-Scholes equation in a suitable form:
We are looking for a function V in a Hilbert space H such that for all $\phi \in H$

$$\left(\frac{\partial^\alpha V}{\partial t^\alpha}, \phi\right) + \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}, \phi\right) + \left(rS\frac{\partial V}{\partial S}, \phi\right) - (rV, \phi) = 0,$$

where $(\cdot, \cdot)$ represents the dot product in H.

Let $J : H \to \mathbb{R}$ defined by:

$$J(V) = \frac{1}{2}\left(\left(\frac{\partial^\alpha V}{\partial t^\alpha}, V\right) + \frac{1}{2}\left(\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}, V\right) + \left(rS\frac{\partial V}{\partial S}, V\right) - (rV, V)\right).$$

This functional must be minimized to find the solution V.

The appropriate Hilbert space H can be $H = H_0^1([0, T])$.

Using a suitable inequality for fractional derivatives, the Poincaré inequality and other functional inequality we have:

$$\left(\frac{\partial^\alpha V}{\partial t^\alpha}, V\right) = \int_0^T \frac{\partial^\alpha V(t)}{\partial t^\alpha}\overline{V(t)}dt \geq c_1\|V\|_H^2$$

$$\frac{1}{2}\left(\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}, V\right) \geq c_2\|V\|_H^2$$

$$\left(rS\frac{\partial V}{\partial S}, V\right) \geq c_3\|V\|_H^2$$

where c_1 is a positive constant which depends on the order α, c_2 and c_3 are positive constants that depend on σ and S.

$$-(rV, V) = -r\|V\|_{L^2}^2.$$

Combining these terms we get:

$$J(V) \geq (c_1 + c_2 + c_3 - r)\,\|V\|_H^2$$

If $C = c_1 + c_2 + c_3 - r$ is positive, then J is coercive.

4.3. **Continuity of J.** Consider the difference $J(V) - J(W)$: We have:

$$|J(V) - J(W)| = \left| \frac{1}{2} \left(\left(\frac{\partial^\alpha V}{\partial t^\alpha} - \frac{\partial^\alpha W}{\partial t^\alpha}, V - W \right) + \frac{1}{2} \left(\sigma^2 S^2 \left(\frac{\partial^2 V}{\partial S^2} - \frac{\partial^2 W}{\partial S^2} \right), V - W \right) + \dots \right) \right|.$$

$$\left| \left(\frac{\partial^\alpha V}{\partial t^\alpha} - \frac{\partial^\alpha W}{\partial t^\alpha}, V - W \right) \right| \leq L_1 \|V - W\|_H^2.$$

$$\left| \left(\sigma^2 S^2 \left(\frac{\partial^2 V}{\partial S^2} - \frac{\partial^2 W}{\partial S^2} \right), V - W \right) \right| \leq L_2 \|V - W\|_H^2$$

$$\left| \left(rS \left(\frac{\partial V}{\partial S} - \frac{\partial W}{\partial S} \right), V - W \right) \right| \leq L_3 \|V - W\|_H^2$$

$$|-(r(V - W), V - W)| = r\|V - W\|_{L^2}^2 \leq L_4 \|V - W\|_H^2$$

Combining these results we obtain:

$$|J(V) - J(W)| \leq (L_1 + L_2 + L_3 + L_4) \|V - W\|_H$$

where $L = L_1 + L_2 + L_3 + L_4$ is a positive constant. Thus, the functional J is continuous. J is coercive and continuous, the Lax-Milgram theorem gives the existence and uniqueness of the solution of the fractional Black-Scholes equation (see [3]).

4.4. **A solving method using fractional Fourier transform (FFT).** Let f be a time function. The fractional Fourier transform of order α is defined by:

$$\mathcal{F}^\alpha\{f(t)\} = F^\alpha(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\alpha\omega t} dt$$

Application to the Fractional Black-Scholes equation [18]

The FFT applied to the fractional Black-Scholes equation with respect to t (6) gives:

$$\mathcal{F}^\alpha \left\{ \frac{\partial^\alpha V}{\partial t^\alpha} \right\} + \mathcal{F}^\alpha \left\{ \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right\} + \mathcal{F}^\alpha \left\{ rS \frac{\partial V}{\partial S} \right\} - \mathcal{F}^\alpha \{rV\} = 0$$

By application of the FFT linearity, we have:

$$(i\omega)^\alpha \hat{V}(S,\omega) + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 \hat{V}(S,\omega)}{\partial S^2} + rS \frac{\partial \hat{V}(S,\omega)}{\partial S} - r\hat{V}(S,\omega) = 0 \tag{7}$$

where $\hat{V}(S,\omega)$ is the FFT of $V(S,t)$.

This last equation (7) is a PDE in $\hat{V}(S,\omega)$ often simpler to solve by numerical or analytical methods to find $\hat{V}(S,\omega\ )$.

