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DAFTARDAR-JAFARI METHOD FOR SOLVING NONLINEAR DIFFERENTIAL EQUATIONS INVOLVING TO CAPUTO-FABRIZIO FRACTIONAL OPERATOR

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Abstract. This paper presents the Daftardar-Jafari method (DJ method) for nonlinear fractional partial differential equations of the Caputo-Fabrizio fractional operator. The effectiveness of this method is demonstrated by finding the exact solutions of the fractional equations proposed. The results obtained by DJ method is compared with the results obtained by HPTM. The results reveal that the suggested algorithm is very effective and simple and can be applied for linear and nonlinear problems in sciences and engineering.

Key words and phrases. Daftardar-Jafari method; Burger equation; Caputo-Fabrizio fractional operator.

1.. Introduction

Most of the problems arising in the physical and biological area of science are nonlinear in nature, and it is not always possible to find the exact solution of such problems. These problems become more complicated when they involve fractional derivatives and are modelled through mathematical tools from fractional calculus. Fractional partial differential equations (FPDEs) are tremendous instrument and are widely used to describe many significant phenomena and dynamic processes such as engineering, rheology, acoustic, electrical networks, and viscoelasticity [1–4].

Several analytical and numerical techniques were successfully applied to deal with fractional partial differential equations, the Adomian descomposition methods [8–13], the variational iteration method [14–19], reduce differential transform method [20–25], homotopy perturbation transform methods [26–30], the homotopy perturbation method [31–36], the series expansion method [37–40], and other analytical approaches that could be of interest for the reader are presented in [41–70].

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In this paper, we use the DJ method to solve nonlinear fractional partial differential equations using the fractional operator of Caputo-Fabrizio type. The paper has been organized as follows: The basic definitions of fractional calculus are given in Section 2, analysis of the method used is given in Section 3, several test problems that show the effectiveness of the proposed method are given in Section 4, and finally the conclusion is given in Section 5.

2.. Preliminaries of fractional calculus

Definition 2..1. [71,72]. Let $u \in H^1(0,b), b > 0$, then the time fractional Caputo-Fabrizio fractional differential operator is defined as

$$ {}^{CF}D_t^{\alpha} u(t) = \frac{M(\alpha)}{1-\alpha} \int_0^t exp\left[-\frac{\alpha(t-s)}{1-\alpha}\right] u'(s)ds, \ t \geq 0, \ 0 < \alpha < 1, \tag{2.1} $$

where $M(\alpha)$ is a normalization constant depending on α.

The following are the basic properties of the operator ${}^{CF}D_t^{\alpha}$:

(1) ${}^{CF}D_t^{\alpha} u(t) = u(t)$.
(2) ${}^{CF}D_t^{\alpha}[u(t)+v(t)] = {}^{CF}D_t^{\alpha}u(t) + {}^{CF}D_t^{\alpha}v(t)$.
(3) ${}^{CF}D_t^{\alpha}(c) = 0$, where c is constant .
(4) ${}^{CF}D_t^{\alpha}(t^p) = \dfrac{(2-\alpha)M(\alpha)\Gamma(p+1)}{2(1-\alpha)\Gamma(p-n+2)}\, e^{-\frac{\alpha}{1-\alpha}}\, \dfrac{\sqrt{\pi}}{2}\, erf(\sqrt{-\dfrac{\alpha}{1-\alpha}t}\,)$.

Definition 2..2. [71,72]. The fractional Caputo-Fabrizio fractional integral operator of order $\alpha > 0$ and $t > 0$ is given by:

$$ {}^{CF}I_t^{\alpha} u(t) = \frac{1-\alpha}{M(\alpha)} u(t) + \frac{\alpha}{\alpha M(\alpha)} \int_0^t u(s)ds. \tag{2.2} $$

