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EXISTENCE AND UNIQUENESS OF SOLUTION FOR THE ADVECTION-DIFFUSION EQUATION WITH A FRACTIONAL LAPLACIAN OPERATOR

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Abstract. We consider an advection-diffusion problem involving a fractional Laplace operator of order $0 < s < 1$. First, we give the preliminaries that will be used in the following to prove the existence of the solution to the problem under consideration using Lions' theorem. Finally, we show that this solution is unique.

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \in \mathbb{N} \setminus \{0\}$ be an open bounded domain with a Lipschitz continuous boundary $\partial\Omega$. For $T > 0$, we set $Q := \Omega \times (0, T)$ and $\Sigma := (\mathbb{R}^N \setminus \Omega) \times (0, T)$. Consider the advection-diffusion equation involving the Laplace fractional operator of order $0 < s < 1$:

$$\begin{cases} \dfrac{\partial \Theta}{\partial t} + v.\nabla\Theta + (-\Delta)^s\Theta &= f \quad \text{in} \quad Q, \\ \Theta &= 0 \quad \text{in} \quad \Sigma, \\ \Theta(0) &= \Theta^0 \quad \text{in} \quad \Omega. \end{cases} \tag{1}$$

where $T > 0$, the functions $\Theta^0 \in L^2(\Omega)$, $f \in L^2(Q)$. The function $v \in [L^\infty(Q)]^N$ and the operator $(-\Delta)^s$ $(0 < s < 1)$ denotes the fractional Laplace operator given formally for a suitable function ψ by:

$$(-\Delta)^s\psi = C_{N,s} \, \text{P.V.} \int_{\mathbb{R}^N} \frac{\psi(x) - \psi(y)}{|x - y|^{N+2s}} \, dy,$$

where P.V denotes the principal value and $C_{N,s}$ is a normalization constant depending only on N and s. We refer to Section 2 for more details on fractional Laplacian.

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In what follows we assume that

$$\operatorname{div}(v) = 0 \text{ in } Q. \tag{2}$$

Fractional derivatives are an excellent tool for describing the memory and hereditary effects of various materials and processes [19,31]. They have been successfully applied by many researchers over the past decade in several scientific fields such as biology [22], physics [21], chemistry [23] and even finance [30].

The question of the study of the existence and uniqueness of the solution of the fractional advection-diffusion problem has attracted the attention of many authors and several techniques have been developed to solve such problems. Miloš Japundžić et *al.* [26] are interested in a purely theoretical approach to analyze this type of problem. Using the theory of generalized and uniformly continuous operator semi-groups, the authors established and proved the existence and uniqueness of a solution in a certain Colombeau space (see [27,28]).

Victor Ginting and *al.* [20], were also interested in the theoretical analysis of the same group of problem. Through the analysis of the Fourier transform and using Riemann-Liouville derivatives and the appropriate fractional Sobolev space, they also established the existence and uniqueness of the strong solution of the fractional equation of advection-diffusion-stationary reaction.

As for Jingjun Zhao [29], they presented an efficient finite element method for a fractional Riesz advection-diffusion problem with multiple terms, in space and time. They proved the existence and uniqueness of the weak solution by the Lax-Milgram theorem. In all the papers cited, the authors examined the problem with purely fractional derivatives instead of the classical spatial derivatives.

In this paper, we bring a new approach to this type of problem compared to the existing literature to our knowledge. The difference between the problem under consideration and existing problems is that, in the classical advection-diffusion equation, we replace the classical Laplace operator with a fractional Laplace operator that favours diffusion [25]. Hence the study of existence and uniqueness of the solution. To begin with, we prove the existence of a solution to the problem considered using the Lions' theorem which remains to our knowledge a new technique for the theoretical analysis of such a problem before showing that this solution is unique.

To structure our work properly, the rest of the article is organised as follows. Section 1 is devoted to the introduction and in section 2 we give the preliminaries on fractional calculus. In section 3, using Lions' theorem, we prove the existence of the solution of the advection-diffusion problem involving a fractional Laplace operator in infinite dimension, and that this solution is unique.

2. Preliminaries

In this section we fix some notations and recall some known results as they are needed throughout the paper. These results can be found for example in [1,3,5,6,9–12,14,16,17] and the references therein.

