Asia Pac. J. Math. 2025 **12**:39

ON TOTAL STABILITY ANALYSIS OF CAPUTO FRACTIONAL DYNAMIC EQUATIONS ON TIME SCALE

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Received Feb. 4, 2025

Abstract. This study explores the total stability of Caputo fractional dynamic equations on time scales, bridging discrete and continuous dynamics. By extending the Caputo fractional delta Dini derivative, it introduces T_1-total stability and T_2-total stability criteria to address bounded perturbations in initial conditions and system parameters. The framework unifies and extends existing stability concepts, effectively handling hybrid systems with gradual and abrupt changes. Rigorous proofs and illustrative examples demonstrate the practical implications of the theory, highlighting its applicability.

2020 Mathematics Subject Classification. 34A08; 34A34; 34D20; 34N05.

Key words and phrases. uniform stability, Caputo derivative, Lyapunov function, fractional dynamic equation, time scale.

1. INTRODUCTION

Since the pioneering work on the stability analysis of Caputo fractional differential equations in [1,15], the field has undergone significant advancements see [8–14, 19]. A major milestone was achieved in [17], which introduced a novel generalized derivative, termed the Caputo fractional delta Dini derivative on time scales. This derivative serves as a foundational tool for analyzing the stability of hybrid systems—systems characterized by generalized domains that combine discrete, continuous, or both types of time scales. The introduction of this derivative has spurred the development of diverse stability results for various systems, as evidenced by works such as [16, 20–22]. Notably, in [16], we demonstrated that a fractional system on time scales is uniformly asymptotically stable, a result that underpins the current study.

DOI: 10.28924/APJM/12-39

The theorems presented in this work emphasize the critical importance of uniform asymptotic stability. Specifically, they reveal that if the trivial solution of a system is uniformly asymptotically stable, it retains specific stability properties under different classes of perturbations. This observation forms the basis for exploring total stability, a concept of profound practical relevance.

Broadly defined, total stability ensures that bounded perturbations in both initial conditions and the system's right-hand side result in bounded effects on the solution trajectory. This property is indispensable for the practical implementation of mathematical models, as real-world systems frequently encounter uncertainties, approximations, and perturbations. The analysis of total stability, therefore, plays a pivotal role in validating the robustness of a system, particularly when the governing equations represent approximations of complex natural phenomena.

The present work focuses on the total stability analysis of Caputo fractional dynamic equations on time scales. By extending the definition of stability to incorporate perturbations in both initial conditions and the right-hand side, this work offers a comprehensive framework for assessing the robustness of fractional systems. Leveraging the generalized Caputo fractional derivative on time scales introduced in our previous studies, we develop new stability criteria tailored to nonlinear fractional dynamic equations.

Our methodology not only unifies the stability analysis of fractional differential and difference equations but also provides distinct advantages in addressing hybrid systems. By integrating the dynamics of discrete and continuous domains, the proposed framework bridges gaps in existing stability theories and expands their applicability to a broader range of systems.

The motivation for studying total stability extends well beyond theoretical considerations. Numerous physical, biological, and engineering systems exhibit dynamics that are inherently fractional and subject to perturbations. For instance, in control systems, environmental factors or modeling inaccuracies frequently introduce perturbations, necessitating robust stability guarantees. By establishing a rigorous framework for total stability analysis, this study enhances the reliability and practical applicability of fractional-order models across diverse disciplines.

The remainder of this manuscript is structured as follows: In Section 2 we review essential preliminaries, including definitions, lemmas, the Caputo fractional derivative on time scales, and key concepts of total stability. In Section 3 we introduce the main results, including T_1-total stability and T_2-total stability criteria, along with their proofs. In Section 4 we present illustrative examples demonstrating the practical implications of the developed theory. Finally, in Section 5 we give a summary of the established results and implication.

2. Preliminaries Notes

In this section, we lay the groundwork by introducing key notations and definitions that will be instrumental in developing the main results.

