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EXPLORING Δ-CONTINUOUS AND Δ-IRRESOLUTE MAPPINGS IN TOPOLOGICAL SPACES

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Abstract. In this study, we investigate the concepts of Δ-continuous, Δ-irresolute, Δ-open, and Δ-closed mappings. We establish that every continuous mapping is inherently Δ-irresolute and that every Δ-irresolute mapping is Δ-continuous. However, the converse implications do not necessarily hold. This distinction sets Δ-continuous mappings apart from traditional continuous mappings, particularly since the composition of two Δ-continuous mappings may not always preserve Δ-continuity. We propose several methods for constructing new Δ-continuous (or Δ-irresolute) mappings from existing ones, including pasting-type lemmas specifically tailored for these mappings. Additionally, we present counterexamples to illustrate and clarify these concepts.

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1. Introduction

The study of open-like and closed-like sets in topological spaces has garnered significant attention from researchers over the past few decades. Likewise, numerous variations of continuous-like mappings have been introduced and explored within this framework. A foundational contribution in this area was made by N. Levine, who introduced the concepts of *semi-open sets* and *semi-continuous mappings* in topological spaces [13]. A set $\mathcal{S}$ in a topological space $\mathfrak{X}$ is defined as semi-open if

$$\mathcal{S} \subseteq Cl(Int(\mathcal{S}))$$

[13], where $Cl(\mathcal{A})$ and $Int(\mathcal{A})$ denote the closure and interior of a set $\mathcal{A}$ in $\mathfrak{X}$, respectively. A mapping is semi-continuous if the inverse image of any open set is semi-open [13].

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Building upon this foundation, S. Crossley and S. Hildebrand introduced *irresolute mappings*, which are characterized by the property that the inverse image of any semi-open set is also semi-open [7].

In 1965, Olav Njåstad introduced α*-sets* [21], where a set $\mathcal{S}$ in a topological space $\mathfrak{X}$ is an α-set if

$$\mathcal{S} \subseteq Int(Cl(Int(\mathcal{S})))$$

[21]. This led to the definition of α-irresolute mappings, in which the inverse image of each α-set is also an α-set [15], followed by the development of α-continuous mappings [18].

Further generalizations introduced notions such as θ-open and δ-open sets [30], as well as *pre-open sets* and *pre-continuous mappings* [17]. A set $\mathcal{S}$ in a topological space $\mathfrak{X}$ is pre-open if

$$\mathcal{S} \subseteq Int(Cl(\mathcal{S}))$$

[17] and is classified as semi-preopen if

$$\mathcal{S} \subseteq Cl(Int(Cl(\mathcal{S})))$$

[2]. These advancements prompted the exploration of various types of mappings in topological spaces, including semi-precontinuous, semi-preopen, semi-preclosed, semi-preirresolute, pre-semi-preopen, and pre-semi-preclosed mappings, along with their fundamental properties and characterizations [20]. Additionally, M. Veera Kumar contributed to this field by introducing ψ-continuous and ψ-irresolute mappings, based on a novel class of closed-like sets known as ψ-closed sets [29].

The pursuit of new classes of continuous-like mappings based on open-like and closed-like sets remains an active area of research. Recent notable contributions include [1], [3], [5], [6], [9], [10], [11], [12], [24], [25], and [26].

A set in a topological space is termed Δ*-open* if it is the symmetric difference of two open sets. This concept, first appearing in [22] and [8], is attributed to a preprint by M. Veera Kumar. Correspondingly, the complement of a Δ-open set is defined as Δ*-closed*. These notions, along with related concepts, have been extensively studied by the author in [16].

This paper is structured as follows:

- **Section 2** provides a consolidation of fundamental notions and preliminary results necessary for the subsequent discussions.
- **Section 3** introduces the concept of a Δ-continuous mapping (Definition 3.1) and examines its properties (Theorem 3.4). Necessary conditions for Δ-continuity are explored in Propositions 3.6, 3.7, and 3.8. Example 3.10 demonstrates that the composition of two Δ-continuous mappings may fail to be Δ-continuous, although Proposition 3.11 establishes that composing a continuous mapping with a Δ-continuous mapping results in a Δ-continuous mapping. Additionally, Proposition 3.12 presents techniques for constructing Δ-continuous mappings, while Theorem 3.15 provides a pasting-type lemma. The section concludes with Proposition 3.17, which states that a Δ-continuous mapping into a

product space $\mathfrak{X} \times \mathfrak{Y}$) ensures the Δ-continuity of its coordinate mappings.

- **Section 4** defines Δ-irresolute mappings (Definition 4.1) and establishes that every Δ-irresolute mapping is necessarily Δ-continuous. However, as demonstrated by Example 4.2, the converse does not always hold. This section mirrors Section 3 in its analysis of results related to Δ-irresolute mappings.
- **Section 5** introduces the notions of Δ-open and Δ-closed mappings, illustrating through various examples that these concepts are independent of each other and distinct from Δ-continuous mappings. The section further presents several results concerning Δ-open and Δ-closed mappings.

2. Preliminaries

In this section, we provide basic notions and results related to Δ-open and Δ-closed sets. These results will be used and applied in the subsequent sections.

Recall that for two sets $\mathcal{A}$ and $\mathcal{B}$, their *symmetric difference* is given as

$$\mathcal{A}\Delta\mathcal{B} := (\mathcal{A} - \mathcal{B}) \cup (\mathcal{B} - \mathcal{A}) = (\mathcal{A} \cup \mathcal{B}) - (\mathcal{A} \cap \mathcal{B}).$$

Definition 2.1. ([22] *and* [8]) *A set* $\mathcal{A}$ *in a topological space* $(\mathfrak{X}, \sigma)$ *is called* Δ*-open if there exist open sets* $\mathcal{O}_1$ *and* $\mathcal{O}_2$ *such that*

$$\mathcal{A} = \mathcal{O}_1\Delta\mathcal{O}_2.$$

In a topological space $(\mathfrak{X}, \sigma)$, any open set $\mathcal{O}$ satisfies $\mathcal{O} = \mathcal{O}\Delta\varnothing$, which directly implies that every open set is also Δ-open. However, there exist Δ-open sets that are not necessarily open in the standard topology on $\mathbb{R}$. For instance, the set $(0, 1] \cup [2, 3)$ can be expressed as $(0, 2)\Delta(1, 3)$, demonstrating that it is Δ-open while not being an open set in the usual topology of $\mathbb{R}$.

The complement of a Δ-open set is called Δ-*closed*. We recall a characterization of Δ-open sets.

Theorem 2.2. ([16]) *A set* $\mathcal{A}$ *in a topological space* $(\mathfrak{X}, \sigma)$ *is* Δ*-open if and only if there is an open set* $\mathcal{O}$ *and a closed set* $\mathcal{C}$ *such that* $\mathcal{A} = \mathcal{O} \cap \mathcal{C}$.

Corollary 2.3. ([16]) *A set* $\mathcal{B}$ *in a topological space* $(\mathfrak{X}, \sigma)$ *is* Δ*-closed if and only if there is an open set* $\mathcal{O}$ *and a closed set* $\mathcal{C}$ *such that* $\mathcal{B} = \mathcal{O} \cup \mathcal{C}$.

From Theorem 2.2, it follows that every open and closed set is Δ-open. Furthermore, the finite intersection of Δ-open sets remains Δ-open. However, the union of two Δ-open sets is not necessarily Δ-open, nor is the arbitrary intersection of Δ-open sets.

Similarly, Corollary 2.3 implies that every open and closed set is Δ-closed. Additionally, a finite union of Δ-closed sets is also Δ-closed. However, the intersection of two Δ-closed sets is not necessarily Δ-closed, and an arbitrary union of Δ-closed sets does not always retain the Δ-closed property.

Example 2.4. ([16]) *Let* $\mathfrak{X} = \{a, b, c, d, e\}$ *with a topology*

$$\sigma = \{\phi, \mathfrak{X}, \{a, b, c\}, \{a, b, c, d\}\}.$$

The collection of all Δ*-open sets in* $\mathfrak{X}$ *is*

$$\sigma_{\Delta o} = \{\phi, \mathfrak{X}, \{a, b, c\}, \{a, b, c, d\}, \{d, e\}, \{d\}, \{e\}\}.$$

Clearly, the sets $\{a, b, c\}$ *and* $\{e\}$ *are* Δ*-open, whereas their union is not.*

It is important to observe from the previous example that the collection of all Δ-open sets does not necessarily form a topology in general.

Example 2.5. ([16]) *Let* $\mathbb{Q} = \bigcup_{n=1}^{\infty}\{r_n\}$ *be an enumeration of the rationals. For each* $n \in \mathbb{N}$*, let* $\mathcal{S}_n = \mathbb{R} - \{r_1, r_2, ..., r_n\}$*, then considering* $\mathbb{R}$ *under the standard topology, each* $\mathcal{S}_n$ *is an open set, so it is* Δ*-open. However,* $\bigcap_{n=1}^{\infty} \mathcal{S}_n = \mathbb{R} - \mathbb{Q}$ *is not* Δ*-open.*

It should be noted that the open set $\mathcal{O}$ and the closed set $\mathcal{C}$ in Corollary 2.3 can be chosen to be disjoint. In fact, if $\mathcal{B} = \mathcal{O} \cup \mathcal{C}$, with $\mathcal{O}$ is open and $\mathcal{C}$ is closed, then $\mathcal{B} = \mathcal{O} \cup (\mathcal{C} - \mathcal{O})$ where $\mathcal{C} - \mathcal{O}$ is closed.