First, let's give a rigorous definition of the fractional Laplacian. For $0 < s < 1$, we have:

$$\mathcal{L}^1_s(\mathbb{R}^N) = \left\{ w : \mathbb{R}^N \to \mathbb{R} \text{ measurable and } \int_{\mathbb{R}^N} \frac{|w(x)|}{(1+|x|)^{N+2s}}\,\mathrm{d}x < \infty \right\}.$$

For $w \in \mathcal{L}^1_s(\mathbb{R}^N)$ and , we establish

$$(-\Delta)^s w(x) = C_{N,s} \int_{\{y\in\mathbb{R}^N:\,|x-y|>\varepsilon\}} \frac{w(x)-w(y)}{|x-y|^{N+2s}}\,\mathrm{d}y, \quad x \in \mathbb{R}^N,$$

where $C_{N,s}$ is a normalisation constant given by

$$C_{N,s} = \frac{s2^{2s}\Gamma\left(\frac{2s+N}{2}\right)}{\pi^{\frac{N}{2}}\Gamma(1-s)}. \tag{3}$$

The fractional Laplacian $(-\Delta)^s$ is defined by the following singular integral:

$$(-\Delta)^s w(x) = C_{N,s}\,\text{P.V.} \int_{\mathbb{R}^N} \frac{w(x)-w(y)}{|x-y|^{N+2s}}\,\mathrm{d}y = \lim_{\epsilon\to 0}(-\Delta)^s_\epsilon w(x), \quad x \in \mathbb{R}^N, \tag{4}$$

provided that the limit exists for $x \in \mathbb{R}^N$ almost everywhere. We refer to [1] and their references concerning the class of functions for which the limit in (4) exists almost everywhere for $x \in \mathbb{R}^N$.

Next, we will introduce the function spaces necessary for the theoretical study of our problems. We begin with fractional order Sobolev spaces.

Let $\Omega \subset \mathbb{R}^N$ be an arbitrary open set. Given that $0 < s < 1$ is a real number, we have:

$$H^s(\Omega) = \left\{ w \in L^2(\Omega) : \int_\Omega\int_\Omega \frac{|w(x)-w(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y < \infty \right\},$$

and we give it the norm defined by

$$\|w\|_{H^s(\Omega)} = \left(\int_\Omega |w(x)|^2\,\mathrm{d}x + \int_\Omega\int_\Omega \frac{|w(x)-w(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2}}.$$

We note that:

$$H^s_0(\Omega) = \left\{ w \in H^s(\mathbb{R}^N) :\ w = 0 \ \text{ in } \ \mathbb{R}^N \setminus \Omega \right\}.$$

and we consider the bilinear function $\mathcal{F} : H^s_0(\Omega) \times H^s_0(\Omega) \to \mathbb{R}$ given by

$$\mathcal{F}(\varphi,\psi) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\varphi(x)-\varphi(y))(\psi(x)-\psi(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y. \tag{5}$$

Then $H^s_0(\Omega)$ has the norm

$$\|w\|_{H^s_0(\Omega)} = \left(\mathcal{F}(w,w)\right)^{1/2}, \tag{6}$$

is a Hilbert space (see for example [2, Lemma 7]). Let $H^{-s}(\Omega) = (H^s_0(\Omega))^\star$ be the dual space of $H^s_0(\Omega)$ with respect to the pivot space $L^2(\Omega)$, so that we have the following dense continuous injections (see for example [13]):

$$H^s_0(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-s}(\Omega). \tag{7}$$

Assuming that Ω is bounded and has a continuous Lipschitz limit, we have that $\mathcal{D}(\Omega)$ is dense in $H_0^s(\Omega)$ (for every $0 < s < 1$, see for example [4]), and by [3, Chapter 1], if $0 < s \neq 1/2 < 1$, then

$$H_0^s(\Omega) = \overline{\mathcal{D}(\Omega)}^{H^s(\Omega)},$$

with equivalent norms, where $\mathcal{D}(\Omega)$ denotes the space of all continuously infinitely differentiable functions with compact support in Ω. But if $s = 1/2$, then $H_0^s(\Omega)$ is a proper subspace of $\overline{\mathcal{D}(\Omega)}^{H^s(\Omega)}$.