We apply the inverse fractional Fourier transformation (IFFT) to obtain $V(S,t)$.

$$V(S,t) = \mathcal{F}^{-\alpha}\{\hat{V}(S,\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{V}(S,\omega) e^{i\alpha\omega t} d\omega$$

Simulated asset price trajectories

Consider a simple numerical example:

- $S_0 = 100$: The initial price of the asset is 100.

- $K = 100$: The exercise price of the option is 100.
- $T = 1.0$: The option expires in 1 year.
- $r = 0.05$: The risk-free rate is 5%.
- $\sigma = 0.2$: The variant is 20%.
- $H = 0.75$: The Hurst exponent is 0.75, indicating long-term memory.

We apply the Monte Carlo method which is a numerical simulation technique widely used in financial models to estimate the value of options and other derivative products. When dealing with fractional models, this method can be adapted to generate price trajectories that follow fractional dynamics.We obtain:

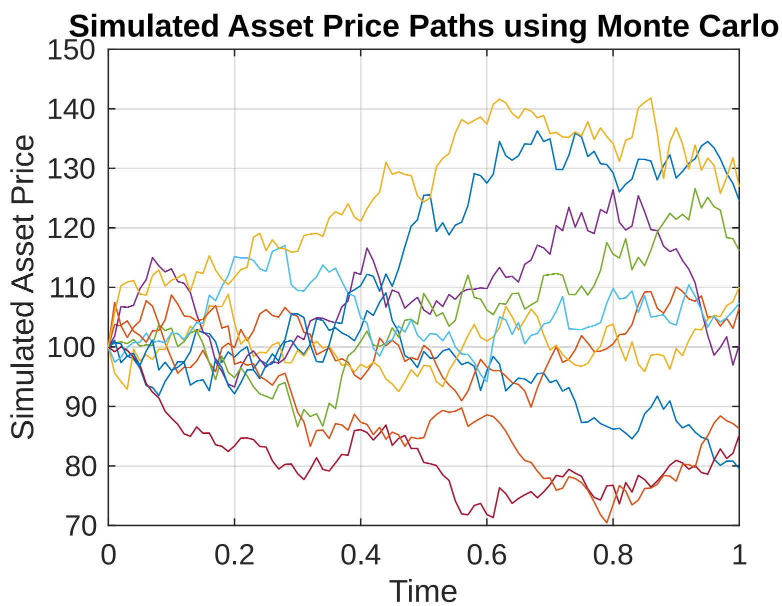

FIGURE 2.

The curves in the figure represent different asset price trajectories simulated using the Fractional Brownian Motion (FBM) model.

Each trajectory shows how the price of an asset changes over time according to specified parameters.

Interpretation

By analyzing the curves generated by the asset price simulation code, several conclusions can be drawn:

1. Price volatility:

The simulated trajectories show significant variability in asset prices over time. This indicates that the market is subject to fluctuations, which is typical of financial assets. The presence of several curves which deviate widely from the others underlines the uncertainty inherent in price movements.

2. Probability of exercising the option:

Given that the strike price K is set at 100, we can observe that certain trajectories exceed this threshold at expiration. This suggests that there is a non-negligible probability that the call option will be exercised, especially if the majority of trajectories are above 100 at the end of the simulation period.

3.The impact of time on option pricing:

The curves show that although the initial price is 100, the trajectories evolve exponentially depending on the opportunity and interest rate parameters. This highlights the importance of time in pricing options, as prices can vary significantly as maturity approaches.

4. Market Scenarios:

The different trajectories illustrate various market scenarios, ranging from upward movements to significant declines. This allows investors to visualize risks and potential opportunities, making strategic decisions easier to make.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] M. Achehboune, A. Sani, O. Elamrania, New Treatment of Multiplicative Perturbation of Black-Scholes Operator, Asia Pac. J. Math. 11 (2024) 92. https://doi.org/10.28924/APJM/11-92.

[2] L.F. Ackert, R. Deaves, Behavioral Finance: Psychology, Decision-Making, and Markets, Cengage Learning, Mason, 2009.

[3] X. An, F. Liu, M. Zheng, V.V. Anh, I.W. Turner, A Space-Time Spectral Method for Time-Fractional Black-Scholes Equation, Appl. Numer. Math. 165 (2021), 152–166. https://doi.org/10.1016/j.apnum.2021.02.009.

[4] R.G. Batogna, A. Atangana, Generalised Class of Time Fractional Black Scholes Equation and Numerical Analysis, Discret. Contin. Dyn. Syst. - S 12 (2019), 435–445. https://doi.org/10.3934/dcdss.2019028.

[5] F. Black, M. Scholes, The Pricing of Options and Corporate Liabilities, J. Polit. Econ. 81 (1973), 637–654. https://doi.org/10.1086/260062.