Definition 2.6. ([16]) *Let* $(\mathfrak{X}, \sigma)$ *be a topological space, and* $x \in \mathfrak{X}$*. A* Δ*-open set containing* x *is called* Δ*-neighborhood. We write* $\Delta N(x)$*.*

Definition 2.7. ([16]) *Let* $(\mathfrak{X}, \sigma)$ *be a topological space, and* $\mathcal{A} \subseteq \mathfrak{X}$*.*
(1) *The union of all* Δ*-open sets contained in* $\mathcal{A}$ *is said to be the* Δ*-interior of* $\mathcal{A}$ *and is denoted by* $\Delta Int(\mathcal{A})$*.*
(2) *The intersection of all* Δ*-closed sets containing* $\mathcal{A}$ *is said to be the* Δ*-closure of* $\mathcal{A}$ *and is denoted by* $\Delta Cl(\mathcal{A})$*.*

Clearly, $\Delta Int(\mathcal{A})$ need not be Δ-open and $\Delta Cl(\mathcal{A})$ need not be Δ-closed. It should be also noted that if $\mathcal{A}$ is Δ-open, then $\Delta Int(\mathcal{A}) = \mathcal{A}$, and if $\mathcal{A}$ is Δ-closed, then $\Delta Cl(\mathcal{A}) = \mathcal{A}$. In either case the converse is not true.

Example 2.8. ([16]) *Let* $\mathbb{Q} = \bigcup_{n=1}^{\infty}\{r_n\}$ *be an enumeration of the rationals. For each* $n \in \mathbb{N}$*, let* $\mathcal{O}_n = (-n, n)$ *and* $\mathcal{C}_n = \{r_1, r_2, ..., r_n\}$*. Then* $\mathcal{A}_n = \mathcal{O}_n \cap \mathcal{C}_n$ *is* Δ*-open set in* $\mathbb{R}$ *under the standard topology. Let* $\mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{A}_n$ *then* $Int(\mathcal{A}) = \mathcal{A}$*, nevertheless* $\mathcal{A} = \mathbb{Q}$ *is not* Δ*-open.*

It is crucial to note that in the proof of [22, Theorem 3], the authors assumed that the Δ-interior of a set is Δ-open. However, the preceding example demonstrates that this assumption does not always hold.

Because each open set is Δ-open and each closed set is Δ-closed, the following result follows directly.

Lemma 2.9. ([16]) *Let* $(\mathfrak{X}, \sigma)$ *be a topological space and* $\mathcal{A} \subseteq \mathfrak{X}$*. Then*

(1) $\Delta Int(\mathcal{A}) \subseteq \Delta Cl(\mathcal{A})$.

(2) $Int(\mathcal{A}) \subseteq \Delta Int(\mathcal{A}) \subseteq \mathcal{A}$.

(3) $\mathcal{A} \subseteq \Delta Cl(\mathcal{A}) \subseteq Cl(\mathcal{A})$.

Proposition 2.10. ([16]) *Let* $(\mathfrak{X}, \sigma)$ *be a topological space and* $\mathcal{A} \subseteq \mathfrak{X}$*. Then* $x \in \Delta Int(\mathcal{A})$*, if and only if, there is a* $\Delta N(x)$*, such that* $\Delta N(x) \subseteq \mathcal{A}$*.*

Basic properties of Δ-interior are summarized in the next proposition.

Proposition 2.11. ([16]) *Let* $(\mathfrak{X}, \sigma)$ *be a topological space and* $\mathcal{A}, \mathcal{B} \subseteq \mathfrak{X}$*. Then*

(1) *If* $\mathcal{A} \subseteq \mathcal{B}$*, then* $\Delta Int(\mathcal{A}) \subseteq \Delta Int(\mathcal{B})$.

(2) $\Delta Int(\mathcal{A}) \cup \Delta Int(\mathcal{B}) \subseteq \Delta Int(\mathcal{A} \cup \mathcal{B})$.

(3) $\Delta Int(\mathcal{A} \cap \mathcal{B}) = \Delta Int(\mathcal{A}) \cap \Delta Int(\mathcal{B})$.

Proposition 2.12. ([16]) *Let* $(\mathfrak{X}, \sigma)$ *be a topological space and* $\mathfrak{Y} \subseteq \mathfrak{X}$*. Then,* $\mathcal{S}$ *is* Δ*-open in* $\mathfrak{Y}$ *if and only if there is a* Δ*-open set* $\mathcal{A}$ *in* $\mathfrak{X}$ *such that* $\mathcal{S} = \mathfrak{Y} \cap \mathcal{A}$*.*

Proposition 2.13. ([16]) *Let* $(\mathfrak{X}, \sigma)$ *be a topological space and* $\mathfrak{Y} \subseteq \mathfrak{X}$*. Then,* $\mathcal{S}$ *is* Δ*-closed in* $\mathfrak{Y}$ *if and only if there is a* Δ*-closed set* $\mathcal{B}$ *in* $\mathfrak{X}$ *such that* $\mathcal{S} = \mathfrak{Y} \cap \mathcal{B}$*.*

Proposition 2.14. ([16]) *Let* $(\mathfrak{X}, \sigma)$ *be a topological space and* $\mathfrak{Y} \subseteq \mathfrak{X}$*. If* $\mathcal{S}$ *is* Δ*-open in* $\mathfrak{Y}$ *and* $\mathfrak{Y}$ *is* Δ*-open in* $\mathfrak{X}$*, then* $\mathcal{S}$ *is* Δ*-open in* $\mathfrak{X}$*.*

3. Δ-continuous mappings

Given topological spaces $\mathfrak{X}$ and $\mathfrak{Y}$, recall that a mapping $f : \mathfrak{X} \to \mathfrak{Y}$ is called *continuous* if for every open set $\mathcal{O}$ in $\mathfrak{Y}$, the preimage $f^{-1}(\mathcal{O})$ is open in $\mathfrak{X}$. Similarly, a mapping $f : \mathfrak{X} \to \mathfrak{Y}$ is termed *semi-continuous* if for every open set $\mathcal{O}$ in $\mathfrak{Y}$, the preimage $f^{-1}(\mathcal{O})$ is semi-open in $\mathfrak{X}$ [13]. Following this pattern, we introduce the following concept.

Definition 3.1. *Let* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *be topological spaces. A mapping* $f : \mathfrak{X} \to \mathfrak{Y}$ *is said to be* Δ*-continuous if for every open set* $\mathcal{O}$ *in* $\mathfrak{Y}$*, the preimage* $f^{-1}(\mathcal{O})$ *is* Δ*-open in* $\mathfrak{X}$*.*

It is evident that every continuous mapping is Δ-continuous. However, the converse does not necessarily hold.

Example 3.2. *Let* $\mathfrak{X} = \{a, b, c\}$*. Consider two topologies on* $\mathfrak{X}$*:*

$$\sigma_1 = \{\varnothing, \mathfrak{X}, \{a\}\}, \quad \sigma_2 = \{\varnothing, \mathfrak{X}, \{a, b\}\}.$$

Define a mapping $f : (\mathfrak{X}, \sigma_1) \to (\mathfrak{X}, \sigma_2)$ *by*

$$f(a) = c, \quad f(b) = b, \quad f(c) = a.$$

The collection of all Δ-open sets in $(\mathfrak{X}, \sigma_1)$ is given by

$$\sigma_{1\Delta o} = \{\varnothing, \mathfrak{X}, \{a\}, \{b, c\}\}.$$

It is clear that f is Δ-continuous. However, the preimage of the set $\{a, b\}$ under f is

$$f^{-1}(\{a, b\}) = \{b, c\},$$

which is not open in $(\mathfrak{X}, \sigma_1)$. Therefore, f is not continuous.

We provide the following characterization for Δ-continuous mappings.

Theorem 3.3. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be topological spaces, and let $f : \mathfrak{X} \to \mathfrak{Y}$. Then f is Δ-continuous if and only if for every closed set $\mathcal{C}$ in $\mathfrak{Y}$, the preimage $f^{-1}(\mathcal{C})$ is Δ-closed in $\mathfrak{X}$.*

Proof. ($\Longrightarrow$) Assume that $f : \mathfrak{X} \to \mathfrak{Y}$ is Δ-continuous. Let $\mathcal{C}$ be a closed set in $\mathfrak{Y}$. Since $\mathfrak{Y} \setminus \mathcal{C}$ is open in $\mathfrak{Y}$, applying the definition of Δ-continuity gives

$$f^{-1}(\mathfrak{Y} \setminus \mathcal{C}) = \mathfrak{X} \setminus f^{-1}(\mathcal{C}),$$

which must be Δ-open in $\mathfrak{X}$. Consequently, $f^{-1}(\mathcal{C})$ is Δ-closed in $\mathfrak{X}$.

($\Longleftarrow$) Conversely, suppose that for every closed set $\mathcal{C}$ in $\mathfrak{Y}$, the preimage $f^{-1}(\mathcal{C})$ is Δ-closed in $\mathfrak{X}$. Let $\mathcal{O}$ be an open set in $\mathfrak{Y}$. Then $\mathfrak{Y} \setminus \mathcal{O}$ is closed in $\mathfrak{Y}$, and by assumption, its preimage

$$f^{-1}(\mathfrak{Y} \setminus \mathcal{O}) = \mathfrak{X} \setminus f^{-1}(\mathcal{O})$$

must be Δ-closed in $\mathfrak{X}$. This implies that $f^{-1}(\mathcal{O})$ is Δ-open in $\mathfrak{X}$, proving that f is Δ-continuous. □

Theorem 3.4. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be topological spaces, and let $f : \mathfrak{X} \to \mathfrak{Y}$. The Δ-continuity of f implies each of the following statements:*

(1) *For each $x \in \mathfrak{X}$ and for every neighborhood $N(f(x))$ of $f(x)$ in $\mathfrak{Y}$, there exists a Δ-neighborhood $\Delta N(x)$ of x in $\mathfrak{X}$ such that $f\big(\Delta N(x)\big) \subseteq N(f(x))$.*

(2) *For every subset $\mathcal{A} \subseteq \mathfrak{X}$, we have*

$$f\big(\Delta Cl(\mathcal{A})\big) \subseteq Cl\big(f(\mathcal{A})\big).$$

(3) *For every subset $\mathcal{B} \subseteq \mathfrak{Y}$, we have*

$$f^{-1}\big(Int(\mathcal{B})\big) \subseteq \Delta Int\big(f^{-1}(\mathcal{B})\big).$$

