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ATOMIC SOLUTION OF ABSTRACT CAUCHY PROBLEM IN ℓ^2 SPACE

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Abstract. In this paper, based on the theory of tensor product of Banach spaces, we find the atomic solutions of the following homogeneous second order abstract Cauchy problem.

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1. Introduction

One of the most attractive abstract differential equations is the abstract Cauchy problem. Various applications of such problems come into play when modeling dynamical systems with spatial diffusions aspects [1]. However, for abstract Cauchy problems with ordinary and fractional orders, there is a wide range of scientific applications. The evolution of the dynamical system can be handled more effectively, in this particular case, utilizing proper infinite dimensional spaces such as Banach or Hilbert spaces with specific abstract properties which can be chosen according to the dynamical system under study. By employing different analytical approaches, many results have been reported in the literature in order to solve abstract Cauchy problems with ordinary and fractional orders [2,3].

In the current article, based on the theory of tensor product of Banach spaces, we are interested on finding the atomic solutions of the following homogeneous second order abstract Cauchy problem

$$u_{ss}(s,t) + u_{tt}(s,t) + 2u_{st}(s,t) + A\left[u_s(s,t) + u_t(s,t)\right] + Bu(s,t) = 0, \quad (1)$$

where $u : [0,1] \times [0,1] \to \ell^2$ is an unknown twice continuously differentiable function and $A, B : \ell^2 \to \ell^2$ are two densely defined linear operators on the range of u.

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2. Preliminaries

It is a well-known fact that one can study the aspects of a given linear space by studying the features of the space of functions defined on that space [4].

In this section, we introduce some definitions and basic results on tensor product theory of Banach spaces [5,6].

Definition 2.1. [7] *Let X and Y be any two Banach spaces and X^* is the dual space of X. For $x \in X$ and $y \in Y$, the operator $T : X^* \to Y$, defined by $Tx^* = x^*(x)y = \langle x, x^* \rangle\, y$, is bounded one rank linear operator. We write $x \otimes y$ for T and such operators are called atoms.*

Theorem 2.1. *The atomic operator T is a bounded linear operator, with*

$$\| T \| = \| x \| \| y \| .$$

Proof.: Suppose that $x_1^*, x_2^* \in X^*$ and $\alpha, \beta \in R$. Then

$$\begin{aligned} T(\alpha x_1^* + \beta x_2^*) &= (\alpha x_1^* + \beta x_2^*)\,(x)\,y \\ &= \alpha x_1^*\,(x)\,y + \beta x_2^*\,(x)\,y \\ &= \alpha\,(x_1^*\,(x)\,y) + \beta\,(x_2^*\,(x)\,y) \\ &= \alpha T x_1^* + \beta T x_2^*. \end{aligned} \tag{2}$$

Therefore, T is a linear operator.

Moreover, for any $x^* \in X^*$, we have $\| Tx^* \| = \|x^*(x)y\| = |x^*(x)|\,\|y\| \le \|x^*\|\,\|x\|\,\|y\|$. Take the supremum over all $x^* \in X^*$ such that $\|x^*\| = 1$ to obtain $\|Tx^*\| \le \|x\|\,\|y\|$. Now, by employing a famous consequence of Hahn-Banach theorem, that is for all $x \in X$ there exists $x^* \in X^*$ such that $\|x^*\| = 1$ and $x^*\,(x) = \|x\|$, we have $\| Tx^* \| = \|x^*(x)y\| = \|x\|\,\|y\|$. Hence, T is bounded and the proof, now, is complete.

In the following we present a nice result regarding the atomic operators.

Lemma 1. [8] *Let $\zeta_1 \otimes \eta_1$ and $\zeta_2 \otimes \eta_2$ be two nonzero atoms in $X \otimes Y$ such that $\zeta_1 \otimes \eta_1 + \zeta_2 \otimes \eta_2 = \zeta_3 \otimes \eta_3$. Then either $\zeta_1 = \zeta_2 = \zeta_3$ or $\eta_1 = \eta_2 = \eta_3$.*

It should be noted that any solution to the abstract Cauchy problem, that is given in (1), that has the form

$$u\,(s,t) = v(s,t) \otimes \omega, \tag{3}$$

is said to be an atomic solution provided that $v(s,t) : [0,1] \times [0,1] \to \mathbb{R}$ and $\omega \in \ell^2$.

Let $L(X^*, Y)$ denotes the space of all bounded (continuous) linear operators from X^* to Y. Then

$$X \otimes Y = span\{x \otimes y : x \in X, y \in Y\}, \tag{4}$$

is a subspace of $L(X^*, Y)$. So, if $T \in X \otimes Y$, then $T = \sum_{i=1}^{n} x_i \otimes y_i$, where $x_i \in X$ and $y_i \in Y$, for all $i = 1, 2, \cdots, n$.