For more information on fractional order Sobolev spaces, we refer to [1,3,11] and their references.

Then, for $\varphi \in H^s(\mathbb{R}^N)$ we introduce the *non-local normal derivative* $\mathcal{N}_s$ given by

$$\mathcal{N}_s\varphi(x) = C_{N,s}\int_\Omega \frac{\varphi(x)-\varphi(y)}{|x-y|^{N+2s}}\,\mathrm{d}y,\ x \in \mathbb{R}^N \setminus \overline{\Omega}, \tag{8}$$

where $C_{N,s}$ is the constant given in (3). *We note that the non-local normal derivative $\mathcal{N}_s$ was first introduced in* [15].

The following integration by parts formula is contained in [8,15]. Let $\varphi \in H^s(\mathbb{R}^N)$ be such that $(-\Delta)^s\varphi \in L^2(\Omega)$ and $\mathcal{N}_s\varphi \in L^2(\mathbb{R}^N \setminus \overline{\Omega})$. Then, for each $\psi \in H^s(\mathbb{R}^N)$, the identity

$$\begin{aligned}\frac{C_{N,s}}{2}\int\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}^N\setminus\Omega)^2} &\frac{(\varphi(x)-\varphi(y))(\psi(x)-\psi(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_\Omega \psi(-\Delta)^s\varphi\,\mathrm{d}x + \int_{\mathbb{R}^N\setminus\Omega}\psi\mathcal{N}_s\varphi\,\mathrm{d}x \end{aligned}\tag{9}$$

holds.

If $C = 0$ in $\mathbb{R}^N \setminus \Omega$ or $\psi = 0$ in $\mathbb{R}^N \setminus \Omega$, then

$$\int\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}^N\setminus\Omega)^2} \frac{(C(x)-C(y))(\psi(x)-\psi(y))}{|x-y|^{N+2s}}\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(C(x)-C(y))(\psi(x)-\psi(y))}{|x-y|^{N+2s}}\mathrm{d}x\,\mathrm{d}y,$$

and the integration by part formula (9) becomes

$$\frac{C_{N,s}}{2}\int\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}^N\setminus\Omega)^2} \frac{(C(x)-C(y))(\psi(x)-\psi(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega \psi(-\Delta)^s C\,\mathrm{d}x \quad \forall C,\psi \in H^s(\mathbb{R}^N) \tag{10}$$

Next, we introduce the classical first-order Sobolev space.

$$H^1(\Omega) = \left\{u \in L^2(\Omega):\ \int_\Omega |\nabla u|^2\,\mathrm{d}x < \infty\right\}$$

which has the standard defined by

$$\|u\|_{H^1(\Omega)} = \left(\int_\Omega |u|^2\,\mathrm{d}x + \int_\Omega |\nabla u|^2\,\mathrm{d}x\right)^{\frac{1}{2}}.$$

In order to study the solvency of (1), we will also need the following space of functions

$$\mathbb{H}_0^1(\Omega) = \left\{w \in H^1(\mathbb{R}^N):\ w \equiv 0 \text{ in } \mathbb{R}^N \setminus \Omega\right\}. \tag{11}$$

We have $\mathbb{H}_0^1(\Omega) \subset H_0^1(\Omega)$ and with the scalar product

$$\int_\Omega \nabla w \cdot \nabla \varphi \, \mathrm{d}x,$$

and the associated norm

$$\|\varphi\|_{\mathbb{H}_0^1(\Omega)} = \|\nabla \varphi\|_{L^2(\Omega)}, \tag{12}$$

$\mathbb{H}_0^1(\Omega)$ is a (real) Hilbert space.