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(2) ${}^{CF}I_t^{\alpha}[u(t)+v(t)] = {}^{CF}I_t^{\alpha}u(t) + {}^{CF}I_t^{\alpha}v(t)$
(3) ${}^{CF}I_t^{\alpha}\left[{}^{CF}D_t^{\alpha}u(t)\right] = u(t) - u(0)$
(4) ${}^{CF}I_t^{\alpha}(c) = \dfrac{2c(1-\alpha)+2\alpha ct}{(2-\alpha)M(\alpha)}$
(5) ${}^{CF}I_t^{\alpha}(t^p) = \dfrac{2(1-\alpha)t^p}{(2-\alpha)M(\alpha)} + \dfrac{2\alpha t^{p+1}}{(2-\alpha)M(\alpha)(p+1)}$

3.. Analysis of DJ method

Let us consider the following nonlinear partial differential equation in the Caputo-Fabrizio sense:

$$ {}^{CF}D_t^{n+\alpha} u(x,t) + R\left[u(x,t)\right] + N\left[u(x,t)\right] = g(x,t). \tag{3.1} $$

with initial conditions

$$ \frac{\partial^k u(x,0)}{\partial t^k} = \phi_k(x), k = 0, 1, ..., m-1, \tag{3.2} $$

where ${}_0^{CF}D_t^{n+\alpha}u(x,t)$ is Caputo-Fabrizio operator of $u(x,t)$, $m-1<n+\alpha\leq m, m\in N$,R is a linear operator , N is an nonlinear operator and g is a source term.

Taking integral of Caput-Fabrizio to both sides of (3.1) we get

$$\left({}^{CF}I_t^{\alpha}\right)\left[D_t^{n+\alpha}u(x,t)\right]+\left({}^{CF}I_t^{\alpha}\right)R\left[u(x,t)\right]+\left({}^{CF}I_t^{\alpha}\right)N\left[u(x,t)\right]=\left({}^{CF}I_t^{\alpha}\right)g(x,t). \tag{3.3}$$

Then, we obtain

$$u(x,t)=\sum_{k=0}^{m-1}u^k(x,0)\frac{x^k}{k!}+^{CF}I_t^{\alpha}g(x,t)-^{CF}I_t^{\alpha}R\left[u(x,t)\right]-^{CF}I_t^{\alpha}N\left[u(x,t)\right]. \tag{3.4}$$

We are looking for a solution $u(x,t)$ of Eq. (3.4) having the series form:

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t) \tag{3.5}$$

The non-linear operator N can be decomposed:

$$N\left[\sum_{n=0}^{\infty}u_n(x,t)\right]=N\left[u_0\right]+\sum_{n=1}^{\infty}\left(N\left[\sum_{i=0}^{n}u_i\right]-N\left[\sum_{i=0}^{n-1}u_i\right]\right) \tag{3.6}$$

In view of (3.5) and (3.9), Eq. (3.4) is equivalent to

$$\begin{aligned}\sum_{n=0}^{\infty}u_n &= \sum_{k=0}^{m-1}u^k(x,0)\frac{x^k}{k!}+^{CF}I_t^{\alpha}g(x,t)-^{CF}I_t^{\alpha}R\left[\sum_{n=0}^{\infty}u_n\right]\\ &\quad -^{CF}I_t^{\alpha}N\left[u_o(x,t)\right]-^{CF}I_t^{\alpha}\left(\sum_{n=1}^{\infty}N\left[\sum_{i=0}^{n}u_i\right]-N\left[\sum_{i=0}^{n-1}u_i\right]\right).\end{aligned} \tag{3.7}$$

Moreover , the relation is defined with recurrence so that

$$\begin{aligned}&u_0(x,t)=\sum_{k=0}^{m-1}u^k(x,0)\frac{x^k}{k!}+{}^{CF}I_t^{\alpha}g(x,t),\\ &u_1(x,t)=-^{CF}I_t^{\alpha}R\left[u_0\right]-{}^{CF}I_t^{\alpha}N\left[u_0(x,t)\right],\\ &u_{n+1}(x,t)=-^{CF}I_t^{\alpha}R\left[u_n\right]-{}^{CF}I_t^{\alpha}\left(N\left[\sum_{i=0}^{n}u_i\right]-N\left[\sum_{i=0}^{n-1}u_i\right]\right),\quad n=1,2,\dots\end{aligned} \tag{3.8}$$