Definition 2.1. ([5]) *For $t \in \mathbb{T}$, we define the forward jump operator as $\wp(t) = \inf\{s \in \mathbb{T} : s > t\}$ and the backward jump operator as $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$. $t \in \mathbb{T}$ is said to be right scattered (r-s) if $\wp(t) > t$, left scattered (l-s) if $\rho(t) < t$, right dense (r-d) if $\wp(t) = t$ and left dense (l-d) if $\rho(t) = t$. A function $\kappa(t) = \wp(t) - t$ is called the graininess function.*

Definition 2.2 ([5]). *A function $\mathfrak{G} : \mathbb{T} \to \mathbb{R}$ is said to be r-d continuous (C_{rd}) if it is continuous at all right-dense points of $\mathbb{T}$, and its left-sided limits exist and are finite at all left-dense points of $\mathbb{T}$.*

Definition 2.3 ([5]). *If a function $\mathfrak{K} \in C[[0, j], [0, \infty)]$ is strictly increasing on $[0, j]$ with $\mathfrak{K}(0) = 0$, then it is called a class $\mathcal{K}$ function.*

Definition 2.4 ([5]). *Let $a, b \in \mathbb{T}$ and $\mathfrak{F} \in C_{rd}$. The integral on a time scale $\mathbb{T}$ is defined as follows:*

(i) *If $\mathbb{T} = \mathbb{R}$, then*

$$\int_a^b \mathfrak{F}(t)\Delta t = \int_a^b \mathfrak{F}(t)dt.$$

where $\int_a^b \mathfrak{F}(t)dt$ is the usual Riemann integral from calculus.

(ii) *If $[a, b]$ consists of only isolated points, then*

$$\int_a^b \mathfrak{F}(t)\Delta t = \begin{cases} \sum_{t\in[a,b)} \kappa(t)\mathfrak{F}(t) & \text{if} \quad a < b \\ 0 & \text{if} \quad a = b \\ -\sum_{t\in[b,a)} \kappa(t)\mathfrak{F}(t) & \text{if} \quad a > b. \end{cases}$$

(iii) *If there exists a point $\wp(t) > t$, then*

$$\int_t^{\wp(t)} \mathfrak{F}(s)\Delta s = \kappa(t)f(t).$$

Definition 2.5 ([6]). *A continuous function $\mathcal{V} : \mathbb{R}^n \to \mathbb{R}$ with $\mathcal{V}(0) = 0$ is called positive definite(negative definite) on the domain D if there exists a function $\phi \in \mathcal{K}$ such that $\phi(|x|) \le \mathcal{V}(x)$ $(\phi(|x|) \le -\mathcal{V}(x))$ for $x \in D$.*

Definition 2.6. [4] *Let $h \in C^{\alpha}_{rd}[\mathbb{T}, \mathbb{R}^n]$, the Grunwald-Letnikov fractional delta derivative is given by*

$$^{GL}\Delta^{\alpha}_{0}h(t) = \lim_{\mu\to 0+} \frac{1}{\mu^{\alpha}} \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha} C_r[h(\sigma(t) - r\mu)]. \quad t \ge t_0, \tag{1}$$

and the Grunwald-Letnikov fractional delta dini derivative is given by

$$^{GL}\Delta^{\alpha}_{0+}h(t) = \limsup_{\mu\to 0+} \frac{1}{\mu^{\alpha}} \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha} C_r[h(\sigma(t) - r\mu)]. \quad t \ge t_0, \tag{2}$$

where $0 < \alpha < 1$, ${}^{\alpha}C_r = \frac{\alpha(\alpha-1)\ldots(\alpha-r+1)}{r!}$, *and* $[\frac{(t-t_0)}{\mu}]$ *denotes the integer part of the fraction* $\frac{(t-t_0)}{\mu}$.

Observe that if the domain is $\mathbb{R}$, *then* (2) *becomes*

$$^{GL}\Delta^{\alpha}_{0+}h(t) = \limsup_{d\to 0+} \frac{1}{d^{\alpha}} \sum_{r=0}^{[\frac{(t-t_0)}{d}]} (-1)^{r\alpha}C_r[h(t-rd)], \quad t \geq t_0.$$

Remark 2.1. [17] *Given that* $\lim\limits_{N\to\infty} \sum_{r=0}^{N}(-1)^{r\alpha}C_r = 0$ *where* $\alpha \in (0,1)$, *and* $\lim\limits_{\mu\to 0^+}[\frac{(t-t_0)}{\mu}] = \infty$ *then it is easy to see that*

$$\lim_{\mu\to 0^+} \sum_{r=1}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha}C_r = -1. \tag{3}$$