Proof. (1) Let $x \in \mathfrak{X}$ and let $N(f(x))$ be a neighborhood of $f(x)$ in $\mathfrak{Y}$. Since f is Δ-continuous, the preimage $f^{-1}\big(N(f(x))\big)$ is Δ-open in $\mathfrak{X}$, and since $x \in f^{-1}\big(N(f(x))\big)$, we take $\Delta N(x) = f^{-1}\big(N(f(x))\big)$. Thus,

$$f\big(\Delta N(x)\big) = f\Big(f^{-1}\big(N(f(x))\big)\Big) \subseteq N(f(x)).$$

(2) Let $\mathcal{A} \subseteq \mathfrak{X}$. Since $Cl(f(\mathcal{A}))$ is closed in $\mathfrak{Y}$, Theorem 3.3 ensures that $f^{-1}\Big(Cl(f(\mathcal{A}))\Big)$ is Δ-closed in $\mathfrak{X}$. Moreover, since $\mathcal{A} \subseteq f^{-1}\Big(Cl(f(\mathcal{A}))\Big)$, we conclude that

$$\Delta Cl(\mathcal{A}) \subseteq f^{-1}\Big(Cl(f(\mathcal{A}))\Big).$$

Therefore, applying f yields

$$f\big(\Delta Cl(\mathcal{A})\big) \subseteq Cl\big(f(\mathcal{A})\big).$$

(3) Let $\mathcal{B} \subseteq \mathfrak{Y}$. Since $Int(\mathcal{B})$ is open in $\mathfrak{Y}$, its preimage $f^{-1}\big(Int(\mathcal{B})\big)$ is Δ-open in $\mathfrak{X}$ by the Δ-continuity of f. Moreover, since $f^{-1}\big(Int(\mathcal{B})\big) \subseteq f^{-1}(\mathcal{B})$, it follows that

$$f^{-1}\big(Int(\mathcal{B})\big) \subseteq \Delta Int\big(f^{-1}(\mathcal{B})\big).$$

□

None of the statements in Theorem 3.4 assures Δ-continuity.

Example 3.5. *Let $\mathfrak{X} = \{a, b, c\}$ with the topologies*

$$\sigma_1 = \{\varnothing, \mathfrak{X}, \{a\}, \{a, b\}\}, \quad \sigma_2 = \{\varnothing, \mathfrak{X}, \{a\}\}.$$

Define a mapping $f : (\mathfrak{X}, \sigma_1) \to (\mathfrak{X}, \sigma_2)$ by

$$f(a) = a, \quad f(b) = b, \quad f(c) = a.$$

The collection of all Δ-open sets in $(\mathfrak{X}, \sigma_1)$ is given by

$$\sigma_{1\Delta o} = \{\varnothing, \mathfrak{X}, \{a\}, \{a, b\}, \{b, c\}, \{c\}, \{b\}\}.$$

The set $\{a\}$ is open in $(\mathfrak{X}, \sigma_2)$, but its preimage under f is

$$f^{-1}(\{a\}) = \{a, c\},$$

which is not Δ-open in $(\mathfrak{X}, \sigma_1)$. Thus, f is not Δ-continuous.

However, it is easy to verify that statement (1) in Theorem 3.4 is satisfied. Furthermore, in $(\mathfrak{X}, \sigma_1)$, we observe that

$$\Delta Cl(\mathcal{A}) = \Delta Int(\mathcal{A}) = \mathcal{A}$$

for any subset $\mathcal{A} \subseteq \mathfrak{X}$. Hence, statements (2) and (3) in Theorem 3.4 are also trivially satisfied.

An extra condition is needed for each statement in Theorem 3.4 to guarantee the Δ-continuity.

Proposition 3.6. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be topological spaces, and let $f : \mathfrak{X} \to \mathfrak{Y}$. Suppose that for each $x \in \mathfrak{X}$ and every neighborhood $N(f(x))$ of $f(x)$ in $\mathfrak{Y}$, there exists a Δ-neighborhood $\Delta N(x)$ of x in $\mathfrak{X}$ such that $f\big(\Delta N(x)\big) \subseteq N\big(f(x)\big)$. Further, assume that for any subset $\mathcal{A} \subseteq \mathfrak{X}$, the Δ-interior $\Delta Int(\mathcal{A})$ is Δ-open in $\mathfrak{X}$. Then f is Δ-continuous.*

Proof. Let $\mathcal{O}$ be an open set in $\mathfrak{Y}$, and let $x \in f^{-1}(\mathcal{O})$. Since $\mathcal{O}$ is a neighborhood of $f(x)$, there exists a Δ-neighborhood $\Delta N(x)$ of x in $\mathfrak{X}$ such that $f(\Delta N(x)) \subseteq \mathcal{O}$. This implies that $\Delta N(x) \subseteq f^{-1}(\mathcal{O})$. Consequently, we obtain

$$f^{-1}(\mathcal{O}) = \bigcup_{x \in f^{-1}(\mathcal{O})} \Delta N(x) = \Delta Int\big(f^{-1}(\mathcal{O})\big).$$

Since $\Delta Int\big(f^{-1}(\mathcal{O})\big)$ is assumed to be Δ-open in $\mathfrak{X}$, it follows that $f^{-1}(\mathcal{O})$ is Δ-open in $\mathfrak{X}$. Hence, f is Δ-continuous. □

Proposition 3.7. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be topological spaces, and let $f : \mathfrak{X} \to \mathfrak{Y}$. Suppose that for every subset $\mathcal{A} \subseteq \mathfrak{X}$, we have*

$$f\big(\Delta Cl(\mathcal{A})\big) \subseteq Cl\big(f(\mathcal{A})\big).$$

Further, assume that for any subset $\mathcal{A} \subseteq \mathfrak{X}$, the Δ-closure $\Delta Cl(\mathcal{A})$ is Δ-closed in $\mathfrak{X}$. Then f is Δ-continuous.

Proof. Let $\mathcal{C}$ be a closed set in $\mathfrak{Y}$ and set $\mathcal{A} = f^{-1}(\mathcal{C})$. Then, $\mathcal{A} \subseteq \mathfrak{X}$ and by assumption, we have

$$f\big(\Delta Cl(\mathcal{A})\big) \subseteq Cl\big(f(\mathcal{A})\big).$$

That is,

$$f\Big(\Delta Cl\big(f^{-1}(\mathcal{C})\big)\Big) \subseteq Cl\Big(f\big(f^{-1}(\mathcal{C})\big)\Big) \subseteq Cl(\mathcal{C}) = \mathcal{C}.$$

Thus, applying the preimage under f, we obtain

$$\Delta Cl\big(f^{-1}(\mathcal{C})\big) \subseteq f^{-1}\bigg(f\Big(\Delta Cl\big(f^{-1}(\mathcal{C})\big)\Big)\bigg) \subseteq f^{-1}(\mathcal{C}).$$

This implies that

$$f^{-1}(\mathcal{C}) = \Delta Cl\big(f^{-1}(\mathcal{C})\big),$$

which is Δ-closed in $\mathfrak{X}$. By Theorem 3.3, it follows that f is Δ-continuous. □

Proposition 3.8. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be topological spaces, and let $f : \mathfrak{X} \to \mathfrak{Y}$. Suppose that for every subset $\mathcal{B} \subseteq \mathfrak{Y}$, we have*

$$f^{-1}\big(Int(\mathcal{B})\big) \subseteq \Delta Int\big(f^{-1}(\mathcal{B})\big).$$

Further, assume that for any subset $\mathcal{A} \subseteq \mathfrak{X}$, the Δ-interior $\Delta Int(\mathcal{A})$ is Δ-open in $\mathfrak{X}$. Then f is Δ-continuous.

Proof. Let $\mathcal{O}$ be an open subset of $\mathfrak{Y}$. Since the interior of an open set is itself, we have $Int(\mathcal{O}) = \mathcal{O}$, and thus

$$f^{-1}(\mathcal{O}) = f^{-1}\big(Int(\mathcal{O})\big).$$

By assumption, it follows that

$$f^{-1}\big(Int(\mathcal{O})\big) \subseteq \Delta Int\big(f^{-1}(\mathcal{O})\big).$$

Consequently, we obtain

$$f^{-1}(\mathcal{O}) = \Delta Int\big(f^{-1}(\mathcal{O})\big).$$

Since $\Delta Int\big(f^{-1}(\mathcal{O})\big)$ is assumed to be Δ-open in $\mathfrak{X}$, it follows that $f^{-1}(\mathcal{O})$ is Δ-open in $\mathfrak{X}$. Therefore, f is Δ-continuous. □

Next we consider some methods of constructing Δ-continuous mappings, but first we recall the following result on continuous mappings in topological spaces.

Proposition 3.9. ([19]) *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be topological spaces. Then*
(1) If $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ is given by $f(x) = y_0$ where $y_0 \in \mathfrak{Y}$ is a fixed element, then f is continuous.
(2) If $\mathcal{A}$ is a subspace of $\mathfrak{X}$, the inclusion mapping $i_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathfrak{X}$ is continuous.
(3) Let $\pi_1 : \mathfrak{X} \times \mathfrak{Y} \to \mathfrak{X}$ and $\pi_2 : \mathfrak{X} \times \mathfrak{Y} \to \mathfrak{Y}$ be projections onto the first and second factors, respectively. Then π_1 and π_2 are continuous.

The composition of two Δ-continuous mappings need not be Δ-continuous as illustrated in the following example.

Example 3.10. *Let $\mathfrak{X} = \{a, b, c, d\}$ with the topology*

$$\sigma = \{\varnothing, \mathfrak{X}, \{a\}, \{a, b\}, \{a, b, c\}\}.$$

The collection of all Δ-open sets in $\mathfrak{X}$ is

$$\sigma_{\Delta o} = \{\varnothing, \mathfrak{X}, \{a\}, \{a, b\}, \{a, b, c\}, \{b, c, d\}, \{c, d\}, \{d\}, \{b\}, \{c\}, \{b, c\}\}.$$

Define a mapping $f : \mathfrak{X} \to \mathfrak{X}$ by

$$f(a) = d, \quad f(b) = b, \quad f(c) = a, \quad f(d) = c.$$

It is easy to check that f is Δ-continuous. Nevertheless, the composition $f \circ f$ is not Δ-continuous. Indeed, we compute:

$$(f \circ f)^{-1}(\{a, b, c\}) = f^{-1}\big(f^{-1}(\{a, b, c\})\big) = f^{-1}(\{b, c, d\}) = \{a, b, d\},$$

which is not Δ-open in $\mathfrak{X}$. Thus, $f \circ f$ fails to be Δ-continuous.