In the following, we will assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with a regular $\partial\Omega$ boundary. Under this assumption, we have the following dense continuous injection for every $0 < s < 1$ (see for example [1,3]):

$$\mathbb{H}_0^1(\Omega) \hookrightarrow H_0^s(\Omega). \tag{13}$$

This means that there exists $C_{st} > 0$ such that

$$\|C\|_{H_0^s(\Omega)} \leq C_{st} \|\nabla C\|_{L^2(\Omega}. \tag{14}$$

Therefore, using (7), we obtain the continuous injection for $0 < s < 1$:

$$\mathbb{H}_0^1(\Omega) \hookrightarrow H_0^s(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-s}(\Omega) \hookrightarrow \left(\mathbb{H}_0^1(\Omega)\right)^\star. \tag{15}$$

We conclude this section by giving the following result. Let $T > 0$ be a real number and $\mathbb{X}^\star$ be the dual of $\mathbb{X}$. We define

$$\mathbb{W}((0,T);\mathbb{X}) = \left\{ \zeta \in L^2((0,T);\mathbb{X}) : \frac{\partial \zeta}{\partial t} \in L^2\left((0,T);\mathbb{X}^\star\right) \right\}. \tag{16}$$

Then $\mathbb{W}((0,T);\mathbb{X})$ has the norm given by

$$\|\psi\|^2_{\mathbb{W}((0,T);\mathbb{X})} = \|\psi\|^2_{L^2((0,T);\mathbb{X})} + \left\| \frac{\partial \psi}{\partial t} \right\|^2_{L^2\left((0,T);\mathbb{X}^\star\right)}, \; \forall \psi \in \mathbb{W}((0,T);\mathbb{X}), \tag{17}$$

is a Hilbert space. Moreover, if $\mathbb{Y}$ is a Hilbert space which can be identified with its dual $\mathbb{Y}^\star$ and we have the continuous injections $\mathbb{X} \hookrightarrow \mathbb{Y} = \mathbb{Y}^\star \hookrightarrow \mathbb{X}^\star$, then using [7] we have the continuous injection

$$\mathbb{W}((0,T);\mathbb{X}) \hookrightarrow \mathcal{C}([0,T];\mathbb{Y}). \tag{18}$$

We recall the following result from [18, page 37] which will be useful in proving the existence results for the solution to our problem:

Theorem 2.1. *Let $\left(F, \|\cdot\|_F\right)$ be a Hilbert space. Let Φ be a subspace of F with a pre-Hilbert scalar product $(((\cdot,\cdot)))$. Furthermore, $E : F \times \Phi \to \mathbb{C}$ is a sesquilinear form. Assume that the following assumptions hold:*

(1) *The injection $\Phi \hookrightarrow F$ is continuous, i.e. there exists a constant $C_{ste1} > 0$ such that*

$$\|\varphi\|_F \leq C_{ste1} |||\varphi||| \; \forall \; \varphi \; in \; \Phi. \tag{19}$$

(2) *For any $\varphi \in \Phi$, the application $u \mapsto E(u, \varphi)$ is continuous on F.*

(3) *There exists a constant* $C_{ste2} > 0$ *such that*

$$|E(\varphi,\varphi)| \geq C_{ste2}|||\varphi|||^2 \text{ for all } \varphi \in \Phi. \tag{20}$$

(4) *If* $\varphi \mapsto L(\varphi)$ *is a semi-linear form on* Φ*, then there exists a function* $u \in F$ *satisfying*

$$E(u,\varphi) = L(\varphi) \text{ for all } \varphi \in \Phi. \tag{21}$$

3. Results for the existence and uniqueness of the solution

In this section we focus on the existence and uniqueness of the solution of (1).

Definition 3.1. *Let* $0 < s < 1$*,* $f \in L^2((0,T);H^{-s}(\Omega))$ *and* $\Theta^0 \in L^2(\Omega)$*. Let also* $\mathcal{F}$ *be defined in* (5) *and* $v \in [L^\infty(Q)]^N$ *such that* (2) *is true. We will say that* (1) *has a weak solution* $\Theta \in L^2((0,T),H_0^s(\Omega))$*, if the following equality holds:*

$$\begin{aligned} \int_0^T \langle f(t),\rho(t)\rangle_{H^{-s}(\Omega),H_0^s(\Omega)}\,dt + \int_\Omega \Theta^0\rho(0)dx &= -\int_0^T \left\langle \Theta(t), \frac{\partial\rho(t)}{\partial t}\right\rangle_{H_0^s(\Omega),H^{-s}(\Omega)} dt \\ &- \int_Q v\cdot\nabla\rho\,\Theta\,dxdt \\ &+ \frac{C_{N,s}}{2}\int_0^T \mathcal{F}(\Theta(t),\rho(t))dt, \end{aligned} \tag{22}$$