Also, since the Caputo and Riemann-Liouville formulations coincide when $h(t_0) = 0$, *then we have that*

$$\limsup_{\mu\to 0^+} \frac{1}{\mu^{\alpha}} \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha}C_r =^{RL} \Delta^{\alpha}(1) = \frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}, \quad t \geq t_0. \tag{4}$$

Statement of the Problem

Consider the Caputo fractional dynamic system of order α with $0 < \alpha < 1$

$$\begin{aligned} {}^{C}\Delta^{\alpha}\chi &= \mathfrak{f}(t,\chi),\ t \in \mathbb{T}, \\ \chi(t_0) &= \chi_0,\ t_0 \geq 0, \end{aligned} \tag{5}$$

where $\mathfrak{f} \in C_{rd}[\mathbb{T}\times\mathbb{R}^n, \mathbb{R}^n]$, $\mathfrak{f}(t,0) \equiv 0$ and ${}^{C}\Delta^{\alpha}\chi$ is the Caputo fractional delta derivative of $\chi \in \mathbb{R}^n$ of order α with respect to $t \in \mathbb{T}$. Let $\chi(t) = \chi(t,t_0,\chi_0) \in C^{\alpha}_{rd}[\mathbb{T},\mathbb{R}^n]$ (the fractional derivative of order α of $\chi(t)$ exist and it is rd-continuous) be a solution of (5) and assume the solution exists and is unique (results on existence and uniqueness of (5) are contained in [2,7,18,19,23,24]), this work aims to investigate the T_1-total stability and T_2-total stability of the system (5).

Also, consider the comparison system of the form

$$^{C}\Delta^{\alpha}u = g(t,u),\ u(t_0) = u_0 \geq 0, \tag{6}$$

where $u \in \mathbb{R}_+$, $g : \mathbb{T}\times\mathbb{R}_+ \to \mathbb{R}_+$ and $g(t,0) \equiv 0$. For this work, we will assume that the function $g \in [\mathbb{T}\times\mathbb{R}_+, \mathbb{R}_+]$, is such that for any initial data $(t_0,u_0) \in \mathbb{T}\times\mathbb{R}_+$, the system (6) with $u(t_0) = u_0$ has a unique solution $u(t) = u(t;t_0,u_0) \in C^{\alpha}_{rd}[\mathbb{T},\mathbb{R}_+]$ see [2,25–28].

Definition 2.7. *Let* $[t_0,T)_{\mathbb{T}} = [t_0,T)\cap\mathbb{T}$ *where* $T > t_0$. *Then a function* $V(t,\chi) \in C_{rd}[\mathbb{T}\times\mathbb{R}^n,\mathbb{R}_+]$ *then we define the generalized Caputo fractional dini derivative relative to* (5) *as follows: given* $\epsilon > 0$, *there exists a* *neighborhood $P(\epsilon)$ of $t \in \mathbb{T}$ such that*

$$\frac{1}{\mu^q}\Big\{V(\sigma(t),\chi(\sigma(t)) - V(t_0,\chi_0) - \sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^{r+1}({}^qC_r)[V(\sigma(t)-r\mu,\chi(\sigma(t)-\mu^q f(t,\chi(t)) - V(t_0,\chi_0)]\Big\} < {}^C\Delta_+^\alpha V(t,\chi)+\epsilon,$$

for each $s \in P(\epsilon)$ and $s > t$, where $\mu = \sigma(t) - t$ and $\chi(t) = \chi(t,t_0,\chi_0)$ is any solution of (5).

Definition 2.8. [3,17] *Using Definition 2.7, we state the Caputo fractional delta derivative and Dini derivative of $V(t,\chi)$ as*

$$ {}^C\Delta_+^\alpha V(t,\chi) = \limsup_{\mu\to 0^+}\frac{1}{\mu^\alpha}\Bigg[\sum_{r=0}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^\alpha C_r)[V(\sigma(t)-r\mu,\chi(\sigma(t)) - \mu^\alpha f(t,\chi(t)) - V(t_0,\chi_0)]\Bigg], \quad (7)$$

and can be expanded as

$$ {}^C\Delta_+^\alpha V(t,\chi) = \limsup_{\mu\to 0^+}\frac{1}{\mu^\alpha}\Big\{V(\sigma(t),\chi(\sigma(t)) - V(t_0,\chi_0) - \sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^{r+1}({}^\alpha C_r)[V(\sigma(t)-r\mu,\chi(\sigma(t)) - \mu^\alpha f(t,\chi(t)) - V(t_0,\chi_0)]\Big\}, \quad (8)$$

where $t \in \mathbb{T}$, and $\chi,\chi_0 \in \mathbb{R}^n$, $\mu = \sigma(t) - t$ and $\chi(\sigma(t)) - \mu^\alpha f(t,\chi) \in \mathbb{R}^n$.