Next, we demonstrate that the composition of a Δ-continuous mapping and a continuous mapping results in a Δ-continuous mapping.

Proposition 3.11. *Let $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}$, and $\mathfrak{W}$ be topological spaces. Consider the mappings $f : \mathfrak{X} \to \mathfrak{Y}$, $g : \mathfrak{Y} \to \mathfrak{Z}$, and $h : \mathfrak{Z} \to \mathfrak{W}$.*

(1) *If f is continuous and g is Δ-continuous, then $g \circ f$ is Δ-continuous.*
(2) *If g is Δ-continuous and h is continuous, then $h \circ g$ is Δ-continuous.*

Proof. (1) Assume that f is continuous and g is Δ-continuous. Let $\mathcal{G}$ be an open set in $\mathfrak{Z}$. Since g is Δ-continuous, it follows that $g^{-1}(\mathcal{G})$ is Δ-open in $\mathfrak{Y}$. By Theorem 2.2, we can express $g^{-1}(\mathcal{G})$ as the intersection of an open set $\mathcal{O}$ and a closed set $\mathcal{C}$ in $\mathfrak{Y}$, i.e.,

$$g^{-1}(\mathcal{G}) = \mathcal{O} \cap \mathcal{C}.$$

Since f is continuous, its preimage preserves openness and closedness, meaning that $f^{-1}(\mathcal{O})$ is open in $\mathfrak{X}$ and $f^{-1}(\mathcal{C})$ is closed in $\mathfrak{X}$. Consequently, we obtain

$$(g \circ f)^{-1}(\mathcal{G}) = f^{-1}\big(g^{-1}(\mathcal{G})\big) = f^{-1}(\mathcal{O} \cap \mathcal{C}) = f^{-1}(\mathcal{O}) \cap f^{-1}(\mathcal{C}).$$

By Theorem 2.2, this intersection is Δ-open in $\mathfrak{X}$, proving that $g \circ f$ is Δ-continuous.

(2) Suppose that g is Δ-continuous and h is continuous. Let $\mathcal{G}$ be an open set in $\mathfrak{W}$. Since h is continuous, we know that $h^{-1}(\mathcal{G})$ is open in $\mathfrak{Z}$. The Δ-continuity of g then ensures that

$$g^{-1}\big(h^{-1}(\mathcal{G})\big)$$

is Δ-open in $\mathfrak{Y}$. Hence, we obtain

$$(h \circ g)^{-1}(\mathcal{G}) = g^{-1}\big(h^{-1}(\mathcal{G})\big),$$

which is Δ-open in $\mathfrak{Y}$. Therefore, $h \circ g$ is Δ-continuous.

□

Proposition 3.12. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be topological spaces.*

(1) *If $f : \mathfrak{X} \to \mathfrak{Y}$ is Δ-continuous and $\mathcal{A}$ is a subspace of $\mathfrak{X}$, then the restriction of f to $\mathcal{A}$, denoted by $f|_{\mathcal{A}} : \mathcal{A} \to \mathfrak{Y}$, is Δ-continuous.*

(2) *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be Δ-continuous. If $\mathcal{B}$ is a subspace of $\mathfrak{Y}$ such that $f(\mathfrak{X}) \subseteq \mathcal{B}$, then the mapping $f : \mathfrak{X} \to \mathcal{B}$ is Δ-continuous.*

(3) *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be Δ-continuous. If $\mathfrak{Z}$ is a topological space containing $\mathfrak{Y}$ as a subspace, then the mapping $h : \mathfrak{X} \to \mathfrak{Z}$ obtained by extending the codomain of f is Δ-continuous.*

Proof. (1) The mapping $f|_{\mathcal{A}}$ can be expressed as the composition $f|_{\mathcal{A}} = f \circ i_{\mathcal{A}}$, where $i_{\mathcal{A}}$ is the inclusion mapping $i_{\mathcal{A}} : \mathcal{A} \to \mathfrak{X}$. By Proposition 3.9(2) and Proposition 3.11(1), it follows that $f|_{\mathcal{A}}$ is Δ-continuous.

(2) Suppose that $f : \mathfrak{X} \to \mathfrak{Y}$ is Δ-continuous and that $\mathcal{B}$ is a subspace of $\mathfrak{Y}$ such that $f(\mathfrak{X}) \subseteq \mathcal{B}$. Let $\mathcal{G}$ be an open set in $\mathcal{B}$. Then, by the definition of the subspace topology, there exists an open set $\mathcal{O}$ in $\mathfrak{Y}$ such that $\mathcal{G} = \mathcal{B} \cap \mathcal{O}$. The preimage under f satisfies:

$$f^{-1}(\mathcal{G}) = f^{-1}(\mathcal{B} \cap \mathcal{O}) = f^{-1}(\mathcal{B}) \cap f^{-1}(\mathcal{O}).$$

Since $f(\mathfrak{X}) \subseteq \mathcal{B}$, we have $f^{-1}(\mathcal{B}) = \mathfrak{X}$, so that

$$f^{-1}(\mathcal{G}) = \mathfrak{X} \cap f^{-1}(\mathcal{O}) = f^{-1}(\mathcal{O}).$$

Since f is Δ-continuous, $f^{-1}(\mathcal{O})$ is Δ-open in $\mathfrak{X}$, and hence $f : \mathfrak{X} \to \mathcal{B}$ is Δ-continuous.

(3) Assume that $f : \mathfrak{X} \to \mathfrak{Y}$ is Δ-continuous and that $\mathfrak{Z}$ is a topological space containing $\mathfrak{Y}$ as a subspace. Define the mapping $h : \mathfrak{X} \to \mathfrak{Z}$ by extending the codomain of f. This mapping can be expressed as the composition

$$h = i_{\mathfrak{Y}} \circ f,$$

where $i_{\mathfrak{Y}} : \mathfrak{Y} \to \mathfrak{Z}$ is the inclusion mapping. By Proposition 3.9(2) and Proposition 3.11(2), it follows that h is Δ-continuous.

□

We recall the usual *pasting lemma* for continuous mappings in topological spaces

Theorem 3.13. ([19]) *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be topological spaces. Let $\mathfrak{X} = \mathcal{A} \cup \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are closed in $\mathfrak{X}$. Let $f : \mathcal{A} \to \mathfrak{Y}$ and $g : \mathcal{B} \to \mathfrak{Y}$ be continuous. Assume $f(x) = g(x)$ for each $x \in \mathcal{A} \cap \mathcal{B}$ and let $h : \mathfrak{X} \to \mathfrak{Y}$ be a mapping defined by $h(x) = f(x)$ for $x \in \mathcal{A}$, and $h(x) = g(x)$ for $x \in \mathcal{B}$. Then, h is continuous.*

Proof. Let $\mathcal{C}$ be a closed subset of $\mathfrak{Y}$. Then $h^{-1}(\mathcal{C}) = f^{-1}(\mathcal{C}) \cup g^{-1}(\mathcal{C})$. Since f is continuous, $f^{-1}(\mathcal{C})$ is closed in $\mathcal{A}$, and so it is closed in $\mathfrak{X}$. Similarly, $g^{-1}(\mathcal{C})$ is closed in $\mathcal{B}$, and so in $\mathfrak{X}$. Hence, $h^{-1}(\mathcal{C}) = f^{-1}(\mathcal{C}) \cup g^{-1}(\mathcal{C})$ is closed in $\mathfrak{X}$. Therefore, h is continuous. □

The proof of Theorem 3.13 relies on the following classical result: *Let $\mathfrak{Y}$ be a subspace of $\mathfrak{X}$. If $\mathcal{C}$ is closed in $\mathfrak{Y}$ and $\mathfrak{Y}$ is closed in $\mathfrak{X}$, then $\mathcal{C}$ is closed in $\mathfrak{X}$.* However, this conclusion does not necessarily hold in the context of Δ-closedness.

Example 3.14. *Let $\mathfrak{X} = \{a, b, c, d, e\}$ be a topological space with topology*

$$\sigma = \{\varnothing, \mathfrak{X}, \{a, b, c\}, \{a, b, c, d\}\}.$$

The collection of all Δ-open sets in $\mathfrak{X}$ is

$$\sigma_{\Delta o} = \{\varnothing, \mathfrak{X}, \{a, b, c\}, \{a, b, c, d\}, \{d, e\}, \{d\}, \{e\}\}.$$

The collection of all Δ-closed sets in $\mathfrak{X}$ is

$$\sigma_{\Delta c} = \{\varnothing, \mathfrak{X}, \{a, b, c\}, \{a, b, c, d\}, \{a, b, c, e\}, \{d, e\}, \{e\}\}.$$

Consider the subspace $\mathfrak{Y} = \{a, b, c, d\}$ and the set $\mathcal{C} = \{d\}$. Clearly, $\mathfrak{Y}$ is Δ-closed in $\mathfrak{X}$. Additionally, since

$$\mathcal{C} = \mathfrak{Y} \cap \{d, e\},$$

Proposition 2.13 ensures that $\mathcal{C}$ is Δ-closed in $\mathfrak{Y}$. However, $\mathcal{C}$ is not Δ-closed in $\mathfrak{X}$, demonstrating that Δ-closedness is not necessarily transitive in this setting.

Motivated by Theorem 3.13 and Example 3.14 we propose the following pasting-type lemma for Δ-continuous mappings.