Then k-term approximate solution of Eq. (3.1) is given by:

$$u(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+\cdots \tag{3.9}$$

4.. Illustrative examples

Example 4..1. *Let us consider the following nonlinear Burger equation in the Caputo-Fabrizio sense*

$${}^{CF}D_t^{\alpha}u(x,t)+u\frac{\partial u}{\partial x}=\frac{\partial^2 u}{\partial x^2},0<\alpha\leq 1 \tag{4.1}$$

subject to the initial condition

$$u(x,0)=x. \tag{4.2}$$

Taking integral Caputo-Fabrizio to both sides of (4.1), *we obtain*

$$u(x,t) = u(,0)x +^{CF} I_t^{\alpha}\left[\frac{\partial^2 u}{\partial x^2}\right] -^{CF} I_t^{\alpha}\left[u\frac{\partial u}{\partial x}\right] \tag{4.3}$$

Thus according to Eq. (3.8), *the approximate solution can be obtained:*

$$\begin{aligned} u_0(x,t) &= u(x,0) \\ u_1(x,t) &= {}^{CF}I_t^{\alpha}\left[\frac{\partial^2 u_0(x,t)}{\partial x^2}\right] -^{CF} I_t^{\alpha}\left[u_0(x,t)\frac{\partial u_0(x,t)}{\partial x}\right] \\ u_2(x,t) &= {}^{CF}I_t^{\alpha}\left[\frac{\partial^2 u_1(x,t)}{\partial x^2}\right] -^{CF} I_t^{\alpha}\left[(u_0+u_1)\frac{\partial (u_0+u_1)}{\partial x} - u_0\frac{\partial u_0}{\partial x}\right] \\ &\vdots \end{aligned} \tag{4.4}$$

By the above algorithms, we obtain:

$$\begin{aligned} u_0(x,t) &= x \\ u_1(x,t) &= -x(1-\alpha+\alpha t) \\ u_2(x,t) &= x\left(2\alpha^2 - 4\alpha + \alpha^2 t^2 - 4\alpha^2 t + 4\alpha t + 2\right) \\ &\vdots \end{aligned} \tag{4.5}$$

and so on.

Thererfore, the series solution $u(x,t)$ *of Eq.* (4.1) *is given by*

$$u(x,t) = x - x(1-\alpha+\alpha t) + x\left(2\alpha^2 - 4\alpha + \alpha^2 t^2 - 4\alpha^2 t + 4\alpha t + 2\right) - \cdots \tag{4.6}$$

If we put $\alpha \longrightarrow 1$ *in Eq.* (4.6), *we get the exact solution*

$$\begin{aligned} u(x,t) &= x - xt + xt^2 - \cdots \\ &= x\sum_{k=0}^{\infty}(-t)^k \\ &= \frac{x}{1+t}. \end{aligned} \tag{4.7}$$

Example 4..2. Consider the following nonlinear KdV equation in the Caputo-Fabrizio sense

$${}^{CF}D_t^{\alpha}u(x,t) + u\frac{\partial u(x,t)}{\partial x} + u(x,t)\frac{\partial^3 u(x,t)}{\partial x^3} = 0, 0 < \alpha \leq 1 \tag{4.8}$$

subject to the initial condition

$$u(x,0) = x. \tag{4.9}$$

Taking integral Caputo-Fabrizio to both sides of (4.8), we obtain

$$u(x,t) = u(x,0) -^{CF} I_t^{\alpha}\left[u(x,t)\frac{\partial u(x,t)}{\partial x} + u(x,t)\frac{\partial^3 u(x,t)}{\partial x^3}\right] \tag{4.10}$$

Thus according to Eq. (3.8), the approximate solution can be obtained:

$$\begin{aligned} u_0(x,t) &= u(x,0) \\ u_1(x,t) &= -^{CF}I_t^{\alpha}\left[u_0(x,t)\frac{\partial u_0(x,t)}{\partial x} + u_0(x,t)\frac{\partial^3 u_0(x,t)}{\partial x^3}\right] \\ u_2(x,t) &= -^{CF}I_t^{\alpha}\left[(u_0+u_1)\frac{\partial (u_0+u_1)}{\partial x} - u_0\frac{\partial u_0}{\partial x} + (u_0+u_1)\frac{\partial^3 (u_0+u_1)}{\partial x^3} - u_0\frac{\partial^3 u_0}{\partial x^3}\right] \quad (4.11) \\ &\vdots \end{aligned}$$

