Asia Pac. J. Math. 2025 **12**:32

CHARACTERIZATIONS OF WEAKLY s-(τ_1, τ_2)-CONTINUOUS FUNCTIONS

NAPASSANAN SRISARAKHAM[1], SUPUNNEE SOMPONG[2], CHAWALIT BOONPOK[1,*]

[1]Mathematics and Applied Mathematics Research Unit, Department of Mathematics, Faculty of Science, Mahasarakham University, Maha Sarakham, 44150, Thailand

[2]Department of Mathematics and Statistics, Faculty of Science and Technology, Sakon Nakhon Rajbhat University, Sakon Nakhon, 47000, Thailand

[*]Corresponding author: chawalit.b@msu.ac.th

Received Nov. 25, 2024

Abstract. This paper deals with the concept of weakly s-(τ_1, τ_2)-continuous functions. Moreover, several characterizations of weakly s-(τ_1, τ_2)-continuous functions are discussed.

2020 Mathematics Subject Classification. 54C08; 54E55.

Key words and phrases. $\tau_1\tau_2$-open set; $\tau_1\tau_2$-closed set; weakly s-(τ_1, τ_2)-continuous function.

1. Introduction

The field of the mathematical science which goes under the name of topology is concerned with all questions directly or indirectly related to continuity. In topology, there has been recently significant interest in characterizing and investigating the characterizations of some weak forms of continuity for functions. As weak forms of continuity in topological spaces, weak continuity [25], quasicontinuity [26], semi-continuity [24] and almost continuity in the sense of Husain [18] are well-known. Lee [23] studied the concept of semiconnected functions. Kohli [21] introduced the notion of s-continuous functions and investigated some characterizations of semilocally connected spaces in terms of s-continuous functions. The notion of s-continuity as a generalization of continuity and semiconnectedness. In [8], the present authors studied some properties of (Λ, sp)-open sets, $r(\Lambda, sp)$-open sets, $s(\Lambda, sp)$-open sets, $p(\Lambda, sp)$-open sets, $\alpha(\Lambda, sp)$-open sets, $\beta(\Lambda, sp)$-open sets and $b(\Lambda, sp)$-open sets. Viriyapong and Boonpok [36] investigated several characterizations of (Λ, sp)-continuous functions by utilizing the notions of (Λ, sp)-open sets and (Λ, sp)-closed sets. Dungthaisong et al. [17] introduced and studied the concept of $g_{(m,n)}$-continuous functions. Duangphui et al. [16] introduced and investigated the notion of $(\mu, \mu')^{(m,n)}$-continuous functions. Furthermore, some characterizations of almost (Λ, p)-continuous functions,

DOI: 10.28924/APJM/12-32

strongly $\theta(\Lambda, p)$-continuous functions, almost strongly $\theta(\Lambda, p)$-continuous functions, $\theta(\Lambda, p)$-continuous functions, weakly (Λ, b)-continuous functions, $\theta(\star)$-precontinuous functions, $\star$-continuous functions, θ-$\mathscr{I}$-continuous functions, almost (g, m)-continuous functions and pairwise weakly M-continuous functions were presented in [33], [34], [1], [29], [6], [7], [5], [10], [12] and [13], respectively. Kohli [20] introduced the concepts of s-regular spaces and completely s-regular spaces and proved that s-regularity and complete s-regularity are preserved under certain s-continuous functions. In [2], the present authors introduced and investigated the concept of (τ_1, τ_2)-continuous functions. Moreover, some characterizations of almost (τ_1, τ_2)-continuous functions, weakly (τ_1, τ_2)-continuous functions, slightly $(\tau_1, \tau_2)s$-continuous functions, slightly (τ_1, τ_2)-continuous functions, $\delta(\tau_1, \tau_2)$-continuous functions, faintly (τ_1, τ_2)-continuous functions, quasi $\theta(\tau_1, \tau_2)$-continuous functions and weakly quasi (τ_1, τ_2)-continuous functions were investigated in [3], [4], [28], [31], [27], [32], [30] and [15], respectively. Kong-ied et al. [22] introduced and studied the notion of almost quasi (τ_1, τ_2)-continuous functions. Khampakdee et al. [19] introduced and investigated the concept of almost weakly (τ_1, τ_2)-continuous functions. Quite recently, Chiangpradit et al. [14] introduced and studied the notion of s-(τ_1, τ_2)-continuous functions. In this paper, we introduce the concept of weakly s-(τ_1, τ_2)-continuous functions. We also investigate several characterizations of weakly s-(τ_1, τ_2)-continuous functions.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [11] if $A = \tau_1$-Cl$(\tau_2$-Cl$(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [11] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [11] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 1. [11] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) $A \subseteq \tau_1\tau_2$-*Cl*(A) *and* $\tau_1\tau_2$-*Cl*$(\tau_1\tau_2$-*Cl*$(A)) = \tau_1\tau_2$-*Cl*(A).
(2) *If* $A \subseteq B$, *then* $\tau_1\tau_2$-*Cl*$(A) \subseteq \tau_1\tau_2$-*Cl*$(B)$.
(3) $\tau_1\tau_2$-*Cl*(A) *is* $\tau_1\tau_2$-*closed.*
(4) A *is* $\tau_1\tau_2$-*closed if and only if* $A = \tau_1\tau_2$-*Cl*(A).
(5) $\tau_1\tau_2$-*Cl*$(X - A) = X - \tau_1\tau_2$-*Int*$(A)$.

A bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-*connected* [11] if X cannot be written as the union of two nonempty disjoint $\tau_1\tau_2$-open sets. A subset A of a bitopological space (X, τ_1, τ_2) is called $(\tau_1, \tau_2)r$-*open* [37] (resp. $(\tau_1, \tau_2)s$-*open* [9], $(\tau_1, \tau_2)p$-*open* [9], $(\tau_1, \tau_2)\beta$-*open* [9]) if $A = \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$ (resp. $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A))$, $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$, $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A)))$). The complement of a $(\tau_1, \tau_2)r$-open (resp. $(\tau_1, \tau_2)s$-open, $(\tau_1, \tau_2)p$-open, $(\tau_1, \tau_2)\beta$-open) set is called $(\tau_1, \tau_2)r$-*closed* (resp. $(\tau_1, \tau_2)s$-*closed*, $(\tau_1, \tau_2)p$-*closed*, $(\tau_1, \tau_2)\beta$-*closed*). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\alpha(\tau_1, \tau_2)$-*open*) [35] if $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A)))$. The complement of an $\alpha(\tau_1, \tau_2)$-open set is called $\alpha(\tau_1, \tau_2)$-*closed*. Let A be a subset of a bitopological space (X, τ_1, τ_2). A point $x \in X$ is called a $(\tau_1, \tau_2)\theta$-*cluster point* [37] of A if $\tau_1\tau_2\text{-Cl}(U) \cap A \neq \emptyset$ for every $\tau_1\tau_2$-open set U containing x. The set of all $(\tau_1, \tau_2)\theta$-cluster points of A is called the $(\tau_1, \tau_2)\theta$-*closure* [37] of A and is denoted by $(\tau_1, \tau_2)\theta\text{-Cl}(A)$. A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)\theta$-*closed* [37] if $(\tau_1, \tau_2)\theta\text{-Cl}(A) = A$. The complement of a $(\tau_1, \tau_2)\theta$-closed set is said to be $(\tau_1, \tau_2)\theta$-*open*. The union of all $(\tau_1, \tau_2)\theta$-open sets of X contained in A is called the $(\tau_1, \tau_2)\theta$-*interior* [37] of A and is denoted by $(\tau_1, \tau_2)\theta\text{-Int}(A)$.

Lemma 2. [37] *For a subset A of a bitopological space (X, τ_1, τ_2), the following properties hold:*

(1) *If A is $\tau_1\tau_2$-open in X, then $\tau_1\tau_2\text{-Cl}(A) = (\tau_1, \tau_2)\theta\text{-Cl}(A)$.*

(2) *$(\tau_1, \tau_2)\theta\text{-Cl}(A)$ is $\tau_1\tau_2$-closed in X.*

3. Characterizations of weakly s-(τ_1, τ_2)-continuous functions

In this section, we introduce the notion of weakly s-(τ_1, τ_2)-continuous functions. Furthermore, several characterizations of weakly (τ_1, τ_2)-continuous functions are discussed.

