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INDEPENDENT RINGS DOMINATIONS IN SOME GRAPHS

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Abstract. A subset $S \subseteq V(G)$ is said to be an independent rings dominating set of a graph G if it is both an independent dominating set of G and a rings dominating set of G. The minimum cardinality of an independent rings dominating set is called an independent rings domination number of G and is denoted by $\gamma_{iri}(G)$. An independent rings dominating set S of G is called γ_{iri}-set of G. In this paper, the authors give characterizations of an independent rings dominating set of some graphs and graphs obtained from the join and corona of two graphs. Furthermore, the independent rings domination numbers of these graphs are determined, and the graphs with no independent rings dominating sets are investigated.

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1. Introduction

The study of domination in graph is believed to have started in the late 1950's and 1960's, beginning with Claude Berge [3] in 1958 in his book that introduced the coefficient of external stability which is now known today as domination in graphs, and with Oystein Ore [12] when he introduced the terms "dominating set" and "domination number" in his book which was published in 1962. The idea of independent dominating set arose in chessboard problem posed by de Jaenisch [10] in 1862 about finding the minimum number of mutually non-attacking queens that can be placed on a chessboard so that each square of the chessboard is attacked by at least one of the queens.

This foundation became the mainstream of many graph theorists to explore a lot of notions in dominations and independent dominations. Caay and Durog [7] introduced the study of independent equitable domination in graphs. This variant of domination introduced the type of independent

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domination for which it considers that for every vertex outside of an independent dominating set, there exists one in the such set adjacent to it such that the difference of their degree is at most 1. Additionally, Caay and Palahang [8] explored the concept of independent perfect domination in graphs where they introduced that for an independent dominating set to be perfect, every vertex outside of such set is dominated exactly one vertex from an independent dominating set. There are still many papers exploring the notions of different variants of independent domination in graphs, and interested readers can refer to [2] and [5].

Another variant of domination that is quite interesting is the rings domination in graphs which was introduced in 2022 by Abed and Al-Harere [1]. This interesting notion of domination explains that if a dominating set is taken out from the entire graph, the remaining vertices must be adjacent to at least two of the other. This interesting notion was extendedx further by Caay [4] when he introduced equitable rings domination in graphs, which provides additional property of being equitable, Caay and Dondoyano [6] when they introduced perfect rings domination in graphs, which provides additional property of being perfect dominating set, and Necesito and Caay [11] when they introduced the notion of rings convex domination in graphs, which provides the property that a rings dominating set must also be a convex domination set.

In this study, we introduce the concept of independent rings domination in graphs. We define the dominating set to be an independent dominating set and a rings dominating set. In this paper, we explore many properties and examine the characterizations of independent rings domination in many simple graphs and those that are formed by binary operations.

2. Preliminaries and Some Theoretic Notions

This study only considers simple connected graphs. That is, the graphs do not have loops and multiple edges. A pair $G = (V(G), E(G))$ is called a *graph* (on V). The elements of $V(G)$ are called the *vertices* of G, and the elements of $E(G)$ are called the *edges* of G.

We define the *neighborhood* of $v \in (G)$, denoted by $N_G(v)$, as the set

$$N_G(v) := \{u \in V(G) \ : \ uv \in E(G)\}.$$

Given a subset $S \subseteq V(G)$, the sets

$$N_G(S) = N(S) = \bigcup_{v \in S} N_G(v) \ \text{ and } \ N_G[S] = N[S] = S \cup N(S)$$

are the *open neighborhood* and *closed neighborhood* of S, respectively.

Given a vertex $v \in V(G)$, we define the *degree of* v, denoted by $\deg(v)$, to be the number of edges incident to v. The *maximum degree of* G, denoted by $\Delta(G)$, is the degree of the vertex in G having the maximum degree, and the *minimum degree of* G, denoted by $\delta(G)$, is the degree of the vertex in G having the minimum degree. A vertex v in G having the degree equals one less than the order G is said to be a *universal vertex* of G.

The *join* $G+H$ of the two graphs G and H is the graph with vertex set

$$V(G+H) = V(G) + V(H),$$

and the edge set

$$E(G+H) = E(G) \cup (H) \cup \{uv \ : \ u \in V(G), v \in V(H)\}.$$

The *corona* $G \circ H$ of two graphs G and H is the graph obtained by taking one copy of G of order n and n copies of H, and then joining the ith vertex of G to every vertex in the ith copy of H.