By the above algorithms, we obtain:

$$\begin{aligned} u_0(x,t) &= x \\ u_1(x,t) &= -x(1-\alpha+\alpha t) \\ u_2(x,t) &= (x-\alpha x-\alpha^2 x+\alpha^2 x) \\ &\quad +(2\alpha^2 x-3\alpha^2 x+\alpha x)t+(-\alpha^2 x+2\alpha^3 x)t^2-\frac{1}{3}\alpha^3 x t^3 \quad (4.12) \\ &\vdots \end{aligned}$$

and so on.

Thererfore, the series solution $u(x,t)$ of Eq. (4.8) is given by

$$\begin{aligned} u(x,t) &= x-x(1-\alpha+\alpha t)+(x-\alpha x-\alpha^2 x+\alpha^2 x) \\ &\quad +(2\alpha^2 x-3\alpha^2 x+\alpha x)t+(-\alpha^2 x+2\alpha^3 x)t^2-\frac{1}{3}\alpha^3 x t^3+\cdots \quad (4.13) \end{aligned}$$

If we put $\alpha \longrightarrow 1$ in Eq. (4.13), we get the exact solution

$$\begin{aligned} u(x,t) &= x-xt+xt^2-\cdots \\ &= x\sum_{k=0}^{\infty}(-t)^k \\ &= \frac{x}{1+t}. \quad (4.14) \end{aligned}$$

Example 4..3. *Consider the following heat-like equation in the Caputo-Fabrizio sense*

$$^{CF}D_t^{\alpha}u(x,y,t) = \frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2},\ 0<\alpha\leq 1 \quad (4.15)$$

where $0\leq x,y\leq 2\pi$, $t>0$, *with the initial condition*

$$u(x,y,0) = sin(x)sin(y). \quad (4.16)$$

Taking integral Caputo-Fabrizio to both sides of (4.15), *we obtain*

$$u(x,y,t) = u(x,y,0) +^{CF} I_t^{\alpha}\left[\frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2}\right] \quad (4.17)$$

Thus according to Eq. (3.8), the approximate solution can be obtained:

$$\begin{aligned} u_0(x,y,t) &= u(x,y,0) \\ u_1(x,y,t) &= {}^{CF}I_t^{\alpha}\left[\frac{\partial^2 u_0(x,y,t)}{\partial x^2}+\frac{\partial^2 u_0(x,y,t)}{\partial y^2}\right] \\ u_2(x,y,t) &= {}^{CF}I_t^{\alpha}\left[\frac{\partial^2 u_1(x,y,t)}{\partial x^2}+\frac{\partial^2 u_1(x,y,t)}{\partial y^2}\right] \\ &\vdots \end{aligned} \tag{4.18}$$

By the above algorithms, we obtain:

$$\begin{aligned} u_0(x,y,t) &= sin(x)sin(y) \\ u_1(x,y,t) &= {}^{CF}I_t^{\alpha}\left[-2sin(x)sin(y)\right] \\ &= -2(1-\alpha)sin(x)sin(y) - 2\alpha\int_0^t sin(x)sin(y)\,ds \\ &= -2(1-\alpha)sin(x)sin(y) - 2\alpha t\; sin(x)sin(y) \\ &= -2(1-\alpha+\alpha t)sin(x)sin(y) \\ u_2(x,t) &= {}^{CF}I_t^{\alpha}\left[4sin(x)sin(y)(1-\alpha+\alpha t)\right] \\ &= 4sin(x)sin(y)\left[\left(1-2\alpha+\alpha^2\right)\left(2\alpha-2\alpha^2\right)t+\frac{1}{2}\alpha^2t^2\right] \\ &\vdots \end{aligned} \tag{4.19}$$

and so on.