For any $T \in X \otimes Y$, one can define many norms. Among these norms is the injective norm which is denoted by $\|.\|_\vee$ and defined as

$$\|T\|_\vee = \left\| \sum_{i=1}^{n} x_i \otimes y_i \right\|_\vee = \sup_{\|x^*\|=1} \left\| \sum_{i=1}^{n} \langle x^*, x_i \rangle \, y_i \right\| = \sup_{\substack{\|x^*\|=1 \\ \|y^*\|=1}} \left| \sum_{i=1}^{n} x^* \left(x_i\right) \langle y_i, y^* \rangle \right|, \tag{5}$$

where $x^* \in X^*$ and $y^* \in Y^*$. The normed space $(X \otimes Y, \|.\|_\vee)$ need not to be complete. We let $X \overset{\vee}{\otimes} Y$ denotes the completion of $X \otimes Y$ under the injective norm.

A nice and useful result associated with the injective norm is the following theorem.

Theorem 2.2. [5] *Let E be a compact Hausdorff space and X be a Banach space. Then $C\left(E, X\right)$ is isometrically isomorphic to $C\left(E\right) \overset{\vee}{\otimes} X$. Equivalently, for any two compact metric spaces M and N, we have*

$$C\left(M \times N\right) = C\left(M\right) \overset{\vee}{\otimes} C\left(N\right) = C\left(M, C\left(N\right)\right) \tag{6}$$

Corollary 1. [5] *Let X be a Banach space and I be any compact interval. Then $C\left(I, X\right)$is isometrically isomorphic to $C\left(I\right) \overset{\vee}{\otimes} X$.*

Let $C^2\left(\left[0,1\right] \times \left[0,1\right], \ell^2\right)$ be the space of twice continuously differentiable functions defined on $\left[0,1\right] \times \left[0,1\right]$ to ℓ^2. Then by Theorem (2.2), for any solution $u \in C^2\left(\left[0,1\right] \times \left[0,1\right], \ell^2\right)$ to (1), we may consider

$$u(s,t) = P\left(s\right) Q\left(t\right) \omega, \tag{7}$$

where $P\left(s\right), Q\left(t\right) : \left[0,1\right] \to \mathbb{R}$ and $\omega \in \ell^2$.

3. Main Results

In this section, we present an atomic solution (3) to the abstract Cauchy problem (1).

Theorem 3.1. *Consider the following homogeneous second order abstract Cauchy problem in two variables s and t*

$$\begin{gathered} u_{ss}(s,t) + u_{tt}(s,t) + 2u_{st}(s,t) + A\left[u_s(s,t) + u_t(s,t)\right] + Bu(s,t) = 0, \\ u\left(0,0\right) = x_\circ,\ u_s\left(0,0\right) = x_1,\ u_t\left(0,0\right) = x_2, \end{gathered} \tag{8}$$

where $u : \left[0,1\right] \times \left[0,1\right] \to \ell^2$ is an unknown super nice (in the sense that it satisfies all necessary relevant properties) twice differentiable function, $A, B : \ell^2 \to \ell^2$ are two densely defined linear operators on the range of u, and $x_i \in \ell^2$ for $i = 0, 1, 2$. Then (8) has the following atomic solutions

$$\begin{aligned} u_1(s,t) = &\quad e^{s+t}\omega, &&\quad \textit{in case } A2\omega = B\omega = -2\omega, \\ u_2(s,t) = &\quad e^{s}\left(2 - e^{-t}\right)\omega, &&\quad \textit{in case } A\omega = -\omega \textit{ and } B\omega = 0, \end{aligned}$$

$$u_3(s,t) = e^{-2s+t}\left(3e^s-2\right)\omega, \quad \text{in case } A\omega=\omega \text{ and } B\omega=0,$$
$$u_4(s,t) = e^{s+t}\omega, \quad \text{in case } B\omega=2\omega,$$
$$u_5(s,t) = e^{t}\left(2-e^{-s}\right)\omega, \quad \text{in case } A\omega=-\omega \text{ and } B\omega=0.$$