If $\mathbb{T}$ is discrete and $V(t,\chi(t))$ is continuous at t, the Caputo fractional delta Dini derivative of the Lyapunov function in discrete times, is given by:

$$ {}^C\Delta_+^\alpha V(t,\chi) = \frac{1}{\mu^\alpha}\Bigg[\sum_{r=0}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^\alpha C_r)(V(\sigma(t),\chi(\sigma(t))) - V(t_0,\chi_0))\Bigg], \quad (9)$$

and if $\mathbb{T}$ is continuous, that is $\mathbb{T} = \mathbb{R}$, and $V(t,\chi(t))$ is continuous at t, we have that

$$ {}^C\Delta_+^\alpha V(t,\chi) = {}^C D_+^\alpha V(t,\chi) = \limsup_{\kappa\to 0^+}\frac{1}{\kappa^\alpha}\Big\{V(t,\chi(t)) - V(t_0,x_0) - \sum_{r=1}^{[\frac{t-t_0}{\kappa}]}(-1)^{r+1}({}^\alpha C_r)[V(t-r\kappa,\chi(t)) - \kappa^\alpha f(t,\chi(t)) - V(t_0,\chi_0)]\Big\}. \quad (10)$$

Notice that (10) is the same in [1] where $\kappa > 0$.

Consider a perturbed form of (5)

$$ {}^C\Delta^\alpha\chi = \mathfrak{f}(t,\chi) + \mathfrak{M}(t,\chi),\ \chi(t_0) = \chi_0 \geq 0, \quad (11)$$

where $\chi \in \mathbb{S}_\rho$, $\mathfrak{f},\mathfrak{M} \in C_{rd}[\mathbb{T}\times\mathbb{S}_\rho,\mathbb{R}^n]$ and $\mathfrak{f}(t,0) \equiv 0$. For this work, we will assume that a a unique solution $\chi(t) = \chi(t;t_0,\chi_0)$ of (11) exist for any initial data $(t_0,\chi_0) \in \mathbb{T}\times\mathbb{S}_\rho$, see [2,19].

Definition 2.9. *The trivial solution $\chi = 0$ of system (5) is said to be:*

(T_1) *T_1-totally stable or stable with respect to permanent perturbations if, for every $\epsilon > 0$, $t_0 \in \mathbb{T}$, we can find two positive numbers $\delta_1 = \delta_1(\epsilon)$ and $\delta_2 = \delta_2(\epsilon)$ such that for every solution $\chi(t; t_0, \chi_0)$ of the perturbed Caputo fractional dynamic system (11), the inequality*

$$\|\chi(t; t_0, \chi_0)\| < \epsilon, \quad t \geq t_0,$$

holds, whenever

$$\|\chi\| < \delta_1,$$

and

$$\|\mathfrak{M}(t, \chi) < \delta_2 \quad \text{for } \|\chi\| < \epsilon, \ t \in \mathbb{T}. \tag{12}$$

(T_2) *T_2-totally stable or stable with respect to permanent perturbations bounded in the mean if, for every $\epsilon > 0$, $t_0 \in \mathbb{T}$, and $\mathfrak{T} > 0$, there exist two positive numbers $\delta_1 = \delta_1(\epsilon)$ and $\delta_2 = \delta_2(\epsilon)$ such that for every solution $\chi(t; t_0, \chi_0)$ of the perturbed Caputo fractional dynamic system (11), the inequality*

$$\|\chi(t; t_0, \chi_0)\| < \epsilon, \quad t \geq t_0,$$

holds, provided

$$\|\chi_0\| < \delta_1, \quad \|\mathfrak{M}(t, \chi) < \gamma(t), \quad \text{for } \|\chi\| \leq \epsilon, \ t \in \mathbb{T} \tag{13}$$

and

$$\int_t^{t+\mathfrak{T}} \gamma(s) \Delta s < \delta_2. \tag{14}$$

Lemma 2.1. [17] *Assume that*

(i) *$g \in C_{rd}[\mathbb{T} \times \mathbb{R}^n_+, \mathbb{R}^n_+]$ and $g(t, u)\mu$ is non-decreasing in u.*