Definition 1. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be weakly s-(τ_1, τ_2)-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$ and having $\sigma_1\sigma_2$-connected complement, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be weakly s-(τ_1, τ_2)-continuous if f is weakly s-(τ_1, τ_2)-continuous at each point x of X.*

Theorem 1. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *f is weakly s-(τ_1, τ_2)-continuous;*

(2) *$f^{-1}(V) \subseteq \tau_1\tau_2\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$ for every $\sigma_1\sigma_2$-open set V of Y having $\sigma_1\sigma_2$-connected complement;*

(3) *$\tau_1\tau_2\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(K))) \subseteq f^{-1}(K)$ for every $\sigma_1\sigma_2$-connected $\sigma_1\sigma_2$-closed set K of Y;*

(4) *$\tau_1\tau_2\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(B))$ for every subset B of Y having the $\sigma_1\sigma_2$-connected $\sigma_1\sigma_2$-closure;*

(5) *$f^{-1}(\sigma_1\sigma_2\text{-Int}(B)) \subseteq \tau_1\tau_2\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B))))$ for every subset B of Y such that $Y - \sigma_1\sigma_2\text{-Int}(B)$ is $\sigma_1\sigma_2$-connected;*

(6) *$\tau_1\tau_2$-Cl$(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(V))$ for every $\sigma_1\sigma_2$-open set V of Y having the $\sigma_1\sigma_2$-connected $\sigma_1\sigma_2$-closure;*

(7) *$\tau_1\tau_2$-Cl$(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(V))$ for every $\sigma_1\sigma_2$-open set V of Y having the $\sigma_1\sigma_2$-connected $\sigma_1\sigma_2$-closure;*

(8) *$\tau_1\tau_2$-Cl$(f^{-1}(\sigma_1\sigma_2$-Int$(K))) \subseteq f^{-1}(K)$ for every $\sigma_1\sigma_2$-connected $(\sigma_1, \sigma_2)r$-closed set K of Y.*

Proof. $(1) \Rightarrow (2)$: Let V be any $\sigma_1\sigma_2$-open set of Y having $\sigma_1\sigma_2$-connected complement and $x \in f^{-1}(V)$. Then, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$. Therefore, we have $x \in U \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$. Since U is $\tau_1\tau_2$-open, we have

$$x \in \tau_1\tau_2\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$$

and hence $f^{-1}(V) \subseteq \tau_1\tau_2\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$.

$(2) \Rightarrow (3)$: Let K be any $\sigma_1\sigma_2$-connected $\sigma_1\sigma_2$closed set of Y. Then, $Y - K$ is $\sigma_1\sigma_2$-open in Y having $\sigma_1\sigma_2$-connected complement. By (2), we have

$$\begin{aligned} X - f^{-1}(K) &= f^{-1}(Y - K) \\ &\subseteq \tau_1\tau_2\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y - K))) \\ &= \tau_1\tau_2\text{-Int}(f^{-1}(Y - \sigma_1\sigma_2\text{-Int}(K))) \\ &= \tau_1\tau_2\text{-Int}(X - f^{-1}(\sigma_1\sigma_2\text{-Int}(K))) \\ &= X - \tau_1\tau_2\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(K))) \end{aligned}$$

and hence $\tau_1\tau_2\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(K))) \subseteq f^{-1}(K)$.

$(3) \Rightarrow (4)$: Let B be any subset of Y having the $\sigma_1\sigma_2$-connected $\sigma_1\sigma_2$-closure. Then, $\sigma_1\sigma_2\text{-Cl}(B)$ is $\sigma_1\sigma_2$-closed $\sigma_1\sigma_2$-connected in Y and by (3), $\tau_1\tau_2\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(B))$.

$(4) \Rightarrow (5)$: Let B be any subset of Y such that $Y - \sigma_1\sigma_2\text{-Int}(B)$ is $\sigma_1\sigma_2$-connected. Then by (4), we have

$$\begin{aligned} &X - \tau_1\tau_2\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B)))) \\ &= \tau_1\tau_2\text{-Cl}(X - f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B)))) \\ &= \tau_1\tau_2\text{-Cl}(f^{-1}(Y - \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B)))) \\ &= \tau_1\tau_2\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(Y - B))) \\ &\subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y - B)) \\ &= X - f^{-1}(\sigma_1\sigma_2\text{-Int}(B)). \end{aligned}$$

Thus, $f^{-1}(\sigma_1\sigma_2\text{-Int}(B)) \subseteq \tau_1\tau_2\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B))))$.