Proof.: Let u be an atomic solution to (8). Hence by (7)

$$u(s,t) = P(s)\,Q(t)\,\omega, \tag{9}$$

where $P(s), Q(t) : [0,1] \to \mathbb{R}$ and $\omega \in \ell^2$. Further (9) can be written in the form

$$u = P \otimes Q \otimes \omega. \tag{10}$$

Also, by (10) , and recalling initial conditions in (8), we have $u(0,0) = v(0,0)\omega = P(0)\,Q(0)\,\omega = x_\circ$. Consequently, since $x_\circ = \omega$, we can assume

$$P(0) = Q(0) = 1. \tag{11}$$

Similarly, $\frac{\partial u}{\partial s}(0,0) = \frac{\partial v}{\partial s}(0,0)\,\omega = P'(0)\,Q(0)\,\omega = x_1$ and $\frac{\partial u}{\partial t}(0,0) = \frac{\partial v}{\partial t}(0,0)\,\omega = P(0)\,Q'(0)\,\omega = x_2$. Thus,

$$x_\circ = x_1 = x_2 = \omega,$$

and hence,

$$P(0) = P'(0) = Q(0) = Q'(0) = 1. \tag{12}$$

Now, let us substitute (10) into the abstract Cauchy equation (8) to have the form

$$\begin{aligned} & P''(s)\,Q(t)\,\omega + P(s)\,Q''(t)\,\omega + 2P'(s)\,Q'(t)\,\omega \\ & +A\left[P'(s)\,Q(t)\,\omega + P(s)\,Q'(t)\,\omega\right] + BP(s)\,Q(t)\,\omega \\ = \;& 0. \end{aligned} \tag{13}$$

Equivalently,

$$P''(s)\,Q(t)\,\omega + P(s)\left[Q''(t)\,\omega + Q'(t)\,A\omega + Q(t)\,B\omega\right] + P'(s)\left[2Q'(t)\,\omega + Q(t)\,A\omega\right] = 0, \tag{14}$$

which can be written as,

$$P''(s)\,Q(t)\,\omega + P(s)\left[Q''(t)\,\omega + Q'(t)\,A\omega + Q(t)\,B\omega\right] = P'(s)\left[-2Q'(t)\,\omega - Q(t)\,A\omega\right]. \tag{15}$$

According to Lemma (2.1), equation (15) admits that the sum of two atoms is an atom. Thus, we have the following two cases:

$$\begin{aligned} &\textbf{Case (i)}: && P''(s) = P(s) = P'(s)\,, \\ &\textbf{Case (ii)}: && Q(t)\,\omega = Q''(t)\,\omega + Q'(t)\,A\omega + Q(t)\,B\omega = -2Q'(t)\,\omega - Q(t)\,A\omega. \end{aligned} \tag{16}$$

Let us handle each case individually.

Case (i):: By using initial conditions in (12), this case has the following solution

$$P(s) = e^{s}. \tag{17}$$

Now, we substitute (17) into (15) to get

$$Q''(t)\,\omega + Q'(t)\,(A\omega + 2\omega) = Q(t)\,(-A\omega - B\omega - \omega)\,. \tag{18}$$

Again by Lemma (2.1), equation (18) admits that the sum of two atoms is an atom. Thus, we have the following two sub-cases:

$$\begin{aligned} &\text{Sub-case (a)}: \quad Q''(t) = Q'(t) = Q(t)\,, \\ &\text{Sub-case (b)}: \quad \omega = A\omega + 2\omega = -A\omega - B\omega - \omega. \end{aligned} \tag{19}$$

Sub-case (a): Using initial conditions in (12), sub-case (a) has the following solution

$$Q(t) = e^{t}. \tag{20}$$

Now, we substitute (20) into (18) to get $A2\omega + B\omega + 4\omega = 0$. Therefore, if we consider $A2\omega = -2\omega$ and $B\omega = -2\omega$, then an atomic solution to (8) can be obtained. Hence, referring to (10), (17), and (20) the first atomic solution associated with sub-case (a) of case (i) is given by

$$u_1(s,t) = e^{s+t}\omega, \tag{21}$$

provided that $A2\omega = B\omega = -2\omega$.

Sub-case (b): The equation $\omega = A\omega + 2\omega = -A\omega - B\omega - \omega$, can be reduced into the following two equations

$$A\omega = -\omega \text{ and } B\omega = 0. \tag{22}$$

Now, we substitute (22) into (18) to get

$$Q''(t) + Q'(t) = 0. \tag{23}$$

But, by using initial conditions in (12), equation (23) gives

$$Q(t) = 2 - e^{-t}. \tag{24}$$

Hence, referring to (10), (17), and (24) the second atomic solution to (8) that is associated with sub-case (b) of case (i) is given by

$$u_2(s,t) = e^{s}\left(2 - e^{-t}\right)\omega, \tag{25}$$

provided that $A\omega = -\omega$ and $B\omega = 0$.