Thererfore, the series solution $u(x,t)$ of Eq. (4.15) is given by

$$\begin{aligned} u(x,t) &= sin(x)sin(y) - 2(1-\alpha+\alpha t)sin(x)sin(y) \\ &\quad +4sin(x)sin(y)\left[\left(1-2\alpha+\alpha^2\right)\left(2\alpha-2\alpha^2\right)t+\frac{1}{2}\alpha^2t^2\right]+\cdots \end{aligned} \tag{4.20}$$

If we put $\alpha \longrightarrow 1$ in Eq. (4.20), we get the exact solution

$$\begin{aligned} u(x,t) &= sin(x)sin(y)\left(1-2t+\frac{(2t)^2}{4}-\cdots\right) \\ &= sin(x)sin(y)e^{-2t} \end{aligned} \tag{4.21}$$

Example 4..4. Consider the nonlinear system of time-fractional differential equation in the Caputo-Fabrizio operator:

$$\begin{aligned} &{}^{CF}D_t^{\alpha}u(x,t) - u_{xx} - 2uu_x + (uv)_x = 0 \\ &{}^{CF}D_t^{\beta}v(x,t) - v_{xx} - 2vv_x + (uv)_x = 0, \end{aligned} \tag{4.22}$$

where $0 < \alpha, \beta \leq 1$ and the initial conditions are

$$\begin{aligned} u(x,0) &= sin(x) \\ v(x,0) &= sin(x). \end{aligned} \tag{4.23}$$

Taking ${}^{CF}I_t^{\alpha}$ and ${}^{CF}I_t^{\beta}$ to both sides of (4.22) respectively, we get

$$\begin{aligned} u(x,t) &= u(x,0) + {}^{CF}I_t^{\alpha}\left[\frac{\partial^2 u}{\partial x^2} + 2u\frac{\partial u}{\partial x} - \frac{\partial}{\partial x}(uv)\right] \\ v(x,t) &= v(x,0) + {}^{CF}I_t^{\beta}\left[\frac{\partial^2 v}{\partial x^2} + 2v\frac{\partial v}{\partial x} - \frac{\partial}{\partial x}(uv)\right]. \end{aligned} \tag{4.24}$$

Thus according to Eq. (3.8), the approximate solution can be obtained:

$$\begin{aligned} u_0(x,t) &= u(x,0) \\ v_0(x,t) &= v(x,0) \end{aligned} \tag{4.25}$$

$$\begin{aligned} u_1(x,t) &= {}^{CF}I_t^{\alpha}\left[\frac{\partial^2 u_0}{\partial x^2} + 2u_0\frac{\partial u_0}{\partial x} - \frac{\partial}{\partial x}(u_0 v_0)\right] \\ v_1(x,t) &= {}^{CF}I_t^{\beta}\left[\frac{\partial^2 v_0}{\partial x^2} + 2v_0\frac{\partial v_0}{\partial x} - \frac{\partial}{\partial x}(u_0 v_0)\right] \end{aligned} \tag{4.26}$$

$$\begin{aligned} u_2(x,t) &= {}^{CF}I_t^{\alpha}\left[\frac{\partial^2 u_1}{\partial x^2} + 2(u_0+u_1)\frac{\partial (u_0+u_1)}{\partial x} - \frac{\partial}{\partial x}((u_0+u_1)(v_0+v_1))\right] \\ &\quad - {}^{CF}I_t^{\alpha}\left[2u_0\frac{\partial u_0}{\partial x} - \frac{\partial}{\partial x}(u_0 v_0)\right] \\ v_2(x,t) &= {}^{CF}I_t^{\beta}\left[\frac{\partial^2 v_1}{\partial x^2} + 2(v_0+v_1)\frac{\partial (v_0+v_1)}{\partial x} - \frac{\partial}{\partial x}((u_0+u_1)(v_0+v_1))\right] \\ &\quad - {}^{CF}I_t^{\alpha}\left[2v_0\frac{\partial v_0}{\partial x} - \frac{\partial}{\partial x}(u_0 v_0)\right] \\ &\vdots \end{aligned} \tag{4.27}$$