For all $\rho \in H(Q)$ *where*

$$H(Q) = \left\{\rho \in L^2((0,T),\mathbb{H}_0^1(\Omega));\ \frac{\partial\rho}{\partial t} \in L^2((0,T);H^{-s}(\Omega)),\ \rho(T)=0 \ \text{ dans } \Omega\right\}. \tag{23}$$

Remark 3.1. *Note that if* $\rho \in L^2((0,T),\mathbb{H}_0^1(\Omega))$ *and* $\dfrac{\partial\rho}{\partial t} \in L^2((0,T);H^{-s}(\Omega))$*, then* $\rho \in \mathbb{W}((0,T),H_0^s(\Omega))$*, it then follows from* (18) *that* $\rho \in \mathcal{C}([0,T];L^2(\Omega))$*.*

Theorem 3.1. *For* $0 < s < 1$*,* $f \in L^2((0,T);H^{-s}(\Omega))$ *and* $\Theta^0 \in L^2(\Omega)$ *let* $\mathcal{F}$ *be defined in* (5) *and* $v \in [L^\infty(Q)]^N$ *such that* (2) *is true. Then there is a unique weak solution* $\Theta \in L^2((0,T),H_0^s(\Omega)) \cap \mathcal{C}([0,T];L^2(\Omega))$ *to* (1) *in the sense of the definition* 3.1 *such that*

$$\sup_{s\in[0,T]} \|\Theta(s)\|^2_{L^2(\Omega)} \leq \frac{2}{C_{N,s}}\|f\|^2_{L^2((0,T);H^{-s}(\Omega))} + \|\Theta^0\|^2_{L^2(\Omega)} \tag{24}$$

and

$$\|\Theta\|^2_{L^2((0,T);H_0^s(\Omega))} \leq \frac{16}{C_{N,s}^2}\|f\|^2_{L^2((0,T);H^{-s}(\Omega))} + \frac{8}{C_{N,s}}\|\Theta^0\|^2_{L^2(\Omega)} \tag{25}$$

Proof. We will proceed in three stages:

Step 1: We use Theorem 2.1 to prove the existence of $\Theta \in L^2((0,T);H_0^s(\Omega))$.

Recall that the norm on $L^2((0,T);H_0^s(\Omega))$ is given by

$$\int_0^T \|\varphi(t)\|^2_{H_0^s(\Omega)}dt = \int_0^T \mathcal{F}(\varphi(t),\varphi(t))dt$$

, where $\mathcal{F}$ is defined in (5). Using the injection (13), we define the norm on $H(Q)$ by

$$\|\Theta\|^2_{H(Q)} \quad = \quad \int_0^T \|\Theta(t)\|^2_{H_0^s(\Omega)}dt + \|\Theta(0)\|^2_{L^2\Omega)}, \ \forall\Theta \in H(Q). \tag{26}$$

Then, for all $Theta \in H(Q)$,

$$\|\Theta\|_{L^2((0,T);H_0^s(\Omega))} \leq \|\Theta\|_{H(Q)}.$$

This shows that the injection $H(Q) \hookrightarrow L^2((0,T);H_0^s(\Omega))$ is continuous.

Now for $\rho \in H(Q)$, consider the bilinear form $E(\cdot,\cdot)$ defined on $L^2((0,T);H_0^s(\Omega)) \times H(Q)$ by

$$\begin{aligned} E(\Theta,\rho) \quad &= \quad -\int_0^T \left\langle \Theta(t), \frac{\partial\rho(t)}{\partial t} \right\rangle_{H_0^s(\Omega),H^{-s}(\Omega)} dt \\ &- \quad \int_Q v\cdot\nabla\rho\,\Theta\,dxdt + \frac{C_{N,s}}{2}\int_0^T \mathcal{F}((t),\rho(t))dt, \end{aligned} \tag{27}$$

where $\mathcal{F}(\cdot,\cdot)$ is given by (5).