Theorem 3.15. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be topological spaces. Suppose that $\mathfrak{X} = \mathcal{A} \cup \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are Δ-open in $\mathfrak{X}$. Let $f : \mathcal{A} \to \mathfrak{Y}$ and $g : \mathcal{B} \to \mathfrak{Y}$ be Δ-continuous mappings such that $f(x) = g(x)$ for all $x \in \mathcal{A} \cap \mathcal{B}$. Define a mapping $h : \mathfrak{X} \to \mathfrak{Y}$ by*

$$h(x) = \begin{cases} f(x), & \text{if } x \in \mathcal{A}, \\ g(x), & \text{if } x \in \mathcal{B}. \end{cases}$$

Assume further that the union of any two Δ-open sets in $\mathfrak{X}$ is again Δ-open. Then, h is Δ-continuous.

Proof. Let $\mathcal{O}$ be an open subset of $\mathfrak{Y}$. Then, by the definition of h, we have

$$h^{-1}(\mathcal{O}) = f^{-1}(\mathcal{O}) \cup g^{-1}(\mathcal{O}).$$

Since f is Δ-continuous, $f^{-1}(\mathcal{O})$ is Δ-open in $\mathcal{A}$. By Proposition 2.14, this implies that $f^{-1}(\mathcal{O})$ is Δ-open in $\mathfrak{X}$. Similarly, since g is Δ-continuous, $g^{-1}(\mathcal{O})$ is Δ-open in $\mathcal{B}$, and thus it is also Δ-open in $\mathfrak{X}$.

By assumption, the union of any two Δ-open sets in $\mathfrak{X}$ is Δ-open. Therefore,

$$h^{-1}(\mathcal{O}) = f^{-1}(\mathcal{O}) \cup g^{-1}(\mathcal{O})$$

is Δ-open in $\mathfrak{X}$. Consequently, h is Δ-continuous. □

The condition in Theorem 3.15 that the union of any two Δ-open sets is again Δ-open cannot be omitted.

Example 3.16. *Let $\mathfrak{X} = \{a, b, c, d, e\}$ be a topological space with topology*

$$\sigma = \{\varnothing, \mathfrak{X}, \{a, b, c\}, \{a, b, c, d\}\}.$$

The collection of all Δ-open sets in $\mathfrak{X}$ is

$$\sigma_{\Delta o} = \{\varnothing, \mathfrak{X}, \{a, b, c\}, \{a, b, c, d\}, \{d, e\}, \{d\}, \{e\}\}.$$

Clearly, the sets $\{a, b, c\}$ and $\{e\}$ are Δ-open, but their union $\{a, b, c, e\}$ is not Δ-open.

Consider the subspaces $\mathcal{A} = \{a, b, c\}$ and $\mathcal{B} = \{d, e\}$. Let $\sigma_{\mathcal{A}}$ and $\sigma_{\mathcal{B}}$ be the induced topologies on $\mathcal{A}$ and $\mathcal{B}$, respectively.

Define a mapping $f : (\mathcal{A}, \sigma_{\mathcal{A}}) \to (\mathfrak{X}, \sigma)$ by

$$f(a) = a, \quad f(b) = b, \quad f(c) = c.$$

Clearly, f is continuous, so it is also Δ-continuous.

Next, define a mapping $g : (\mathcal{B}, \sigma_{\mathcal{B}}) \to (\mathfrak{X}, \sigma)$ by

$$g(d) = e, \quad g(e) = d.$$

Then, g is Δ-continuous.

Now, consider the mapping $h : (\mathfrak{X}, \sigma) \to (\mathfrak{X}, \sigma)$ as defined in Theorem 3.15. For the open set $\mathcal{O} = \{a, b, c, d\}$, we compute

$$h^{-1}(\mathcal{O}) = f^{-1}(\mathcal{O}) \cup g^{-1}(\mathcal{O}).$$

Since $f^{-1}(\mathcal{O}) = \{a, b, c\}$ and $g^{-1}(\mathcal{O}) = \{e\}$, we obtain

$$h^{-1}(\mathcal{O}) = \{a, b, c\} \cup \{e\} = \{a, b, c, e\}.$$

However, $\{a, b, c, e\}$ is not Δ-open in $\mathfrak{X}$. Thus, h is not Δ-continuous, demonstrating that the assumption on the union of Δ-open sets in Theorem 3.15 is necessary.

Proposition 3.17. *Let $\mathfrak{X}, \mathfrak{Y}$, and $\mathfrak{Z}$ be topological spaces. Suppose $f_1 : \mathfrak{X} \to \mathfrak{Y}$ and $f_2 : \mathfrak{X} \to \mathfrak{Z}$ are mappings, and define $f : \mathfrak{X} \to \mathfrak{Y} \times \mathfrak{Z}$ by*

$$f(x) = (f_1(x), f_2(x)).$$

If f is Δ-continuous on $\mathfrak{X}$, then both f_1 and f_2 are Δ-continuous on $\mathfrak{X}$.

Proof. Let $\pi_1 : \mathfrak{Y} \times \mathfrak{Z} \to \mathfrak{Y}$ and $\pi_2 : \mathfrak{Y} \times \mathfrak{Z} \to \mathfrak{Z}$ be the natural projection mappings onto the first and second coordinates, respectively. Then, we can express f_1 and f_2 as compositions:

$$f_1(x) = \pi_1(f(x)) = (\pi_1 \circ f)(x), \quad f_2(x) = \pi_2(f(x)) = (\pi_2 \circ f)(x).$$

By Proposition 3.9(3), the projection maps π_1 and π_2 are continuous. Since f is Δ-continuous, applying Proposition 3.11(2) ensures that the compositions $\pi_1 \circ f$ and $\pi_2 \circ f$ are Δ-continuous. Hence, both f_1 and f_2 are Δ-continuous. □

It is well known that open and closed sets are preserved under homoeomorphisms. Similar conclusion holds for Δ-open and Δ-closed sets.

Proposition 3.18. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a homeomorphism. Then:*

(1) *The image of a Δ-open set under f is Δ-open in $\mathfrak{Y}$.*
(2) *The inverse image of a Δ-open set under f is Δ-open in $\mathfrak{X}$.*
(3) *The image of a Δ-closed set under f is Δ-closed in $\mathfrak{Y}$.*
(4) *The inverse image of a Δ-closed set under f is Δ-closed in $\mathfrak{X}$.*

Proof. (1) Let $\mathcal{A}$ be a Δ-open set in $\mathfrak{X}$. By Theorem 2.2, there exist an open set $\mathcal{O}$ and a closed set $\mathcal{C}$ in $\mathfrak{X}$ such that

$$\mathcal{A} = \mathcal{O} \cap \mathcal{C}.$$

Since f is a bijection, we obtain

$$f(\mathcal{A}) = f(\mathcal{O} \cap \mathcal{C}) = f(\mathcal{O}) \cap f(\mathcal{C}).$$

As f is a homeomorphism, the image of $\mathcal{O}$ under f is open in $\mathfrak{Y}$, and the image of $\mathcal{C}$ is closed in $\mathfrak{Y}$. By Theorem 2.2, it follows that $f(\mathcal{A})$ is Δ-open in $\mathfrak{Y}$.

(2) Let $\mathcal{B}$ be a Δ-open set in $\mathfrak{Y}$. By Theorem 2.2, there exist an open set $\mathcal{G}$ and a closed set $\mathcal{F}$ in $\mathfrak{Y}$ such that

$$\mathcal{B} = \mathcal{G} \cap \mathcal{F}.$$

Taking the preimage under f, we obtain

$$f^{-1}(\mathcal{B}) = f^{-1}(\mathcal{G} \cap \mathcal{F}) = f^{-1}(\mathcal{G}) \cap f^{-1}(\mathcal{F}).$$

Since f is a homeomorphism, $f^{-1}(\mathcal{G})$ is open in $\mathfrak{X}$ and $f^{-1}(\mathcal{F})$ is closed in $\mathfrak{X}$. By Theorem 2.2, it follows that $f^{-1}(\mathcal{B})$ is Δ-open in $\mathfrak{X}$.

(3) The proof for the image of a Δ-closed set follows similarly by applying Corollary 2.3.

(4) The proof for the inverse image of a Δ-closed set follows analogously using Corollary 2.3.

□

4. Δ-IRRESOLUTE MAPPINGS

Given topological spaces $\mathfrak{X}$ and $\mathfrak{Y}$, a mapping $f : \mathfrak{X} \to \mathfrak{Y}$ is called *irresolute* if for every semi-open set $\mathcal{O}$ in $\mathfrak{Y}$, the preimage $f^{-1}(\mathcal{O})$ is semi-open in $\mathfrak{X}$ [7]. Following this concept, we introduce the following notion in terms of Δ-open sets.

Definition 4.1. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be topological spaces. A mapping $f : \mathfrak{X} \to \mathfrak{Y}$ is said to be Δ-irresolute if for every Δ-open set $\mathcal{S}$ in $\mathfrak{Y}$, the preimage $f^{-1}(\mathcal{S})$ is Δ-open in $\mathfrak{X}$.*

Combining Definition 4.1 and Proposition 3.18, we get that each homeomorphism is Δ-irresolute. Evidently each Δ-irresolute mapping is Δ-continuous. The converse need not be true.

Example 4.2. *Let $\mathfrak{X} = \{a, b, c\}$ be a finite set equipped with the topologies*

$$\sigma_1 = \{\varnothing, \mathfrak{X}, \{a\}, \{a, b\}\} \quad \textit{and} \quad \sigma_2 = \{\varnothing, \mathfrak{X}, \{b\}, \{b, c\}\}.$$

Define a mapping $f : (\mathfrak{X}, \sigma_1) \to (\mathfrak{X}, \sigma_2)$ by

$$f(a) = c, \quad f(b) = b, \quad f(c) = a.$$

The collection of all Δ-open sets in $(\mathfrak{X}, \sigma_1)$ is given by

$$\sigma_{1\Delta_o} = \{\varnothing, \mathfrak{X}, \{a\}, \{a, b\}, \{b, c\}, \{c\}, \{b\}\}.$$

Similarly, the collection of all Δ-open sets in $(\mathfrak{X}, \sigma_2)$ is

$$\sigma_{2\Delta_o} = \{\varnothing, \mathfrak{X}, \{b\}, \{b, c\}, \{a, c\}, \{a\}, \{c\}\}.$$

It is evident that f is Δ-continuous. However, the set $\{a, c\}$ is Δ-open in $(\mathfrak{X}, \sigma_2)$, but its preimage under f is

$$f^{-1}(\{a, c\}) = \{a, c\},$$

which is not Δ-open in $(\mathfrak{X}, \sigma_1)$. Thus, f is not Δ-irresolute.