Case (ii):: This case has the following three sub-cases:

$$\begin{aligned} &\textbf{Sub-case (a)}: && Q(t)\,\omega = Q''(t)\,\omega + Q'(t)\,A\omega + Q(t)\,B\omega, \\ &\textbf{Sub-case (b)}: && Q(t)\,\omega = -2Q'(t)\,\omega - Q(t)\,A\omega, \\ &\textbf{Sub-case (c)}: && Q''(t)\,\omega + Q'(t)\,A\omega + Q(t)\,B\omega = -2Q'(t)\,\omega - Q(t)\,A\omega. \end{aligned} \tag{26}$$

Sub-case (a): Here, the corresponding equation can be rewritten as

$$Q(t)\,[\omega - B\omega] = Q''(t)\,\omega + Q'(t)\,A\omega. \tag{27}$$

By Lemma (2.1), equation (27) admits that the sum of two atoms is an atom. Thus, we have either $Q''(t) = Q'(t) = Q(t)$ or $\omega - B\omega = \omega = A\omega$. Using initial conditions in (12), $Q''(t) = Q'(t) = Q(t)$ implies

$$Q(t) = e^{t}. \tag{28}$$

Moreover, $\omega - B\omega = \omega = A\omega$ implies that

$$A\omega = \omega \text{ and } B\omega = 0. \tag{29}$$

Now, we substitute (28) and (29) into (15) to get

$$P''(s) + 2P(s) = -3P'(s). \tag{30}$$

Hence, by related initial conditions from (12), we have

$$P(s) = e^{-2s}\,(3e^{s} - 2). \tag{31}$$

Hence, referring to (10), (28), and (31) the third atomic solution to (8) that is associated with sub-case (a) of case (ii) is given by

$$u_3(s,t) = e^{-2s+t}\,(3e^{s} - 2)\,\omega, \tag{32}$$

provided that $A\omega = \omega$ and $B\omega = 0$.

Sub-case (b): The corresponding equation can be rewritten as

$$Q(t)\,[\omega + A\omega] = Q'(t)\,[-2\omega]. \tag{33}$$

Hence, by referring to Lemma (2.1), we have either $Q'(t) = Q(t)$ or $\omega + A\omega = -2\omega$. Using initial conditions in (12), $Q'(t) = Q(t)$ implies

$$Q(t) = e^{t}. \tag{34}$$

Moreover, $\omega + A\omega = -2\omega$ implies that

$$A\omega = -3\omega. \tag{35}$$

Now, we substitute (34) and (35) into (15) to get

$$P''(s)\,\omega + P(s)\,[-2\omega + B\omega] = P'(s)\,\omega. \tag{36}$$

Thus, by Lemma (2.1), either $P''(s) = P(s) = P'(s)$ which implies that

$$P(s) = e^{s}, \tag{37}$$

or $\omega + [-2\omega + B\omega] = \omega$, which admits that $B\omega = 2\omega$. Hence, referring to (10), (34), and (37) the fourth atomic solution associated with sub-case (b) of case (ii) is given by

$$u_4(s,t) = e^{s+t}\omega, \tag{38}$$

provided that $B\omega = 2\omega$ that is ω is an eigenvector of B with eigenvalue 2.

Sub-case (c): The corresponding equation can be rewritten as

$$Q''(t)\,\omega + Q'(t)\,[A\omega + 2\omega] = Q(t)\,[-B\omega - A\omega]\,. \tag{39}$$

By Lemma (2.1), equation (39) admits that the sum of two atoms is an atom. Thus, we have either $Q''(t) = Q'(t) = Q(t)$ or $\omega = A\omega + 2\omega = -B\omega - A\omega$. Using initial conditions in (12), $Q''(t) = Q'(t) = Q(t)$ implies

$$Q(t) = e^{t}. \tag{40}$$

Also, $\omega - B\omega = \omega = A\omega$ implies that

$$A\omega = -\omega \text{ and } B\omega = 0. \tag{41}$$

Now, we substitute (40) and (41) into (15) to get

$$P''(s)\,\omega = -P'(s)\,\omega. \tag{42}$$

Consequently, $P''(s) + P'(s) = 0$. Thus,

$$P(s) = \left(2 - e^{-s}\right). \tag{43}$$

Now, referring to (10), (40), and (43) the fifth atomic solution associated with sub-case (c) of case (ii) is given by

$$u_5(s,t) = e^{t}\left(2 - e^{-s}\right)\omega, \tag{44}$$

provided that $A\omega = -\omega$ and $B\omega = 0$. In other words, ω is an eigenvector of A with eigenvalue -1.

Therefore, the proof is complete.

4. Conclusions

This paper, we introduce an analytic method to determine the atomic solutions to a homogeneous second order abstract Cauchy problem in two variables. The unknown function assumed to be defined from $[0, 1] \times [0, 1]$ to the Banach space ℓ^2. The procedure was performed by combining some results from tensor product theory of Banach spaces. We could determine five different atomic solutions to the underline abstract Cauchy problem.

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