$(5) \Rightarrow (1)$: Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$ and having $\sigma_1\sigma_2$-connected complement. By (5), we have

$$x \in f^{-1}(V) = f^{-1}(\sigma_1\sigma_2\text{-Int}(V)) \subseteq \tau_1\tau_2\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))).$$

Then, there exists a $\tau_1\tau_2$-open set U of X containing x such that

$$x \in U \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)).$$

Thus, $f(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$ and hence f is weakly s-(τ_1, τ_2)-continuous.

$(4) \Rightarrow (6)$ and $(6) \Rightarrow (7)$: The proofs are obvious.

$(7) \Rightarrow (8)$: Let K be any $\sigma_1\sigma_2$-connected $(\sigma_1, \sigma_2)r$-closed set of Y.
Then, we have $K = \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))$ is $\sigma_1\sigma_2$-connected and by (7), $\tau_1\tau_2\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(K))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))) = f^{-1}(K)$.

$(8) \Rightarrow (3)$: Let K be any $\sigma_1\sigma_2$-connected $\sigma_1\sigma_2$-closed set of Y. Since K is $\sigma_1\sigma_2$-connected, we have $\sigma_1\sigma_2\text{-Int}(K)$ is $\sigma_1\sigma_2$-connected and hence $\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))$ is $\sigma_1\sigma_2$-connected. Let $H = \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))$. Then, H is a $(\sigma_1, \sigma_2)r$-closed $\sigma_1\sigma_2$-connected set of Y and

$$\sigma_1\sigma_2\text{-Int}(H) = \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))) = \sigma_1\sigma_2\text{-Int}(K).$$

By (8), we have

$$\begin{aligned}\tau_1\tau_2\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(K))) &= \tau_1\tau_2\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(H))) \\ &\subseteq f^{-1}(H) \subseteq f^{-1}(K).\end{aligned}$$

□

Theorem 2. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *f is weakly s-(τ_1, τ_2)-continuous;*
(2) *$\tau_1\tau_2$-Cl$(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(V))$ for every $(\sigma_1, \sigma_2)\beta$-open set V of Y having the $\sigma_1\sigma_2$-connected $\sigma_1\sigma_2$-closure;*
(3) *$\tau_1\tau_2$-Cl$(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(V))$ for every $(\sigma_1, \sigma_2)s$-open set V of Y having the $\sigma_1\sigma_2$-connected $\sigma_1\sigma_2$-closure.*

Proof. $(1) \Rightarrow (2)$: This follows from Theorem 1(4).

$(2) \Rightarrow (3)$: The proof is obvious since every $(\sigma_1, \sigma_2)s$-open set is $(\sigma_1, \sigma_2)\beta$-open.

$(3) \Rightarrow (1)$: Since every $\sigma_1\sigma_2$-open set is $(\sigma_1, \sigma_2)s$-open, the proof follows from Theorem 1(7). □

Theorem 3. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *f is weakly s-(τ_1, τ_2)-continuous;*

(2) *$\tau_1\tau_2$-Cl$(f^{-1}(\sigma_1\sigma_2$-Int$((\sigma_1,\sigma_2)\theta$-Cl$(B)))) \subseteq f^{-1}((\sigma_1,\sigma_2)\theta$-Cl$(B))$ for every subset B of Y having the $\sigma_1\sigma_2$-connected $(\sigma_1,\sigma_2)\theta$-closure;*

(3) *$\tau_1\tau_2$-Cl$(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(B)))) \subseteq f^{-1}((\sigma_1,\sigma_2)\theta$-Cl$(B))$ for every subset B of Y having the $\sigma_1\sigma_2$-connected $(\sigma_1,\sigma_2)\theta$-closure.*

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y having the $\sigma_1\sigma_2$-connected $(\sigma_1,\sigma_2)\theta$-closure. Then, $(\sigma_1,\sigma_2)\theta$-Cl$(B)$ is $\sigma_1\sigma_2$-connected $(\sigma_1,\sigma_2)\theta$-closure and by Theorem1,

$$\tau_1\tau_2\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}((\sigma_1,\sigma_2)\theta\text{-Cl}(B)))) \subseteq f^{-1}((\sigma_1,\sigma_2)\theta\text{-Cl}(B)).$$

(2) $\Rightarrow$ (3): The proof is obvious since $\sigma_1\sigma_2$-Cl$(B) \subseteq (\sigma_1,\sigma_2)\theta$-Cl$(B)$ for every subset B of Y.