(ii) *$V \in C_{rd}[\mathbb{T} \times \mathbb{R}^N, \mathbb{R}^N_+]$ be locally Lipschitzian in the second variable such that*

$${}^{C}\Delta^{\alpha}_{+} V(t, \chi) \leq g(t, V(t, \chi)), (t, \chi) \in \mathbb{T} \times \mathbb{R}^N. \tag{15}$$

(iii) *$z(t) = z(t; t_0, u_0)$ is the maximal solution of (6) existing on $\mathbb{T}$.*

Then

$$V(t, \chi(t)) \leq z(t), \quad t \geq t_0, \tag{16}$$

provided that

$$V(t_0, \chi_0) \leq u_0, \tag{17}$$

where $\chi(t) = \chi(t; t_0, \chi_0)$ is any solution of (5), $t \in \mathbb{T}$, $t \geq t_0$.

Lemma 2.2 (Uniform asymptotic Stability)**.** [16] *Assume the following*

1. *$g \in C_{rd}[\mathbb{T} \times \mathbb{R}_+, \mathbb{R}_+]$ and $g(t, u)$ is non-decreasing in u with $g(t, u) \equiv 0$,*

2. $V(t,\chi(t)) \in C_{rd}[\mathbb{T}\times\mathbb{R}^N,\mathbb{R}_+]$ *be such that*
   (i) *V is locally Lipschitzian in χ with $V(t,0)\equiv 0$*
   (ii) *$b(\|\chi\|) \le V(t,\chi) \le a(\|\chi\|)$ where $a,b\in\mathcal{K}$*
   (iii) *For any points $t,t_0\ge 0$ and $\chi,\chi_0\in R^N$, the inequality*

$$^{C\mathbb{T}}D_+^{\alpha}V^{\Delta}(t,x) \le -c(\|\chi\|).$$

Then the trivial solution $x=0$ of the FrDE (5) is uniformly asymptotically stable.

3. Main Result

In this section, we give the stability criteria for the fractional dynamic system (5) with respect to permanent perturbations and perturbations bounded in the mean.

Theorem 3.1 (T_1-Total Stability)**.** *Assume that*

1. $\mathfrak{f}(t,\chi)\in C_{rd}[\mathbb{T}\times\mathbb{R}^N,\mathbb{S}_\rho]$ *is also Lipschitzian such that*
   (i) $\|\mathfrak{f}(t,\chi)-\mathfrak{f}(t,\upsilon)\le\mathcal{L}(t)\|\chi-\upsilon\|$, *where $(t,\chi),(t,\upsilon)\in\mathbb{T}\times\mathbb{S}_\rho$ and $\mathcal{L}>0$ is the Lipschitz constant,*
   (ii) $\left|\int_t^{t+u}\mathcal{L}(s)\right| \le \daleth|u|$ *where $\daleth>0$,*
2. $g\in C_{rd}[\mathbb{T}\times\mathbb{R}_+,\mathbb{R}_+]$ *and $g(t,u)$ is non-decreasing in u with $g(t,u)\equiv 0$,*
3. *Lemma 2.2 holds,*

then the trivial solution $\chi=0$ of (5) is T_1-totally stable.

Proof. Choose δ_1 such that for a given $\epsilon>0$, $\delta_1(\epsilon)$,

$$b(\epsilon) > a(\delta_1), \tag{18}$$

holds.

Let $\chi(t)=\chi(t;t_0,\chi_0)$ be any solution of (11), and set $h(t)=V(t,\chi(t))$. Since $V(t,0)=0$ and $V\in C_{rd}$, it implies V is continuous at the origin, such that given $\epsilon>0$, we can find a $\delta_1(\epsilon)>0$ such that for $\chi_0\in\mathbb{S}_\rho$, we have that

$$\|\chi_0\|<\delta_1 \implies V(t_0,\chi_0)<\epsilon. \tag{19}$$

From condition 2(ii) of Lemma 2.2 and (19), we deduce that $h(t_0) < a(\delta_1)$. Then, we can then comfortably claim that:

$$h(t) < b(\epsilon)\ t\ge t_0. \tag{20}$$

If this claim were false, then we would be able to find two numbers $t_2>t_1>t_0$:

$$h(t_1)=a(\delta_1),\quad h(t_1)=b(\epsilon),$$

and for $t_1\le t\le t_2$,

$$h(t)\ge a(\delta_1).$$

Implying that

$$^{C}\Delta_{+}^{\alpha}h(t_1) \geq 0. \tag{21}$$

Clearly, for $t_0 \leq t \leq t_2$, we obtain

$$b\|\chi(t\|) \leq h(t) \leq b(\epsilon).$$

So that

$$\|\chi(t)\| \leq \epsilon, \quad \text{whenever } t \in [t_0, t_2]. \tag{22}$$

By the Lipschitzian property of $V(t, \chi)$ and from condition 2(iii) of Lemma 2.2, we state that

$$^{C}\Delta_{+}^{\alpha}h(t_1) \leq -c(a(\delta_1)) + L\|\mathfrak{M}(t_1, \chi(t_1))\|. \tag{23}$$

Setting $\delta^* = \delta^*(\epsilon) = \frac{c(a(\delta_1(\epsilon)))}{L}$, then in-view of (12), (23), becomes

$$^{C}\Delta_{+}^{\alpha}h(t_1) < 0,$$

contradicting (21) and validating (20). Then we can immediately conclude that the trivial solution of (5) is T_1-totally stable. □

Theorem 3.2. *Assume that Theorem 3.1 holds, then the trivial solution of (5) is T_2-totally stable.*

Proof. Given a positive ϵ, we can choose a $\delta_1(\epsilon)$ such that

$$\|\chi_0\| < \delta_1.$$

Set $h(t) = V(t, x(t))$, where $x(t) = x(t; t_0, x_0)$ is any solution of (11). As in Theorem 3.1, for $h(t_0) = a(\delta_1)$, we claim that (20) is true.

If this claim where false, we would be a time $t_1 > t_0$ such that $m(t) \leq b(\epsilon)$ for $t \in [t_0, t_1]$, so that

$$\|\chi(t)\| \leq \epsilon < \delta(\delta_*), \ t_0 \leq t \leq t_1.$$

Choose $\delta^* > 0$, such that $\delta^*(\epsilon)$ and

$$\delta^*(\epsilon) < b(\epsilon) - B^{-}[B(a(\delta_1))]/L, \tag{24}$$

where

$$B(u) - B(u_0) = \int_{u_0}^{u} \frac{1}{c(s)} ds,$$

and B^{-} is the inverse of B. By the assumption that Theorem 3.1 holds, which would immediately imply that conditions 2(ii) and 2(iii) of Lemma 2.2 is also applicable, we can deduce that

$$^{C}\Delta_{+}^{\alpha}V(t, \chi(t)) \leq -c[V(t, \chi(t))] + L\|\mathfrak{M}(t, \chi(t))\|, \ t \in [t_0, t_1]. \tag{25}$$

Set $m(t) = V(t, x(t)) - \wp$ where $\wp = L\int_{t_0}^{t} \|\mathfrak{M}(s, \chi(s))\| ds$.

So that

$$^{C}\Delta_{+}^{\alpha}m(t) \leq -c(m(t)).$$

Since $c \in \mathcal{K}$ and $m(t) \geq V(t, \chi(t))$, we obtain

$$m(t) \leq B^{-}[B(V(t_0, \chi_0)) - (t - t_0)], \ t \in [t_0, t_1].$$

Then, it is obvious to see that

$$V(t, \chi(t)) \leq B^{-}[B(V(t_0, \chi_0)) - (t - t_0)] + \wp, \ t \in [t_0, t_1].$$

Since the trivial solution of (5) is uniformly asymptotically stable from Lemma 2.2 that is $\|\chi(t)\| \leq \epsilon$ for $t_0 \leq t \leq t_0 + T$, $V(t_0, \chi_0) < a(\delta_1)$ and from (13), (14), and (24), we obtain

$$b(\epsilon) \leq V(t_0 + T, \chi(t_0 + T)) \leq B^{-}[(Ba(\delta_1)) - T] + L\delta^{*} < b(\epsilon). \tag{26}$$

Equation (26) is a contradiction, so it immediately implies that our initial assumption of (20) is true. So that we can confidently conclude that the trivial solution of (5) is T_2-totally stable. □