Definition 2.1. A subset $S \subseteq V(G)$ is an independent set if for any two distinct elements $u, v \in S$, $uv \notin E(G)$.

Lemma 2.2. *Let $u \in V(G)$. Then $\{u\}$ is an independent set of G.*

Definition 2.3. [3] A subset $S \subseteq V(G)$ is a dominating set of G if for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $uv \in E(G)$. That is, $N[S] = V(G)$. The minimum cardinality of a dominating set S of G is called the domination number of G, and is denoted by $\gamma(G)$.

Definition 2.4. [9] A dominating set S of G is an independent dominating set of G if it is both independent set and a dominating set. The independent domination number of G, denoted by $\gamma_i(G)$, is the smallest cardinality of an independent dominating set of G.

Remark 2.5. The following are the remarks of a rings dominating set of a graph.

i. If S is a dominating set (respectively, independent dominating set) of G and $u \in S$ with $uv \in E(G)$ for some $v \in V(G) \setminus S$, we say that u dominates (respectively, independent dominates) v or v is dominated (respectively, independent dominated) by u. Furthermore, S dominates (respectively, independent dominates) $V(G) \setminus S$.

ii. If a dominating set S has the cardinality equals $\gamma(G)$, then S is said to be γ-set. If an independent dominating set S' has the cardinality equals $\gamma_i(G)$, then S' is said to be γ_i-set.

Definition 2.6. [1] A dominating set $S \subset V(G)$ in G is a rings dominating set if each vertex $v \in V(G) \setminus S$ is adjacent to at least two vertices in $V(G) \setminus S$. The minimum cardinality of a rings dominating set S of G is called the rings domination number of G, and is denoted by $\gamma_{ri}(G)$.

Remark 2.7. For a rings dominating set S to any graoh G of order n, we have

i. The order of G is $n \geq 4$.

ii. For each vertex $v \in V(G) \setminus S$, $\deg(v) \geq 3$

iii. $3 \leq |V(G) \setminus S| \leq n-1$

iv. $1 \leq \gamma_{ri}(G) \leq |S| \leq n-3$

Remark 2.8. The following are the remarks of a rings dominating set of a graph.

i. If S is a rings dominating set of G and $u \in S$ with $uv \in E(G)$ for some $v \in V(G) \setminus S$, we say that u rings dominates v or v is rings dominated by u. Furthermore, S rings dominates $V(G) \setminus S$.

ii. If a rings dominating set S of a graph G equals $\gamma_{ri}(G)$, then S is a γ_{ri}-set of G.

Theorem 2.9. [1] *Trees have no rings dominating set.*

In this study, we present the working main definition of the paper.

Definition 2.10. A dominating set $S \subset V(G)$ of a graph G is said to be independent rings dominating set of G if it is both an independent dominating set and rings dominating set of G. The minimum cardinality of an independent rings dominating set of G is called the independent rings domination number of G, and is denoted by $\gamma_{iri}(G)$.

Example 2.11. Consider the graph in Figure 1. Observe that $S = \{u_3, u_5, u_8\}$ dominates the other vertices of G. Since u_3, u_5 and u_8 are not adjacent to each other, S forms an independent dominating set. Observe that $V(G) \setminus S = \{u_1, u_2, u_4, u_6, u_7\}$. Now u_1 is adjacent to u_4 and u_7, u_2 is adjacent to u_1 and u_4, u_4 is adjacent to u_1, u_2 and u_6, the vertext u_6 is adjacent to u_4 and u_7, and the vertex u_7 is adjacent to u_1 and u_6. Thus, S forms a rings dominating set. Thus, S is an independent rings dominating set of G.