(3) $\Rightarrow$ (1): Let K be any $(\sigma_1,\sigma_2)r$-closed $\sigma_1\sigma_2$-connected set of Y. Then , we have

$$(\sigma_1,\sigma_2)\theta\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K)) = \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K)) = K$$

and by (3),

$$\begin{aligned}
&\tau_1\tau_2\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(K)))\\
&= \tau_1\tau_2\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K)))))\\
&\subseteq f^{-1}((\sigma_1,\sigma_2)\theta\text{-Cl}(\sigma_1\sigma_2\text{-Cl}(K)))\\
&= f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Cl}(K)))\\
&= f^{-1}(K).
\end{aligned}$$

Thus, $\tau_1\tau_2$-Cl$(f^{-1}(\sigma_1\sigma_2$-Int$(K))) \subseteq f^{-1}(K)$ and by Theorem 1(8), f is weakly s-(τ_1,τ_2)-continuous. □

The *$\tau_1\tau_2$-frontier* [2] of a subset A of a bitopological space (X,τ_1,τ_2), denoted by $\tau_1\tau_2$-fr(A), is defined by

$$\begin{aligned}
\tau_1\tau_2\text{-fr}(A) &= \tau_1\tau_2\text{-Cl}(A) \cap \tau_1\tau_2\text{-Cl}(X-A)\\
&= \tau_1\tau_2\text{-Cl}(A) - \tau_1\tau_2\text{-Int}(A).
\end{aligned}$$

Theorem 4. *The set of all points $x \in X$ at which a function $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is not weakly s-(τ_1,τ_2)-continuous is identical with the union of the $\tau_1\tau_2$-frontier of the inverse images of the $\sigma_1\sigma_2$-closure of $\sigma_1\sigma_2$-open sets containing $f(x)$ and having $\sigma_1\sigma_2$-connected complement.*

Proof. Let x be a point of X at which f is not weakly s-(τ_1,τ_2)-continuous. Then, there exists a $\sigma_1\sigma_2$-open set V of Y containing $f(x)$ and having $\sigma_1\sigma_2$-connected complement such that

$$U \cap (X - f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))) \neq \emptyset$$

for every $\tau_1\tau_2$-open set U of X containing x. Then, we have

$$x \in \tau_1\tau_2\text{-Cl}(X - f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$$

and hence $x \in \tau_1\tau_2\text{-fr}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$ since

$$x \in f^{-1}(V) \subseteq \tau_1\tau_2\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))).$$

Conversely, suppose that V is a $\sigma_1\sigma_2$-open set of Y containing $f(x)$ and having $\sigma_1\sigma_2$-connected complement such that

$$x \in \tau_1\tau_2\text{-fr}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))).$$

If f is weakly s-(τ_1, τ_2)-continuous at $x \in X$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$; hence

$$U \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)).$$

Thus, $x \in U \subseteq \tau_1\tau_2\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$. This contradicts that

$$x \in \tau_1\tau_2\text{-fr}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))).$$

□

Definition 2. [14] *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be s-(τ_1, τ_2)-continuous at $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$ and having $\sigma_1\sigma_2$-connected complement, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq V$. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be s-(τ_1, τ_2)-continuous if f has this property at each point x of X.*

Lemma 3. [14] *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *f is s-(τ_1, τ_2)-continuous;*
(2) *$f^{-1}(V)$ is $\tau_1\tau_2$-open in X for every $\sigma_1\sigma_2$-open set V of Y having $\sigma_1\sigma_2$-connected complement;*
(3) *$f^{-1}(K)$ is $\tau_1\tau_2$-closed in X for every $\sigma_1\sigma_2$-connected $\sigma_1\sigma_2$-closed set K of Y;*
(4) *$\tau_1\tau_2$-Cl$(f^{-1}(B)) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(B))$ for every subset B of Y having the $\sigma_1\sigma_2$-connected $\sigma_1\sigma_2$-closure;*
(5) *$f^{-1}(\sigma_1\sigma_2$-Int$(B)) \subseteq \tau_1\tau_2$-Int$(f^{-1}(B))$ for every subset B of Y such that $Y - \sigma_1\sigma_2$-Int(B) is $\sigma_1\sigma_2$-connected.*

Theorem 5. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is weakly s-(τ_1, τ_2)-continuous and satisfies $f^{-1}(\sigma_1\sigma_2$-Cl$(V)) \subseteq f^{-1}(V)$ for every $\sigma_1\sigma_2$-open set V of Y having $\sigma_1\sigma_2$-connected complement, then f is s-(τ_1, τ_2)-continuous.*