4. Illustration

Consider the Caputo fractional dynamic system

$$\begin{aligned} {}^{C}\Delta^{\alpha} a(t) &= \qquad 4a - \frac{b^2 + 3c^2}{a} \\ {}^{C}\Delta^{\alpha} b(t) &= -\frac{3a^2 +}{b} + 4b + \frac{c^2}{b} \\ {}^{C}\Delta^{\alpha} c(t) &= \qquad -\frac{2b^2}{c} + 3c, \end{aligned} \tag{27}$$

for $t \geq t_0$, with initial conditions

$$a(t_0) = a_0, \quad b(t_0) = b_0, \quad \text{and} \quad c(t_0) = c_0,$$

where $\chi = (a, b, c)$, and $\mathfrak{f} = (\mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{f}_3)$.

We choose our Lyapunov candidate function to be $V(t, a, b, c) = a^2 + b^2 + c^2$, for $t \in \mathbb{T}$ and $(a, b, c) \in \mathbb{R}^3$. From (8), we compute the Caputo fractional Dini derivative for $V(t, a, b, c) = a^2 + b^2 + c^2$ as follows

$$\begin{aligned} & {}^{C}\Delta_{+}^{\alpha} V(a, b, c) \\ = \ & \limsup_{\mu \to 0^{+}} \frac{1}{\mu^{\alpha}} \Big\{ \big[(a(\sigma(t)))^2 + (b(\sigma(t)))^2 + (c(\sigma(t)))^2\big] - \big[(a_0)^2 + (b_0)^2 + (c_0)^2\big] \\ & + \sum_{r=1}^{[\frac{t-t_0}{\mu}]} (-1)^r ({}^{\alpha}C_r)[(a(\sigma(t)) - \mu^{\alpha}\mathfrak{f}_1)^2 \\ & + (b(\sigma(t)) - \mu^{\alpha}\mathfrak{f}_2)^2 + (c(\sigma(t)) - \mu^{\alpha}\mathfrak{f}_3)^2 - ((a_0)^2 + (b_0)^2) + (c_0)^2] \Big\} \\ = \ & \limsup_{\mu \to 0^{+}} \frac{1}{\mu^{\alpha}} \Big\{ \big[(a(\sigma(t)))^2 + (b(\sigma(t)))^2 + (c(\sigma(t)))^2\big] - \big[(a_0)^2 + (b_0)^2 + (c_0)^2\big] \end{aligned}$$

$$
\begin{aligned}
&+\sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^{\alpha}C_r)[(a(\sigma(t)))^2-2a(\sigma(t))\mu^{\alpha}\mathfrak{f}_1+\mu^{2\alpha}(\mathfrak{f}_1)^2\\
&+(b(\sigma(t)))^2-2b(\sigma(t))\mu^{\alpha}\mathfrak{f}_2+\mu^{2\alpha}(\mathfrak{f}_2)^2+(c(\sigma(t)))^2-2c(\sigma(t))\mu^{\alpha}\mathfrak{f}_3+\mu^{2\alpha}(\mathfrak{f}_3)^2\\
&-((a_0)^2+(b_0)^2+(c_0)^2)]\Big\}\\
=\;&\limsup_{\mu\to 0^+}\frac{1}{\mu^{\alpha}}\Big\{-\sum_{r=0}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^{\alpha}C_r)\left[(a_0)^2+(b_0)^2+(c_0)^2\right]\\
&+\sum_{r=0}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^{\alpha}C_r)\left[(a(\sigma(t)))^2+(b(\sigma(t)))^2+(c(\sigma(t)))^2\right]\\
&-\sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^{\alpha}C_r)\left[2a(\sigma(t))\mu^{\alpha}\mathfrak{f}_1+2b(\sigma(t))\mu^{\alpha}\mathfrak{f}_2+2c(\sigma(t))\mu^{\alpha}\mathfrak{f}_3\right]\\
&+\sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^{\alpha}C_r)\left[\mu^{2\alpha}(\mathfrak{f}_1)^2+\mu^{2\alpha}(\mathfrak{f}_2)^2+\mu^{2\alpha}(\mathfrak{f}_3)^2\right]\Big\}\\
=\;&-\limsup_{\mu\to 0^+}\frac{1}{\mu^{\alpha}}\left\{\sum_{r=0}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^{\alpha}C_r)\left[(a_0)^2+(b_0)^2+(c_0)^2\right]\right\}\\
&+\limsup_{\mu\to 0^+}\frac{1}{\mu^{\alpha}}\left\{\sum_{r=0}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^{\alpha}C_r)\left[(a(\sigma(t)))^2+(b(\sigma(t)))^2+(c(\sigma(t)))^2\right]\right\}\\
&-\limsup_{\mu\to 0^+}\frac{1}{\mu^{\alpha}}\left\{\sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^{\alpha}C_r)\left[2a(\sigma(t))\mu^{\alpha}\mathfrak{f}_1+2b(\sigma(t))\mu^{\alpha}\mathfrak{f}_2+2c(\sigma(t))\mu^{\alpha}\mathfrak{f}_3\right]\right\}.
\end{aligned}
$$