We show that each continuous mapping is Δ-irresolute.

Proposition 4.3. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be topological spaces. If $f : \mathfrak{X} \to \mathfrak{Y}$ is continuous, then it is Δ-irresolute.*

Proof. Let $\mathcal{S}$ be a Δ-open set in $\mathfrak{Y}$. By Theorem 2.2, there exist an open set $\mathcal{O}$ and a closed set $\mathcal{C}$ in $\mathfrak{Y}$ such that

$$\mathcal{S} = \mathcal{O} \cap \mathcal{C}.$$

Taking the preimage under f, we obtain

$$f^{-1}(\mathcal{S}) = f^{-1}(\mathcal{O} \cap \mathcal{C}) = f^{-1}(\mathcal{O}) \cap f^{-1}(\mathcal{C}).$$

Since f is continuous, it follows that $f^{-1}(\mathcal{O})$ is open in $\mathfrak{X}$ and $f^{-1}(\mathcal{C})$ is closed in $\mathfrak{X}$. By Theorem 2.2, their intersection is Δ-open in $\mathfrak{X}$. Therefore, $f^{-1}(\mathcal{S})$ is Δ-open in $\mathfrak{X}$, proving that f is Δ-irresolute. □

Example 3.2 provides an instance of a Δ-irresolute mapping that is not continuous. We have previously established that Δ-continuity can be characterized in terms of closed and Δ-closed sets; see Theorem 3.3. In a similar fashion, we present the following result for Δ-irresolute mappings.

Theorem 4.4. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be topological spaces, and let $f : \mathfrak{X} \to \mathfrak{Y}$. Then f is Δ-irresolute if and only if for every Δ-closed set $\mathcal{C}$ in $\mathfrak{Y}$, the preimage $f^{-1}(\mathcal{C})$ is Δ-closed in $\mathfrak{X}$.*

Proof. ($\Longrightarrow$) Assume that $f : \mathfrak{X} \to \mathfrak{Y}$ is Δ-irresolute. Let $\mathcal{C}$ be a Δ-closed set in $\mathfrak{Y}$. By definition, $\mathfrak{Y} \setminus \mathcal{C}$ is Δ-open in $\mathfrak{Y}$. Since f is Δ-irresolute, it follows that

$$f^{-1}(\mathfrak{Y} \setminus \mathcal{C}) = \mathfrak{X} \setminus f^{-1}(\mathcal{C})$$

is Δ-open in $\mathfrak{X}$. Consequently, $f^{-1}(\mathcal{C})$ is Δ-closed in $\mathfrak{X}$.

($\Longleftarrow$) Conversely, suppose that for every Δ-closed set $\mathcal{C}$ in $\mathfrak{Y}$, the preimage $f^{-1}(\mathcal{C})$ is Δ-closed in $\mathfrak{X}$. Let $\mathcal{O}$ be a Δ-open set in $\mathfrak{Y}$. Then, $\mathfrak{Y} \setminus \mathcal{O}$ is Δ-closed in $\mathfrak{Y}$, and by assumption, its preimage satisfies

$$f^{-1}(\mathfrak{Y} \setminus \mathcal{O}) = \mathfrak{X} \setminus f^{-1}(\mathcal{O}),$$

which is Δ-closed in $\mathfrak{X}$. This implies that $f^{-1}(\mathcal{O})$ is Δ-open in $\mathfrak{X}$, proving that f is Δ-irresolute. □

By similar arguments as Theorem 3.4, Proposition 3.6, Proposition 3.7, and Proposition 3.8, one can prove the next results on Δ-irresolute mappings.

Proposition 4.5. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be topological spaces, and let $f : \mathfrak{X} \to \mathfrak{Y}$. If f is Δ-irresolute, then for each $x \in \mathfrak{X}$ and for every Δ-neighborhood $\Delta N(f(x))$ of $f(x)$ in $\mathfrak{Y}$, there exists a Δ-neighborhood $\Delta N(x)$ of x in $\mathfrak{X}$ such that*

$$f\big(\Delta N(x)\big) \subseteq \Delta N(f(x)).$$

Proposition 4.6. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be topological spaces, and let $f : \mathfrak{X} \to \mathfrak{Y}$. If f is Δ-irresolute and for every subset $\mathcal{B} \subseteq \mathfrak{Y}$, the Δ-closure $\Delta Cl(\mathcal{B})$ is Δ-closed in $\mathfrak{Y}$, then for every subset $\mathcal{A} \subseteq \mathfrak{X}$, we have*

$$f\big(\Delta Cl(\mathcal{A})\big) \subseteq \Delta Cl\big(f(\mathcal{A})\big).$$

Proposition 4.7. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be topological spaces, and let $f : \mathfrak{X} \to \mathfrak{Y}$. If f is Δ-irresolute and for every subset $\mathcal{B} \subseteq \mathfrak{Y}$, the Δ-interior $\Delta Int(\mathcal{B})$ is Δ-open in $\mathfrak{Y}$, then for every subset $\mathcal{B} \subseteq \mathfrak{Y}$, we have*

$$f^{-1}\big(\Delta Int(\mathcal{B})\big) \subseteq \Delta Int\big(f^{-1}(\mathcal{B})\big).$$

Proposition 4.8. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be topological spaces, and let $f : \mathfrak{X} \to \mathfrak{Y}$. Suppose that for every $x \in \mathfrak{X}$ and every Δ-neighborhood $\Delta N(f(x))$ of $f(x)$ in $\mathfrak{Y}$, there exists a Δ-neighborhood $\Delta N(x)$ of x in $\mathfrak{X}$ such that*

$$f\big(\Delta N(x)\big) \subseteq \Delta N(f(x)).$$

Further, assume that for any subset $\mathcal{A} \subseteq \mathfrak{X}$, the Δ-interior $\Delta Int(\mathcal{A})$ is Δ-open in $\mathfrak{X}$. Then f is Δ-irresolute.

Proposition 4.9. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be topological spaces, and let $f : \mathfrak{X} \to \mathfrak{Y}$. Suppose that for every subset $\mathcal{A} \subseteq \mathfrak{X}$, we have*

$$f\big(\Delta Cl(\mathcal{A})\big) \subseteq \Delta Cl\big(f(\mathcal{A})\big).$$

Further, assume that for any subset $\mathcal{A} \subseteq \mathfrak{X}$, the Δ-closure $\Delta Cl(\mathcal{A})$ is Δ-closed in $\mathfrak{X}$. Then f is Δ-irresolute.

Proposition 4.10. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be topological spaces, and let $f : \mathfrak{X} \to \mathfrak{Y}$. Suppose that for every subset $\mathcal{B} \subseteq \mathfrak{Y}$, we have*

$$f^{-1}\big(\Delta Int(\mathcal{B})\big) \subseteq \Delta Int\big(f^{-1}(\mathcal{B})\big).$$

Further, assume that for any subset $\mathcal{A} \subseteq \mathfrak{X}$, the Δ-interior $\Delta Int(\mathcal{A})$ is Δ-open in $\mathfrak{X}$. Then f is Δ-irresolute.

The proof of the following theorem is straightforward and is therefore omitted.

Theorem 4.11. *Let $\mathfrak{X}, \mathfrak{Y}$, and $\mathfrak{Z}$ be topological spaces.*

(1) *If $f : \mathfrak{X} \to \mathfrak{Y}$ is a homeomorphism, then f is Δ-irresolute.*
(2) *If $f : \mathfrak{X} \to \mathfrak{Y}$ is Δ-irresolute and $g : \mathfrak{Y} \to \mathfrak{Z}$ is Δ-irresolute, then the composition $g \circ f : \mathfrak{X} \to \mathfrak{Z}$ is Δ-irresolute.*
(3) *If $f : \mathfrak{X} \to \mathfrak{Y}$ is Δ-irresolute and $\mathcal{A}$ is a subspace of $\mathfrak{X}$, then the restriction $f|_{\mathcal{A}} : \mathcal{A} \to \mathfrak{Y}$ is Δ-irresolute.*
(4) *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be Δ-irresolute. If $\mathcal{B}$ is a subspace of $\mathfrak{Y}$ such that $f(\mathfrak{X}) \subseteq \mathcal{B}$, then the mapping $f : \mathfrak{X} \to \mathcal{B}$ is Δ-irresolute.*

(5) *Let* $f : \mathfrak{X} \to \mathfrak{Y}$ *be* Δ*-irresolute. If* $\mathfrak{Z}$ *is a topological space containing* $\mathfrak{Y}$ *as a subspace, then the mapping* $h : \mathfrak{X} \to \mathfrak{Z}$ *obtained by extending the codomain of* f *is* Δ*-irresolute.*

(6) *Let* $f_1 : \mathfrak{X} \to \mathfrak{Y}$ *and* $f_2 : \mathfrak{X} \to \mathfrak{Z}$ *be mappings. Define* $f : \mathfrak{X} \to \mathfrak{Y} \times \mathfrak{Z}$ *by*

$$f(x) = (f_1(x), f_2(x)).$$

If f *is* Δ*-irresolute, then both* f_1 *and* f_2 *are* Δ*-irresolute.*

In connection with Theorem 3.15 the coming result is easily proved.