Proof. Let V be any $\sigma_1\sigma_2$-open set of Y having $\sigma_1\sigma_2$-connected complement. Since f is weakly s-(τ_1, τ_2)-continuous, by Theorem 1 we have

$$f^{-1}(V) \subseteq \tau_1\tau_2\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))) \subseteq \tau_1\tau_2\text{-Int}(f^{-1}(V))$$

and hence $f^{-1}(V)$ is $\tau_1\tau_2$-open in X. By Lemma 3, f is s-(τ_1, τ_2)-continuous. □

Acknowledgements. This research project was financially supported by Mahasarakham University.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] C. Boonpok, J. Khampakdee, Almost Strong $\theta(\Lambda, p)$-Continuity for Functions, Eur. J. Pure Appl. Math. 17 (2024), 300–309. https://doi.org/10.29020/nybg.ejpam.v17i1.4975.

[2] C. Boonpok, N. Srisarakham, (τ_1, τ_2)-Continuity for Functions, Asia Pac. J. Math. 11 (2024), 21. https://doi.org/10.28924/APJM/11-21.

[3] C. Boonpok, P. Pue-On, Characterizations of Almost (τ_1, τ_2)-Continuous Functions, Int. J. Anal. Appl. 22 (2024), 33. https://doi.org/10.28924/2291-8639-22-2024-33.

[4] C. Boonpok, C. Klanarong, On Weakly (τ_1, τ_2)-Continuous Functions, Eur. J. Pure Appl. Math. 17 (2024), 416–425. https://doi.org/10.29020/nybg.ejpam.v17i1.4976.

[5] C. Boonpok, on Some Spaces via Topological Ideals, Open Math. 21 (2023), 20230118. https://doi.org/10.1515/math-2023-0118.

[6] C. Boonpok, N. Srisarakham, Weak Forms of (Λ, b)-Open Sets and Weak (Λ, b)-Continuity, Eur. J. Pure Appl. Math. 16 (2023), 29–43. https://doi.org/10.29020/nybg.ejpam.v16i1.4571.

[7] C. Boonpok, $\theta(\star)$-Precontinuity, Mathematica, 65 (2023), 31–42. https://doi.org/10.24193/mathcluj.2023.1.04.

[8] C. Boonpok, J. Khampakdee, (Λ, sp)-Open Sets in Topological Spaces, Eur. J. Pure Appl. Math. 15 (2022), 572–588. https://doi.org/10.29020/nybg.ejpam.v15i2.4276.

[9] C. Boonpok, $(\tau_1, \tau_2)\delta$-Semicontinuous Multifunctions, Heliyon, 6 (2020), e05367. https://doi.org/10.1016/j.heliyon.2020.e05367.

[10] C. Boonpok, on Characterizations of $\star$-Hyperconnected Ideal Topological Spaces, J. Math. 2020 (2020), 9387601. https://doi.org/10.1155/2020/9387601.

[11] C. Boonpok, C. Viriyapong, M. Thongmoon, on Upper and Lower (τ_1, τ_2)-Precontinuous Multifunctions, J. Math. Computer Sci. 18 (2018), 282–293. https://doi.org/10.22436/jmcs.018.03.04.

[12] C. Boonpok, Almost (G, m)-Continuous Functions, Int. J. Math. Anal. 4 (2010), 1957–1964.

[13] C. Boonpok, M-Continuous Functions in Biminimal Structure Spaces, Far East J. Math. Sci. 43 (2010), 41–58.

[14] M. Chiangpradit, A. Sama-Ae, C. Boonpok, s-(τ_1, τ_2)-Continuity for Multifunctions, Eur. J. Pure Appl. Math. 18 (2025), 5634. https://doi.org/10.29020/nybg.ejpam.v18i1.5634.

[15] M. Chiangpradit, S. Sompong, C. Boonpok, Weakly Quasi (τ_1, τ_2)-Continuous Functions, Int. J. Anal. Appl. 22 (2024), 125. https://doi.org/10.28924/2291-8639-22-2024-125.

[16] T. Duangphui, C. Boonpok and C. Viriyapong, Continuous Functions on Bigeneralized Topological Spaces, Int. J. Math. Anal. 5 (2011), 1165–1174.