[6] T. Bollerslev, B. Hood, J. Huss, L.H. Pedersen, Risk Everywhere: Modeling and Managing Volatility, Rev. Financ. Stud. 31 (2018), 2729–2773. https://doi.org/10.1093/rfs/hhy041.

[7] O. El-Mennaoui, V. Keyantuo, A. Sani, Fractional Integration of Imaginary Order in Vector-Valued Hölder Spaces, Arch. Math. 120 (2023), 493–506. https://doi.org/10.1007/s00013-023-01841-6.

[8] J.L. Vázquez, The Mathematical Theories of Diffusion: Nonlinear and Fractional Diffusion, in: M. Bonforte, G. Grillo (Eds.), Nonlocal and Nonlinear Diffusions and Interactions: New Methods and Directions, Springer, Cham, 2017: pp. 205–278. https://doi.org/10.1007/978-3-319-61494-6_5.

[9] Y. Egorov, V. Kondratiev, On Spectral Theory of Elliptic Operators, Birkhäuser, Basel, 2012. https://doi.org/10.1007/978-3-0348-9029-8.

[10] P. Del Moral, Feynman-Kac Formulae, Springer New York, 2004. https://doi.org/10.1007/978-1-4684-9393-1.

[11] R.P. Feynman, The Principle of Least Action in Quantum Mechanics, in: Feynman's - A New Approach to Quantum Theory, World Scientific, 2005: pp. 1-69. https://doi.org/10.1142/9789812567635_0001.

[12] J. Fleming, B. Ostdiek, R.E. Whaley, Trading Costs and the Relative Rates of Price Discovery in Stock, Futures, and Option Markets, J. Futures Mark. 16 (1996), 353–387. https://doi.org/10.1002/(SICI)1096-9934(199606)16:4<353::AID-FUT1>3.0.CO;2-H.

[13] A. Golbabai, O. Nikan, T. Nikazad, Numerical Analysis of Time Fractional Black–Scholes European Option Pricing Model Arising in Financial Market, Comput. Appl. Math. 38 (2019), 173. https://doi.org/10.1007/s40314-019-0957-7.

[14] J. Hull, Options, Futures, and Other Derivatives, Prentice Hall, 2009.

[15] A. Ilhan, M. Jonsson, R. Sircar, Singular Perturbations for Boundary Value Problems Arising from Exotic Options, SIAM J. Appl. Math. 64 (2004), 1268–1293. https://doi.org/10.1137/S0036139902420043.

[16] J.P. Fouque, G. Papanicolaou, K.R. Sircar, Derivatives in Financial Markets with Stochastic Volatility, Cambridge University Press, 2000

[17] R. Kangro, R. Nicolaides, Far Field Boundary Conditions for Black–Scholes Equations, SIAM J. Numer. Anal. 38 (2000), 1357–1368. https://doi.org/10.1137/S0036142999355921.

[18] Y.F. Luchko, H. Martinez, J.J. Trujillo, Fractional Fourier Transform and Some of Its Applications, Fract. Calc. Appl. Anal. 11 (2008), 457-470.

[19] R.C. Merton, Theory of Rational Option Pricing, Bell J. Econ. Manag. Sci. 4 (1973), 141-183. https://doi.org/10.2307/3003143.

[20] M. Mazouni, Portfolio Insurance Strategies for CBOE's Volatility Index (VIX) Futures, SSRN (2017). https://ssrn.com/abstract=3015426.

[21] B. Øksendal, Stochastic Differential Equations, Springer, Berlin, Heidelberg, 2013. https://doi.org/10.1007/978-3-642-14394-6.

[22] J. Oliva-Maza, M. Warma, Introducing and Solving Generalized Black-Scholes PDEs through the Use of Functional Calculus, arXiv:2203.15463 [math.AP] (2022). https://doi.org/10.48550/arXiv.2203.15463.

[23] H. Tiemtoré, R.G. Bagré, Dynamics of Dependency: Finite Difference Solutions to the Black-Scholes PDE with Copulas, Gulf J. Math. 19 (2025), 374–391. https://doi.org/10.56947/gjom.v19i1.2576.

[24] H. Windcliff, P. Forsyth, K. Vetzal, Analysis of the Stability of the Linear Boundary Condition for the Black–Scholes Equation, J. Comput. Finance 8 (2004), 65–92. https://doi.org/10.21314/JCF.2004.116.

[25] H. Zhang, F. Liu, I. Turner, Q. Yang, Numerical Solution of the Time Fractional Black–Scholes Model Governing European Options, Comput. Math. Appl. 71 (2016), 1772–1783. https://doi.org/10.1016/j.camwa.2016.02.007.

[26] H. Zhang, M. Zhang, F. Liu, M. Shen, Review of the Fractional Black-Scholes Equations and Their Solution Techniques, Fractal Fract. 8 (2024), 101. https://doi.org/10.3390/fractalfract8020101.