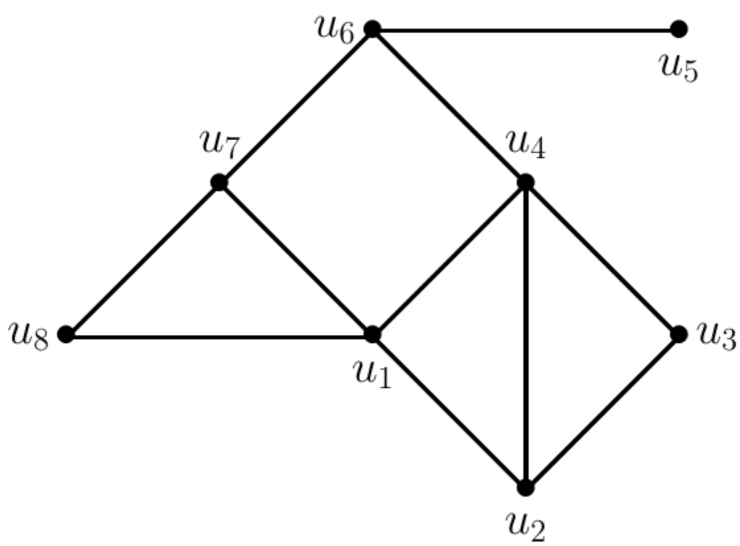

FIGURE 1. Illustration of independent rings domination

3. Independent Rings Dominating Set on Some Graphs

We first present the main condition for which the independent rings dominating set in G exists.

Theorem 3.1. *Let G be any graph. Then G contains a subgraph G' such that $G' \cong C_k$, $k \geq 3$, and $V(G) \backslash V(G')$ forms an independent set if and only if there exists an independent rings dominating set of G.*

Proof. The proof is obvious and directly follows from the definition. □

Remark 3.2. Following Theorem 3.1, if S is an independent rings dominating set of G, then $|V(G) \backslash S| \geq 3$.

Theorem 3.3. *Let G be any graph of order $n \geq 4$ having a universal vertex. Then $\gamma_{iri}(G) = 1$ if and only if $\delta(G) \geq 3$.*

Proof. Assume $\gamma_{iri}(G) = 1$. Then there exists $u \in V(G)$ such that $\{u\}$ is γ_{iri}-set. By Lemma 2.2, $G \setminus u \cong C_k$, $k \geq 3$. Thus, every vertices $v \in G \setminus u$, $\deg(v) \geq 2$. But since u dominates all vertices n $G \setminus u$, it follows that $\deg(v) \geq 3$. Conversely, assume $\deg(v) \geq 3$. If $u \in V(G)$ is a universal in G, then u dominates all other vertices of G. Thus, $\deg_{G \setminus u}(v) \geq 2$, which means that every vertices outside of $\{u\}$ is adjacent to at least two vertices outside of $\{u\}$. This satisfies the definition that $\{u\}$ is γ_{ri}-set of G. By Lemma 2.2, $\{u\}$ is also γ_i-set of G. Hence, $\{u\}$ is a γ_{iri}-set of G. Therefore, $\gamma_{iri}(G) = 1$. □

Example 3.4. Consider the complete graph K_5 in Figure 2. Observe that every vertices of K_5 are universal vertices, and so any singleton subset of $V(K_5)$ is independent set. In particular, $\{u_1\}$ is γ_i-set of K_5, and since every vertices of $V(K_5) \setminus \{u_1\}$ is adjacent to all other vertices of K_5, it follows that $\{u_1\}$ is γ_{iri}-set of K_5. Therefore, $\gamma_{iri}(K_5) = 1$.

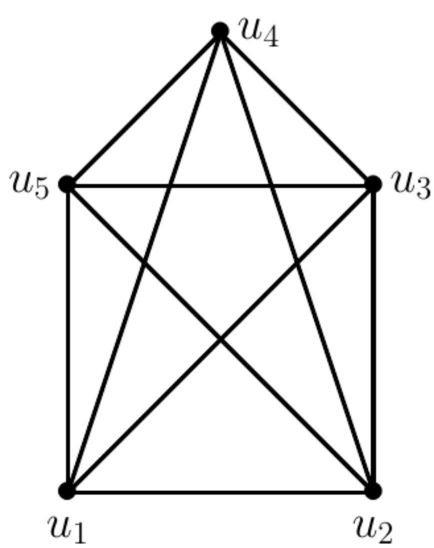

Figure 2. Complete graph of order 5.

Remark 3.5. In generalization of Example 3.4, for any $n \geq 4$, $\gamma_{iri}(K_n) = 1$.

The following claim shows the non-existence, and the proof is obvious.