Using the Cauchy-Schwarz inequality, the continuity of the bilinear form $\mathcal{F}$ and (15), it follows that there exists $C_{ste} > 0$, such that

$$\begin{aligned} |E(\Theta,\rho)| \quad &\leq \quad \|\Theta\|_{L^2((0,T);H_0^s(\Omega))} \left\| \frac{\partial\rho}{\partial t} \right\|_{L^2((0,T);H^{-s}(\Omega))} + \|v\|_{L^\infty(Q)}\|\nabla\rho\|_{L^2(Q)}\|\Theta\|_{L^2(Q)} \\ &+ \quad \frac{C_{N,s}}{2}\|\Theta\|_{L^2((0,T);H_0^s(\Omega))}\|\rho\|_{L^2((0,T);H_0^s(\Omega))} \\ &\leq \quad \|\Theta\|_{L^2((0,T);H_0^s(\Omega))} \left\| \frac{\partial\rho}{\partial t} \right\|_{L^2((0,T);H^{-s}(\Omega))} + C_{ste}\|v\|_{L^\infty(Q)}\|\nabla\rho\|_{L^2(Q)}\|\Theta\|_{L^2((0,T);H_0^s(\Omega))} \\ &+ \quad \frac{C_{N,s}}{2}\|\Theta\|_{L^2((0,T);H_0^s(\Omega))}\|\rho\|_{L^2((0,T);H_0^s(\Omega))} \\ &\leq \quad C_{ste}\|\Theta\|_{L^2((0,T);H_0^s(\Omega))} \end{aligned}$$

with

$$C_{ste} = \left[\left\| \frac{\partial\rho}{\partial t} \right\|_{L^2((0,T);H^{-s}(\Omega))} + C_{ste}\|v\|_{L^\infty(Q)}\|\nabla\rho\|_{L^2(Q)} + \frac{C_{N,s}}{2}\|\rho\|_{L^2((0,T);H_0^s(\Omega))} \right] > 0.$$

So

$$|E(\Theta,\rho)| \leq C_{ste}\|\Theta\|_{L^2((0,T);H_0^s(\Omega))}.$$

Consequently, for all $\rho \in H(Q)$, the function $\Theta \mapsto E(\Theta,\rho)$ is continuous on $L^2((0,T);H_0^s(\Omega))$.

Then, using integration by parts and (2), it follows that for all $\rho \in H(Q)$, we have

$$\begin{aligned} E(\rho,\rho) &= \int_Q \rho \frac{\partial \rho}{\partial t} dt dx + \int_Q v.\nabla \rho \ \rho dt dx + \frac{C_{N,s}}{2} \int_0^T \mathcal{F}(\rho(t),\rho(t)) dt \\ &= \frac{1}{2}\|\rho(0)\|^2_{L^2(\Omega)} + \frac{C_{N,s}}{2}\|\rho\|^2_{L^2((0,T);H^s_0(\Omega))} \\ &\geq \frac{1}{2}\min(1,C_{N,s})\|\rho\|^2_{H(Q)}. \end{aligned}$$

Finally, we define the functional $L(\cdot) : H(Q) \to \mathbb{R}$ by

$$L(\rho) = \int_0^T \langle f(t),\rho(t)\rangle_{H^{-s}(\Omega),H^s_0(\Omega)} dt + \int_\Omega \Theta^0 \rho(0) dx, \ \forall \rho \in H(Q). \tag{28}$$

using Cauchy Schwarz and the continuous injection (7), we get

$$\begin{aligned} |L(\rho)| &\leq \|f\|_{L^2((0,T);H^{-s}(\Omega))}\|\rho\|_{L^2((0,T);H^s_0(\Omega))} + \|\Theta^0\|_{L^2(\Omega)}\|\rho(0)\|_{L^2(Q)} \\ &\leq \left(\|f\|^2_{L^2((0,T);H^{-s}(\Omega))} + \|\Theta^0\|^2_{L^2(\Omega)}\right)^{1/2}\left(\|\rho\|^2_{L^2((0,T);H^s_0(\Omega))} + \|\rho(0)\|^2_{L^2(Q)}\right)^{1/2} \\ &= C_{ste}\|\rho\|_{H(Q)}, \end{aligned}$$

with $C_{ste} = \left(\|f\|^2_{L^2((0,T);H^{-s}(\Omega))} + \|\Theta^0\|^2_{L^2(\Omega)}\right)^{1/2} > 0.$

We deduce that the functional $L(\cdot)$ is continuous on $H(Q)$. Consequently, there exists $Theta \in L^2((0,T);H^s_0(\Omega)))$ such that

$$E(\Theta,\rho) = L(\rho), \quad \forall \varphi \in H(Q). \tag{29}$$

Consequently, we can say that the system (1) has a weak solution $\Theta \in L^2((0,T);H^s_0(\Omega)))$ in the sense of the definition 3.1.