Theorem 4.12. *Let* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *be topological spaces. Suppose that* $\mathfrak{X} = \mathcal{A} \cup \mathcal{B}$*, where* $\mathcal{A}$ *and* $\mathcal{B}$ *are* Δ*-open in* $\mathfrak{X}$*. Let* $f : \mathcal{A} \to \mathfrak{Y}$ *and* $g : \mathcal{B} \to \mathfrak{Y}$ *be* Δ*-irresolute mappings such that* $f(x) = g(x)$ *for all* $x \in \mathcal{A} \cap \mathcal{B}$*. Define a mapping* $h : \mathfrak{X} \to \mathfrak{Y}$ *by*

$$h(x) = \begin{cases} f(x), & \text{if } x \in \mathcal{A}, \\ g(x), & \text{if } x \in \mathcal{B}. \end{cases}$$

Assume further that the union of any two Δ*-open sets in* $\mathfrak{X}$ *is again* Δ*-open. Then,* h *is* Δ*-irresolute.*

5. Δ-open and Δ-closed mappings

Recall that a mapping is called *open* (*closed*) if the image of each open (closed) set is again an open (a closed) set. Similar notions can be defined in terms of Δ-open and Δ-closed sets.

Definition 5.1. *Let* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *be topological spaces. A mapping* $f : \mathfrak{X} \to \mathfrak{Y}$ *is said to be* Δ*-open if for every* Δ*-open set* $\mathcal{S}$ *in* $\mathfrak{X}$*, the image* $f(\mathcal{S})$ *is* Δ*-open in* $\mathfrak{Y}$*.*

Definition 5.2. *Let* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *be topological spaces. A mapping* $f : \mathfrak{X} \to \mathfrak{Y}$ *is said to be* Δ*-closed if for every* Δ*-closed set* $\mathcal{S}$ *in* $\mathfrak{X}$*, the image* $f(\mathcal{S})$ *is* Δ*-closed in* $\mathfrak{Y}$*.*

In connection with Proposition 3.18, it is evident that a homeomorphism is both a Δ-open and a Δ-closed mapping. However, for the mapping f in Example 3.10, the set $\mathcal{S} = \{c, d\}$ is both Δ-open and Δ-closed, but its image under f, given by $f(\mathcal{S}) = \{a, c\}$, is neither Δ-open nor Δ-closed. This provides an example of a mapping that is Δ-continuous but neither Δ-open nor Δ-closed.

Example 5.3. *Let* $\mathfrak{X} = \{a, b, c\}$ *be a topological space with topology*

$$\sigma_1 = \{\varnothing, \mathfrak{X}, \{a, b\}\}.$$

Let $\mathfrak{Y} = \{a, b, c, d\}$ *be a topological space with topology*

$$\sigma_2 = \{\varnothing, \mathfrak{Y}, \{b, d\}, \{c, d\}, \{d\}, \{b, c, d\}\}.$$

Define a mapping $f : (\mathfrak{X}, \sigma_1) \to (\mathfrak{Y}, \sigma_2)$ *by*

$$f(a) = b, \quad f(b) = a, \quad f(c) = d.$$

The collection of all Δ*-open sets in* $(\mathfrak{X}, \sigma_1)$ *coincides with the collection of all* Δ*-closed sets in* $(\mathfrak{X}, \sigma_1)$*, which is*

$$\sigma_{1\Delta o} = \{\varnothing, \mathfrak{X}, \{a, b\}, \{c\}\}.$$

It is straightforward to verify that f *is both a* Δ*-open and a* Δ*-closed mapping. However, the set* $\{b, d\}$ *is open in* $(\mathfrak{Y}, \sigma_2)$*, but its preimage under* f*,*

$$f^{-1}(\{b, d\}) = \{a, c\},$$

is not Δ*-open in* $(\mathfrak{X}, \sigma_1)$*. Thus,* f *is not* Δ*-continuous. Consequently,* f *is neither* Δ*-irresolute nor continuous.*

Next we give an example of a Δ-open mapping that is not Δ-closed and vice versa.

Example 5.4. *Let* $\mathfrak{X} = \{a, b, c, d\}$ *be a topological space with topology*

$$\sigma = \{\varnothing, \mathfrak{X}, \{a, c, d\}, \{c, d\}\}.$$

The collection of all Δ*-open sets in* $(\mathfrak{X}, \sigma)$ *is given by*

$$\sigma_{\Delta o} = \{\varnothing, \mathfrak{X}, \{a, c, d\}, \{c, d\}, \{b\}, \{a, b\}, \{a\}\}.$$

On one hand, define a mapping $f : \mathfrak{X} \to \mathfrak{X}$ *by*

$$f(a) = f(b) = f(c) = f(d) = a.$$

It is easy to verify that f *is* Δ*-open but not* Δ*-closed.*

On the other hand, define a mapping $g : \mathfrak{X} \to \mathfrak{X}$ *by*

$$g(a) = g(b) = b, \quad g(c) = c, \quad g(d) = d.$$

Then, g *is* Δ*-closed. However, the set* $\{a, c, d\}$ *is* Δ*-open, yet its image under* g*,*

$$g(\{a, c, d\}) = \{b, c, d\},$$

is not Δ*-open. Thus,* g *is not* Δ*-open.*

It is evident that the composition of two Δ-open (respectively, Δ-closed) mappings is again a Δ-open (respectively, Δ-closed) mapping.

Theorem 5.5. *Let* $f : \mathfrak{X} \to \mathfrak{Y}$ *and* $g : \mathfrak{Y} \to \mathfrak{Z}$ *be mappings. Assume that the composition* $g \circ f : \mathfrak{X} \to \mathfrak{Z}$ *is* Δ*-open* (*respectively,* Δ*-closed*)*.*

(1) *If* g *is a* Δ*-irresolute injection, then* f *is* Δ*-open* (*respectively,* Δ*-closed*)*.*
(2) *If* f *is a* Δ*-irresolute surjection, then* g *is* Δ*-open* (*respectively,* Δ*-closed*)*.*

Proof. We consider only the case where $g \circ f$ is Δ-open, as the case for Δ-closed mappings follows analogously.

(1) Let $\mathcal{S}$ be a Δ-open set in $\mathfrak{X}$. Since $g \circ f$ is Δ-open, we have

$$(g \circ f)(\mathcal{S}) = g\big(f(\mathcal{S})\big)$$

as a Δ-open set in $\mathfrak{Z}$. Since g is Δ-irresolute, its preimage satisfies

$$g^{-1}\Big(g\big(f(\mathcal{S})\big)\Big)$$

being Δ-open in $\mathfrak{Y}$. Since g is injective, we obtain

$$g^{-1}\Big(g\big(f(\mathcal{S})\big)\Big) = f(\mathcal{S}).$$

Thus, $f(\mathcal{S})$ is Δ-open in $\mathfrak{Y}$, proving that f is Δ-open.

(2) Let $\mathcal{S}$ be a Δ-open set in $\mathfrak{Y}$. Since f is Δ-irresolute, we conclude that

$$f^{-1}(\mathcal{S})$$

is Δ-open in $\mathfrak{X}$. Because $g \circ f$ is Δ-open, its image satisfies

$$(g \circ f)\big(f^{-1}(\mathcal{S})\big)$$

being Δ-open in $\mathfrak{Z}$. Since f is surjective, it follows that

$$(g \circ f)\big(f^{-1}(\mathcal{S})\big) = g\Big(f\big(f^{-1}(\mathcal{S})\big)\Big) = g(\mathcal{S}).$$

Hence, $g(\mathcal{S})$ is Δ-open in $\mathfrak{Z}$, proving that g is Δ-open.

□

Proposition 5.6. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be topological spaces. If $f : \mathfrak{X} \to \mathfrak{Y}$ is Δ-open, then for any subset $\mathcal{A} \subseteq \mathfrak{X}$, we have*

$$f\big(\Delta Int(\mathcal{A})\big) \subseteq \Delta Int\big(f(\mathcal{A})\big).$$

Proof. Assume that f is Δ-open. For any subset $\mathcal{A} \subseteq \mathfrak{X}$, let $y \in f\big(\Delta Int(\mathcal{A})\big)$. Then there exists some $x \in \Delta Int(\mathcal{A})$ such that $y = f(x)$. By Proposition 2.10, there exists a Δ-neighborhood $\Delta N(x)$ of x satisfying

$$x \in \Delta N(x) \subseteq \mathcal{A}.$$

Applying f to both sides, we obtain

$$f(x) \in f\big(\Delta N(x)\big) \subseteq f(\mathcal{A}).$$

Since f is Δ-open, it follows that $f\big(\Delta N(x)\big)$ is Δ-open in $\mathfrak{Y}$. Consequently,

$$f\big(\Delta N(x)\big) \subseteq \Delta Int\big(f(\mathcal{A})\big).$$

Therefore, we conclude that

$$y = f(x) \in \Delta Int\big(f(\mathcal{A})\big),$$

which proves the desired result. □

The following result provides a partial converse to Proposition 5.6.

Proposition 5.7. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be topological spaces, and let $f : \mathfrak{X} \to \mathfrak{Y}$. Suppose that for each subset $\mathcal{A} \subseteq \mathfrak{X}$, we have*

$$f\big(\Delta Int(\mathcal{A})\big) \subseteq \Delta Int\big(f(\mathcal{A})\big),$$

and that for each subset $\mathcal{B} \subseteq \mathfrak{Y}$, the Δ-interior $\Delta Int(\mathcal{B})$ is Δ-open in $\mathfrak{Y}$. Then f is Δ-open.