Proposition 3.6. *Any graph tree T_n has no independent rings dominating set. In particular, P_n and S_n have no independent rings dominating set. Moreover, C_n has no independent rings dominating set.*

Proposition 3.7. *Let $G = K_{P_1}, \cdots, K_{P_k}$ be a complete k-partite graph with $k \geq 4$. Then*

$$\gamma_{iri}(G) = \min\{|P_i| \ : \ P_i \text{ is a vertex partition in } G, \ i = 1, \cdots, k\}.$$

Proof. Let $P_1, \cdots, P_k$ be the k vertex partitions of G. Let a_{ij} be the jth vertex of the partition P_i, $i = 1, \cdots, k$, $j = 1, \cdots, |P_i|$. Note that $a_{st}a_{ij} \in E(G)$, for all $s \neq i$, $t = 1, \cdots, |P_s|$. This means that a vertex a_{st} dominates all vertices of $V(G) \setminus J$, where $J = \{a_{sw} \in P_s \mid w \neq t, \ s = 1, \cdots, k\}$. Hence, $J = P_s$ forms a dominating set of G, for any $s = 1, \cdots, k$. Also, observe that $a_{st}a_{sw} \notin E(G)$, for all $t \neq w$. This means that all vertices in the same partitions are not adjacent to each other. Thus, $J = P_s$ forms an independent set of G. Since $k \geq 4$, it follows that if a_{pr} is dominated by a_{st}, $s \neq p$, $r = 1, \cdots, |P_p|$, it follows that a_{pr} is adjacent to atleast 2 vertices not in P_s and P_p. Hence, $J = P_s$ forms an independent rings dominating set. Since, P_s is arbitrary, any vertex partition of G forms an independent rings dominating set of G. Hence, the vertex partition whose cardinality is the least is the γ_{iri}-set of G. Therefore, $\gamma_{iri}(G) = \min\{|P_i| \ : \ P_i \text{ is a vertex partition of } G, \ i = 1, \cdots, k\}$. □

Proposition 3.8. *Let $G = K_{n,m}$ be any bipartite graphs of vertex partitions G_1 and G_2 of cardinalities $n \geq 3$ and $m \geq 3$, respectively. Let $S \subseteq V(G)$, then only one of the following holds:*

i. *S is a rings dominating set of G, and S is not an independent dominating set of G*
ii. *S is an independent dominating set of G, and S is not a rings dominating set of G.*

Consequently, there does not exist an independent rings dominating set of G.

Proof. Let G_1 and G_2 be vertex partitions of G such that $|G_1| = n \geq 3$ and $|G_2| = m \geq 3$. Let $u_{1i} \in G_1$ and $u_{2j} \in G_2$, $i = 1, \cdots, n$, $j = 1, \cdots, m$. Observe that $u_{1i}u_{1k} \notin E(G)$, for all $i \neq k$, and $u_{2j}u_{2l} \notin E(G)$, for all $j \neq l$. Let $S \subseteq V(G)$.

Now, if S is an independent dominating set, then every vertices in S are not adjacent. Since $u_{1i}u_{1k} \notin E(G)$, for all $i \neq k$, it follows that G is an independent dominating set of G. But $u_{2j}u_{2l} \in E(G)$, implying that $V(G) \setminus G_1$ are all isolated and so every vertices in $V(G) \setminus G_1$ does not have degree at least two in $V(G) \setminus G_1$. Thus, S is not a rings dominating set.

Now if S is a rings dominating set of G. Then $V(G) \setminus S$ has vertices of degree at least two in $V(G) \setminus S$. Let $u \in V(G) \setminus S$, and without loss of generality, $u \in G_1$. So that $\deg(u) \geq 2$, u must be adjacent to at least two vertices in $V(G) \setminus S$, and so we may pick two vertices in G_2, say v_{2s} and v_{2t}. Thus, v_{2s} and v_{2t} must also have at least two adjacent vertices from G_1. Doing this inductively, we have $S = \overline{G_1} \cup \overline{G_2}$, where $\overline{G_1} \subseteq G_1$ and $\overline{G_2} \subseteq G_2$. Note that if $u_{1x} \in G_1$, then u_{1x} is adjacent to all vertices $u_{2y} \in G_2$, $y = 1, \cdots m$. Similarly, if $u_{2y} \in G_2$, then u_{2y} is adjacent to all vertices $u_{1x} \in G_1$, $x = 1, \cdots n$. Hence $u_{1x}, u_{2y} \in S$, for some x and y. But $u_{1x}u_{2y} \in E(G)$ implying that S is an independent dominating set.