By the above algorithms, we obtain:

$$\begin{aligned} u_0(x,t) &= sin(x) \\ v_0(x,t) &= sin(x). \end{aligned} \tag{4.28}$$

$$\begin{aligned} u_1(x,t) &= {}^{CF}I_t^{\alpha}\left[-sin(x) + 2\,sinx\,cosx - 2\,sinx\,cosx\right] \\ &= -sinx(1-\alpha+\alpha t) \\ v_1(x,t) &= {}^{CF}I_t^{\beta}\left[-sin(x) + 2\,sinx\,cosx - 2\,sinx\,cosx\right] \\ &= -sinx(1-\beta+\beta t) \end{aligned} \tag{4.29}$$

$$\begin{aligned}
u_2(x,t) &= {}^{CF}I_t^{\alpha}\left[sin(x)(1-\alpha+\alpha t)\right] \\
&+{}^{CF}I_t^{\alpha}\left[2\alpha^2 sin(x)\,cos(x)\,(1-t)^2 - 2\alpha^2 sin(x)\,cos(x)\,(1-t)^2\right] \\
&-{}^{CF}I_t^{\alpha}\left[2\,sin(x)\,cos(x) - 2\,sin(x)\,cos(x)\right] \\
&= sin(x)\left[(1-\alpha)(1-\alpha+\alpha t)+\alpha(t-\alpha t+\frac{1}{2}\alpha^2 t^2)\right] \\
&= sin(x)\left[(1-\alpha)^2+(2\alpha-2\alpha^2)t+\frac{1}{2}\alpha^2 t^2\right] \\
v_2(x,t) &= {}^{CF}I_t^{\beta}\left[sin(x)(1-\beta+\beta t)\right] \\
&+{}^{CF}I_t^{\beta}\left[2\beta^2 sin(x)\,cos(x)\,(1-t)^2 - 2\beta^2 sin(x)\,cos(x)\,(1-t)^2\right] \\
&-{}^{CF}I_t^{\beta}\left[2\,sin(x)\,cos(x) - 2\,sin(x)\,cos(x)\right] \\
&= sin(x)\left[(1-\beta)(1-\beta+\beta t)+\beta(t-\beta t+\frac{1}{2}\beta^2 t^2)\right] \\
&= sin(x)\left[(1-\beta)^2+(2\beta-2\beta^2)t+\frac{1}{2}\beta^2 t^2\right] \qquad (4.30) \\
&\vdots
\end{aligned}$$

Then the approximate solution of (4.22) is

$$\begin{aligned}
u(x,t) = sin(x)\left[(1-\alpha+\alpha^2)+(\alpha-2\alpha^2)t+\frac{1}{2}\alpha^2 t^2+\cdots\right] \\
v(x,t) = sin(x)\left[(1-\beta+\beta^2)+(\beta-2\beta^2)t+\frac{1}{2}\beta^2 t^2+\cdots\right] \qquad (4.31)
\end{aligned}$$

If we put $\alpha \longrightarrow 1$ and $\beta \longrightarrow 1$ in (4.31), we ireproduce the isolution of the problemi as follows

$$\begin{aligned}
u(x,y,t) &= sin(x)\left(1-t+\frac{t^2}{2!}-\cdots\right) \\
v(x,y,t) &= sin(x)\left(1+t+\frac{t^2}{2!}+\cdots\right) \qquad (4.32)
\end{aligned}$$

This solutioni is equivalenti to the exact isolution in closed iform:

$$\begin{aligned}
u(x,t) &= sin(x)e^{t} \\
v(x,t) &= sin(x)e^{-t}. \qquad (4.33)
\end{aligned}$$

Conclusions

In this work, the DJ method has been successfully applied to finding the approximate solution of nonlinear partial differential equations of the Caputo-Fabrizio fractional operator. The method is very powerful and efficient in finding analytical as well as numerical solutions for wide classes of linear and non-linear local fractional PDEs. The methodology presented has become an important mathematical tool, motivated by the potential use for physicists and engineers working in various areas of the natural sciences.

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