Step 2: We prove the uniqueness of the solution of (1).

Suppose that (1) has two solutions Θ^1 and Θ^2 in $L^2((0,T);H^s_0(\Omega))$ with the same data $v,\ f$ and Θ^0. If $z = \Theta^1 - \Theta^2$, then $z \in L^2((0,T);H^s_0(\Omega))$ is a solution of

$$\left\{ \begin{aligned} \frac{\partial z}{\partial t} + v.\nabla z + (-\Delta)^s z &= 0 \quad \text{in} \quad Q, \\ z &= 0 \quad \text{on} \quad \Sigma, \\ z(0) &= 0 \quad \text{in} \quad \Omega. \end{aligned} \right. \tag{30}$$

If we multiply the first equation in (30) by z and integrate by parts, we obtain that

$$\frac{1}{2}\|z(T)\|^2_{L^2(\Omega)} + \frac{C_{N,s}}{2}\|z\|^2_{L^2((0,T);H^s_0(\Omega))} = 0.$$

This implies that $|z\|^2_{L^2((0,T);H^s_0(\Omega))} = 0$ and therefore $z = 0$ in $\mathbb{R}^N$.This means that $\Theta^1 = \Theta^2$ in $\mathbb{R}^N$.

Step 3: We prove (24) and (25).

If we multiply the first equation of the (1) by Θ and integrate by parts, using the Young inequality and (7), we obtain that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Theta(t)\|^2_{L^2(\Omega)} + \frac{C_{N,s}}{2}\mathcal{F}(\Theta(t),\Theta(t)) &= \langle f(t), \rho(t)\rangle_{H^{-s}(\Omega),H^s_0(\Omega)} \\ &\leq \frac{1}{2\delta}\|f(t)\|^2_{H^{-s}(\Omega)} + \frac{\delta}{2}\|\Theta(t)\|^2_{H^s_0(\Omega)},\end{aligned}$$

with $\delta = \dfrac{C_{N,s}}{2}$.

So

$$\frac{1}{2}\frac{d}{dt}\|\Theta(t)\|^2_{L^2(\Omega)} + \frac{C_{N,s}}{4}\|\Theta(t)\|^2_{H^s_0(\Omega)} \leq \frac{1}{C_{N,s}}\|f(t)\|^2_{H^{-s}(\Omega)}. \tag{31}$$

The integration of (31) over $(0, s)$ with $s \in [0, T]$, gives

$$2\|\Theta(s)\|^2_{L^2(\Omega)} + \frac{C_{N,s}}{4}\int_0^s \|\Theta(t)\|^2_{H^s_0(\Omega)}dt \leq \frac{4}{C_{N,s}}\|f\|^2_{L^2((0,T);H^{-s}(\Omega))} + 2\|\Theta^0\|^2_{L^2(\Omega)}.$$

We deduce that $\displaystyle\sup_{s\in[0,T]}\|\Theta(s)\|^2_{L^2(\Omega)} \leq \frac{2}{C_{N,s}}\|f\|^2_{L^2((0,T);H^{-s}(\Omega))} + \|\Theta^0\|^2_{L^2(\Omega)}$,

and

$$\|\Theta\|^2_{L^2((0,T);H^s_0(\Omega))} \leq \frac{16}{C^2_{N,s}}\|f\|^2_{L^2((0,T);H^{-s}(\Omega))} + \frac{8}{C_{N,s}}\|\Theta^0\|^2_{L^2(\Omega)}.$$

4. Conclusion

Using Lions' theorem, we have studied the existence of a solution in $L^2((0,T);H^s_0(\Omega)))$ for the advection-diffusion equation involving a fractional Laplace operator of order $0 < s < 1$ in dimension $N \in \mathbb{N} \setminus \{0\}$ and we have proved that this solution is unique. Our next objective will be to study the existence and uniqueness of the solution of the fractional acvection-diffusion equation with perturbation in infinite dimension in Hilbert spaces.

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