Therefore, there does not exist an independent rings dominating set in G. □

4. Join and Corona of Graphs

Theorem 4.1. *Let G and H be any connected graphs having γ_{iri}-sets S_1 and S_2, respectively. Then $\gamma_{iri}(G + H) = \min\{|S_1|, |S_2|\}$.*

Proof. Let S_1 and S_2 be γ_{iri}-sets of G and H, respectively. Since every vertices of G is adjacent to every vertices of H on $G + H$, it follows that S_1 and S_2 are dominating sets on $G + H$. Since S_1 and S_2 are independent dominating sets, they are independent dominating sets on $G + H$. Now since S_1 is an independent rings dominating set of G, by Remark 3.2, $|V(G) \setminus S_1| \geq 3$. This means that every vertices of $V(G) \setminus S_1$ is adjacent to at least 3 vertices of $V(G) \setminus S_1$. Hence, every vertices of H is adjacent to at least 3 vertices of $V(G) \setminus S_1$. Therefore, S_1 is an independent rings dominating set of $G + H$. Using the similar argument, it follows that S_2 is an independent rings dominating set of G_+H. Hence, the γ_{iri}-set of $G + H$ is either S_1 or S_2, whichever is smaller. Therefore, $\gamma_{iri}(G + H) = \min\{|S_1|, |S_2|\}$. □

Theorem 4.2. *Let G and H be any connected graphs such that there exists a universal vertex v in H and $\delta(H) \geq 2$. Then $\gamma_{iri}(G \circ H) = |G|$.*

Proof. Let $v \in H$ be a universal vertex of H. Then v is a dominating set of H. By Lemma 2.2, $\{v\}$ is an independent dominating set of H. Now, since $\delta() \geq 2$, it follows that for every $u \neq v$ of H, $\deg_{H\setminus\{v\}}(u) \geq 1$. Let $w_i \in V(G)$ such that w_i is adjacent to all vertices of the ith copy of H. Thus, if $\left\{v^{(i)}\right\}$ be an independent dominating set of the ith copy of H, $v^{(i)}$ also dominates w_i. Hence, for all vertex $u^{(i)} \neq v^{(i)}$ of the ith copy $H^{(i)}$ of H, it follows that $\deg_{G\circ H\setminus\{v^{(i)}\}}\left(u^{(i)}\right) \geq 2$. Since there are $|G|$ copies of H, it follows that $S = \bigcup_{i=1}^{|G|}\left\{v^{(i)}\right\}$ forms an independent rings dominating set of $G \circ H$. Therefore, $\gamma_{iri}(G \circ H) = n\left|\left\{v^{(i)}\right\}\right| = n$. □

In Proposition 3.6, it is obvious that P_n and C_n do not have independent rings dominating set. However, the corona will give an exemption.

Theorem 4.3. *Let G be any graph. If $n \equiv 1 \mod 3$, then $\gamma_{iri}(G \circ P_n) = |G|\left\lceil\frac{n}{3}\right\rceil$.*

Proof. Let G be any graph. If $u_i \in V(G)$ is adjacent to all vertices of the ith copy $P_n^{(i)}$ of P_n, then there exists $v_j^{(i)\in V\left(P_n^{(i)}\right)}$ that dominates u_i, $j = 1, \cdots, n$. Note that for all $w_i \in V(P_n)$, we have

$$\deg(w_i) = \begin{cases} 1 & \text{, if } i = 1, n \\ 2, & \text{, otherwise,} \end{cases}$$

where i is the index order arrangement of vertices of P_n.

If w_1 is dominated, then w_1 is either dominated by w_2. But if w_2 dominates w_1, then w_1 is only adjacent to $u_i \in V(G)$ and its degree outside of a dominating set is 1. This is a contradiction to the definition of a rings dominating set. Thus, w_1 must dominate, and it does not to $u_1 \in V(G)$ and w_2 since w_2 is adjacent to w_1. So that w_1 must rings dominate w_2, w_3 and u_i are not dominators since w_2 must be adjacent to at least two that do not dominate. Similarly, w_3 is adjacent to atleast two which is w_2 and u_i. Thus, w_4 must dominate w_3. This means w_4 is a dominator. So that w_4 is a rings dominator, its adjacent w_5 must be adjacent to w_6 and $u_1 \in V(G)$ which are not dominators. Continuing the process, the chosen vertices must be w_1, w_4, w_7 and so on. if we divide the vertices of P_n by 3, the dominators are the first order of the group. Hence, if $n \equiv 1 \mod 3$, then there are $\left\lceil \frac{n}{3} \right\rceil$ groups formed by grouping 3 vertices such that the last group consists of 1 vertex only. Doing this to $|G|$ copies of H, we have