[17] W. Dungthaisong, C. Boonpok, C. Viriyapong, Generalized Closed Sets in Bigeneralized Topological Spaces, Int. J. Math. Anal. 5 (2011), 1175–1184.

[18] T. Husain, Almost Continuous Mappings, Prace Mat. 10 (1966), 1–7.

[19] C. Boonpok, S. Sompong, J. Khampakdee, Almost Weakly (τ_1, τ_2)-Continuous Functions, Eur. J. Pure Appl. Math. 18 (2025), 5721. https://doi.org/10.29020/nybg.ejpam.v18i1.5721.

[20] J.K. Kohli, S-Continuous Functions and Certain Weak Forms of Regularity and Complete Regularity, Math. Nachr. 97 (1980), 189–196. https://doi.org/10.1002/mana.19800970116.

[21] J.K. Kohli, A Class of Mappings Containing All Continuous and All Semiconnected Mappings, Proc. Amer. Math. Soc. 72 (1978), 175–181. https://doi.org/10.1090/S0002-9939-1978-0493941-9.

[22] B. Kong-ied, S. Sompong, C. Boonpok, Almost Quasi (τ_1, τ_2)-Continuous Functions, Asia Pac. J. Math. 23 (2025), 14. https://doi.org/10.28924/APJM/12-14.

[23] Y.L. Lee, Some Characterizations of Semilocally Connected Spaces, Proc. Amer. Math. Soc. 16 (1965), 1318–1320.

[24] N. Levine, Semi-Open Sets and Semi-Continuity in Topological Spaces, Amer. Math. Mon. 70 (1963), 36–41. https://doi.org/10.1080/00029890.1963.11990039.

[25] N. Levine, A Decomposition of Continuity in Topological Spaces, Amer. Math. Mon. 68 (1961), 44–46. https://doi.org/10.2307/2311363.

[26] S. Marcus, Sur les Fonctions Quasicontinues au Sense de S. Kempisty, Colloq. Math. 8 (1961), 47–53. http://eudml.org/doc/210887.

[27] C. Prachanpol, C. Boonpok, C. Viriyapong, $\delta(\tau_1, \tau_2)$-Continuous Functions, Eur. J. Pure Appl. Math. 17 (2024), 3730–3742. https://doi.org/10.29020/nybg.ejpam.v17i4.5335.

[28] P. Pue-on, S. Sompong, C. Boonpok, Slightly $(\tau_1, \tau_2)s$-Continuous Functions, Int. J. Math. Comput. Sci. 20 (2025), 217–221.

[29] P. Pue-on, C. Boonpok, $\theta(\Lambda, p)$-Continuity for Functions, Int. J. Math. Comput. Sci. 19 (2024), 491–495.

[30] N. Srisarakham, S. Sompong, C. Boonpok, Quasi $\theta(\tau_1, \tau_2)$-Continuous Functions, Eur. J. Pure Appl. Math. 18 (2025), 5722. https://doi.org/10.29020/nybg.ejpam.v18i1.5722.

[31] N. Srisarakham, S. Sompong, C. Boonpok, Slight (τ_1, τ_2)-Continuity for Functions, Int. J. Math. Comput. Sci. 20 (2025), 211–215.

[32] N. Srisarakham, A. Sama-Ae, C. Boonpok, Characterizations of Faintly (τ_1, τ_2)-continuous functions, Eur. J. Pure Appl. Math. 17 (2024), 2753–2762.

[33] N. Srisarakham, C. Boonpok, Almost (Λ, p)-Continuous Functions, Int. J. Math. Comput. Sci. 18 (2023), 255–259.

[34] M. Thongmoon, C. Boonpok, Strongly $\theta(\Lambda, p)$-Continuous Functions, Int. J. Math. Comput. Sci. 19 (2024), 475–479.

[35] N. Viriyapong, S. Sompong, C. Boonpok, (τ_1, τ_2)-Extremal Disconnectedness in Bitopological Spaces, Int. J. Math. Comput. Sci. 19 (2024), 855–860.

[36] C. Viriyapong, C. Boonpok, (Λ, sp)-Continuous Functions, WSEAS Trans. Math. 21 (2022), 380–385. https://doi.org/10.37394/23206.2022.21.45.

[37] C. Viriyapong, C. Boonpok, $(\tau_1, \tau_2)\alpha$-Continuity for Multifunctions, J. Math. 2020 (2020), 6285763. https://doi.org/10.1155/2020/6285763.