Proof. Let $\mathcal{A}$ be a Δ-open set in $\mathfrak{X}$. Then by definition, $\Delta Int(\mathcal{A}) = \mathcal{A}$, so we obtain

$$f(\mathcal{A}) = f\big(\Delta Int(\mathcal{A})\big) \subseteq \Delta Int\big(f(\mathcal{A})\big).$$

However, since $\Delta Int\big(f(\mathcal{A})\big) \subseteq f(\mathcal{A})$, it follows that

$$f(\mathcal{A}) = \Delta Int\big(f(\mathcal{A})\big).$$

By assumption, $\Delta Int\big(f(\mathcal{A})\big) = f(\mathcal{A})$ is Δ-open in $\mathfrak{Y}$. Thus, f is Δ-open. □

Proposition 5.8. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be topological spaces. If $f : \mathfrak{X} \to \mathfrak{Y}$ is Δ-closed, then for any subset $\mathcal{A} \subseteq \mathfrak{X}$, we have*

$$\Delta Cl\big(f(\mathcal{A})\big) \subseteq f\big(\Delta Cl(\mathcal{A})\big).$$

Proof. Assume that f is Δ-closed and let $\mathcal{A} \subseteq \mathfrak{X}$. Then, for any Δ-closed set $\mathcal{C}$ in $\mathfrak{X}$ with $\mathcal{C} \supseteq \mathcal{A}$, we have that $f(\mathcal{C})$ is Δ-closed in $\mathfrak{Y}$ and clearly satisfies $f(\mathcal{C}) \supseteq f(\mathcal{A})$. Therefore, we obtain

$$\Delta Cl\big(f(\mathcal{A})\big) = \bigcap_{\substack{K \supseteq f(\mathcal{A}) \\ K\ \Delta\text{-closed}}} K \subseteq \bigcap_{\substack{\mathcal{C} \supseteq \mathcal{A} \\ \mathcal{C}\ \Delta\text{-closed}}} f(\mathcal{C}) \subseteq f\left(\bigcap_{\substack{\mathcal{C} \supseteq \mathcal{A} \\ \mathcal{C}\ \Delta\text{-closed}}} \mathcal{C}\right) = f\big(\Delta Cl(\mathcal{A})\big).$$

□

The following result provides a partial converse to Proposition 5.8.

Proposition 5.9. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be topological spaces, and let $f : \mathfrak{X} \to \mathfrak{Y}$. Suppose that for every subset $\mathcal{A} \subseteq \mathfrak{X}$, we have*

$$\Delta Cl\big(f(\mathcal{A})\big) \subseteq f\big(\Delta Cl(\mathcal{A})\big),$$

and that for every subset $\mathcal{B} \subseteq \mathfrak{Y}$, the Δ-closure $\Delta Cl(\mathcal{B})$ is Δ-closed in $\mathfrak{Y}$. Then f is Δ-closed.

Proof. Let $\mathcal{A}$ be a Δ-closed set in $\mathfrak{X}$. Then by definition, $\Delta Cl(\mathcal{A}) = \mathcal{A}$, so we obtain

$$\Delta Cl\big(f(\mathcal{A})\big) \subseteq f\big(\Delta Cl(\mathcal{A})\big) = f(\mathcal{A}).$$

However, since $f(\mathcal{A}) \subseteq \Delta Cl\big(f(\mathcal{A})\big)$, it follows that

$$f(\mathcal{A}) = \Delta Cl\big(f(\mathcal{A})\big).$$

By assumption, $\Delta Cl\big(f(\mathcal{A})\big) = f(\mathcal{A})$ is Δ-closed in $\mathfrak{Y}$. Thus, f is Δ-closed. □

Proposition 5.10. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be topological spaces, and let $f : \mathfrak{X} \to \mathfrak{Y}$ be Δ-open. Then for each subset $\mathcal{B} \subseteq \mathfrak{Y}$ and each Δ-closed set $\mathcal{H}$ in $\mathfrak{X}$ containing $f^{-1}(\mathcal{B})$, there exists a Δ-closed set $\mathcal{K}$ in $\mathfrak{Y}$ such that $\mathcal{K} \supseteq \mathcal{B}$ and $f^{-1}(\mathcal{K}) \subseteq \mathcal{H}$.*

Proof. Let $\mathcal{B} \subseteq \mathfrak{Y}$ and let $\mathcal{H}$ be a Δ-closed set in $\mathfrak{X}$ such that $\mathcal{H} \supseteq f^{-1}(\mathcal{B})$. Then $\mathfrak{X} - \mathcal{H}$ is Δ-open in $\mathfrak{X}$, and since f is Δ-open, it follows that $f(\mathfrak{X} - \mathcal{H})$ is Δ-open in $\mathfrak{Y}$. Define

$$\mathcal{K} = \mathfrak{Y} - f(\mathfrak{X} - \mathcal{H}).$$

Clearly, $\mathcal{K}$ is Δ-closed in $\mathfrak{Y}$, and we have

$$\begin{aligned}
\mathcal{H} \supseteq f^{-1}(\mathcal{B}) \Leftrightarrow \mathfrak{X} - \mathcal{H} \subseteq \mathfrak{X} - f^{-1}(\mathcal{B}) \\
&\Rightarrow f(\mathfrak{X} - \mathcal{H}) \subseteq f\big(\mathfrak{X} - f^{-1}(\mathcal{B})\big) \\
&\Leftrightarrow \mathcal{K} = \mathfrak{Y} - f(\mathfrak{X} - \mathcal{H}) \supseteq \mathfrak{Y} - f\big(\mathfrak{X} - f^{-1}(\mathcal{B})\big) \\
&\Rightarrow \mathcal{K} \supseteq \mathfrak{Y} - f\big(\mathfrak{X} - f^{-1}(\mathcal{B})\big) \supseteq \mathcal{B} \\
&\Rightarrow \mathcal{K} \supseteq \mathcal{B}.
\end{aligned}$$

Furthermore, we verify that $f^{-1}(\mathcal{K}) \subseteq \mathcal{H}$:

$$\begin{aligned}
f^{-1}(\mathcal{K}) &= f^{-1}\big(\mathfrak{Y} - f(\mathfrak{X} - \mathcal{H})\big) \\
&= f^{-1}(\mathfrak{Y}) - f^{-1}\big(f(\mathfrak{X} - \mathcal{H})\big) \\
&= \mathfrak{X} - f^{-1}\big(f(\mathfrak{X} - \mathcal{H})\big) \\
&\subseteq \mathfrak{X} - (\mathfrak{X} - \mathcal{H}) = \mathcal{H}.
\end{aligned}$$

□

Proposition 5.11. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be topological spaces, and let $f : \mathfrak{X} \to \mathfrak{Y}$ be Δ-closed. Then for each subset $\mathcal{B} \subseteq \mathfrak{Y}$ and each Δ-open set $\mathcal{U}$ in $\mathfrak{X}$ containing $f^{-1}(\mathcal{B})$, there exists a Δ-open set $\mathcal{V}$ in $\mathfrak{Y}$ such that $\mathcal{V} \supseteq \mathcal{B}$ and $f^{-1}(\mathcal{V}) \subseteq \mathcal{U}$.*

Proof. Let $\mathcal{B} \subseteq \mathfrak{Y}$ and let $\mathcal{U}$ be a Δ-open set in $\mathfrak{X}$ with $\mathcal{U} \supseteq f^{-1}(\mathcal{B})$. Then $\mathfrak{X} - \mathcal{U}$ is Δ-closed in $\mathfrak{X}$, and since f is Δ-closed, it follows that $f(\mathfrak{X} - \mathcal{U})$ is Δ-closed in $\mathfrak{Y}$. Define

$$\mathcal{V} = \mathfrak{Y} - f(\mathfrak{X} - \mathcal{U}).$$

Clearly, $\mathcal{V}$ is Δ-open in $\mathfrak{Y}$, and we have

$$\begin{aligned}
\mathcal{U} \supseteq f^{-1}(\mathcal{B}) \Leftrightarrow \mathfrak{X} - \mathcal{U} \subseteq \mathfrak{X} - f^{-1}(\mathcal{B}) \\
&\Rightarrow f(\mathfrak{X} - \mathcal{U}) \subseteq f\big(\mathfrak{X} - f^{-1}(\mathcal{B})\big) \\
&\Leftrightarrow \mathcal{V} = \mathfrak{Y} - f(\mathfrak{X} - \mathcal{U}) \supseteq \mathfrak{Y} - f\big(\mathfrak{X} - f^{-1}(\mathcal{B})\big)
\end{aligned}$$

$$\Rightarrow \mathcal{V} \supseteq \mathfrak{Y} - f\big(\mathfrak{X} - f^{-1}(\mathcal{B})\big) \supseteq \mathcal{B}$$

$$\Rightarrow \mathcal{V} \supseteq \mathcal{B}.$$

Furthermore, we verify that $f^{-1}(\mathcal{V}) \subseteq \mathcal{U}$:

$$\begin{aligned} f^{-1}(\mathcal{V}) &= f^{-1}\big(\mathfrak{Y} - f(\mathfrak{X} - \mathcal{U})\big) \\ &= f^{-1}(\mathfrak{Y}) - f^{-1}\big(f(\mathfrak{X} - \mathcal{U})\big) \\ &= \mathfrak{X} - f^{-1}\big(f(\mathfrak{X} - \mathcal{U})\big) \\ &\subseteq \mathfrak{X} - (\mathfrak{X} - \mathcal{U}) = \mathcal{U}. \end{aligned}$$

□

6. Conclusion

In this work, we introduced four distinct classes of mappings in topological spaces, each characterized by its interaction with Δ-open and Δ-closed sets.

The first class, Δ-continuous mappings, is defined by the condition that the inverse image of any open set must be Δ-open. A comprehensive characterization of these mappings, incorporating closed and Δ-closed sets, is provided in Theorem 3.3.

Next, we examined Δ-irresolute mappings, which are distinguished by the property that the inverse image of every Δ-open set remains Δ-open. Theorem 4.4 presents a fundamental characterization of Δ-irresolute mappings in terms of Δ-closed sets.

The hierarchical relationships among these classes are encapsulated in the implications

$$\text{continuous mapping} \Rightarrow \Delta\text{-irresolute mapping} \Rightarrow \Delta\text{-continuous mapping}.$$

However, as demonstrated in Example 3.2 and Example 4.2, these implications are not necessarily reversible.

The third and fourth classes-Δ-open and Δ-closed mappings—further enrich this framework. A Δ-open mapping ensures that the image of any Δ-open set remains Δ-open, while a Δ-closed mapping guarantees that the image of every Δ-closed set is also Δ-closed. The distinctions between Δ-open, Δ-closed, and Δ-continuous mappings, as illustrated in Examples 5.3 and 5.4, underscore the intricate relationships among these concepts.

Looking ahead, we intend to extend this study to explore Δ-compactness, Δ-connectedness, and Δ-separation axioms in topological spaces, further expanding the theoretical foundation of Δ-structured mappings.

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