$$\gamma_{iri}(G \circ P_n) = |G| \cdot \left\lceil \frac{n}{3} \right\rceil.$$

This proves the claim. □

Remark 4.4. If $n \equiv 0 \mod 3$ or $n \equiv 2 \mod 3$, then an independent dominating set S_1 does not coincide with rings dominating set S_2. That is, $S_1 \neq S_2$, but $|S_1| = |S_2|$.

To see this, consider $G = \{v\}$ and P_5 in Figure 3. $S_1 = \{v_1, v_3, v_5\}$ is an independent dominating set but not a rings dominating set. Also, $S_2 = \{u, v_1, v_5\}$ is a rings dominating set but not an independent dominating set. Thus, $S_1 \neq S_2$ but $|S_1| = |S_2|$.

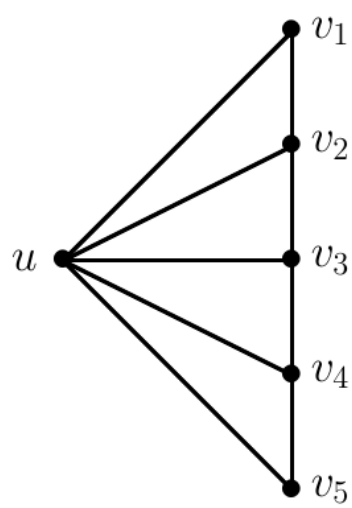

FIGURE 3. $G = \{v\} \circ P_5$.

Theorem 4.5. *Let G be any graph. If $n \equiv 0 \mod 3$, $\gamma_{iri}(G \circ C_n) = |G| \cdot \dfrac{n}{3}$.*

Proof. Let $u_1, u_2, \cdots, u_n$ be an ordered arrangement of the vertices of C_n such that $u_iu_{i+1} \in E(C_n)$, for all $i = 1, \cdots, n$. Consider u_1. Since $u_1u_2 \in E(C_n)$, we may take u_1 to dominate u_2. Since u_2 is adjacent to u_3 and $w_i \in V(G)$, we may take u_3 as vertex dominated by u_4 so that u_1 is a rings dominating vertex. Similarly, u_3 is adjacent to u_2 and $w_i \in V(G)$, and so u_4 must rings dominate u_3. Continuing the process, the vertices that rings dominate the other are u_1, u_4, u_7 and so on. Let P_i be partitions of $V(C_n)$ such that $|P_i| = 3$ and the rings dominators u_1, u_4, u_7 and so on are the first vertex in each P_i. Since $n \equiv 0 \mod 3$, the order of P_i divides the order of C_n. Hence there are $\frac{n}{3}$ copies of P_i and each contains 3 vertices with 1 vertex that rings dominate. Thus, there are $\frac{n}{3}$ rings dominating vertices. Observe that there are $|G$ copies of P_n in $G \circ P_n$, and so we have $\gamma_{iri}(G \circ C_n) = |G| \cdot \dfrac{n}{3}$. This proves the claim. □

Corollary 4.6. *There does not exist independent rings dominating set for $G \circ C_n$ for n not divisible by 3. However, there exists a rings dominating set $G \circ C_n$ and $\gamma_{ri}(G \circ C_n) = |G| \cdot \left\lceil \dfrac{n}{3} \right\rceil$.*

Conclusion

The notion of rings domination in graphs which was introduced by Abed and Al-Harere in [1] is further extended into by adding an additional property that a domination is also independent domination. Thus, this paper is developed into independent rings domination in graphs. In this context, we present the characterizaion of the general form of a graph of which this domination exists. We shows some results of graaphs formed by binary operations such as corona and join of two graphs.

For future development, we recommend to interested researchers to explore more on this concept to some binary operations not mention on this paper, and to some significant unary operations in graphs.

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