Applying (3) and (4) we have

$$
\begin{aligned}
{}^{C}\Delta_{+}^{\alpha}V(a,b,c) \quad &= -\frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}\left((a_0)^2+(b_0)^2+(c_0)^2\right)\\
&+\frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}\left((a(\sigma(t)))^2+(b(\sigma(t)))^2+(c(\sigma(t)))^2)\right)\\
&-[2a(\sigma(t))\mathfrak{f}_1+2b(\sigma(t))\mathfrak{f}_2+2c(\sigma(t))\mathfrak{f}_3]
\end{aligned}
$$

As $t\to\infty$, $\frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}\left[(a(\sigma(t)))^2+(b(\sigma(t)))^2+(c(\sigma(t)))^2\right]\to 0$, then

$$
{}^{C}\Delta_{+}^{\alpha}V(a,b,c) \quad \leq \quad -2[a(\sigma(t))\mathfrak{f}_1+b(\sigma(t))\mathfrak{f}_2+c(\sigma(t))\mathfrak{f}_3]
$$

applying $\chi(\sigma(t))\leq\mu^{C}\Delta^{\alpha}\chi(t)+\chi(t)$

$$
\begin{aligned}
{}^{C}\Delta_{+}^{\alpha}V(a,b,c) \quad &= \quad -2\Big[\mu(t)\mathfrak{f}_1^2+a(t)\mathfrak{f}_1+\mu(t)\mathfrak{f}_2^2+b(t)\mathfrak{f}_2+\mu(t)\mathfrak{f}_3^2+c(t)\mathfrak{f}_3\Big]\\
&= \quad -2\Big[\mu(t)(4a-\frac{b^2+3c^2}{a})^2+a(4a-\frac{b^2+3c^2}{a})+\mu(t)(-\frac{3a^2+}{b}+4b+\frac{c^2}{b})^2
\end{aligned}
$$

$$+b(-\frac{3a^2+}{b}+4b+\frac{c^2}{b})+\mu(t)(-\frac{2b^2}{c}+3c)^2+c(-\frac{2b^2}{c}+3c)\Big]$$

$$= -2\left[4a^2-b^2-3c^2-3a^2+4b^2+c^2-2b^2+3c^2\right]$$

$$-2\mu\left[\left(4a-\frac{b^2+3c^2}{a}\right)^2+\left(-\frac{3a^2+}{b}+4b+\frac{c^2}{b}\right)^2+\left(-\frac{2b^2}{c}+3c\right)^2\right]. \quad (28)$$

If $\mathbb{T}=\mathbb{R}$ we have that $\mu=0$, so that (28) becomes;

$$^{C}\Delta_{+}^{\alpha}V(a,b,c)\leq -2\Big[a^2+b^2+c^2\Big].$$

If $\mathbb{T}=\mathbb{N}_0$ we have that $\mu=1$, so that (28) becomes;

$$\begin{aligned} &= -2\left[a^2+b^2+c^2\right] \\ &\quad -2\left[\left(4a-\frac{b^2+3c^2}{a}\right)^2+\left(-\frac{3a^2+}{b}+4b+\frac{c^2}{b}\right)^2+\left(-\frac{2b^2}{c}+3c\right)^2\right] \\ &\leq -2\left[a^2+b^2+c^2\right]. \end{aligned}$$

Therefore, we conclude that

$$^{C}\Delta_{+}^{\alpha}V(a,b,c)\leq -2\left[V(a,b,c)\right].$$

so that

$$V(t,\chi)=V(t_0,\chi_0)E_{\alpha}(-2t^{\alpha}). \quad (29)$$

Clearly, (29) depicts the total stability of system (27).

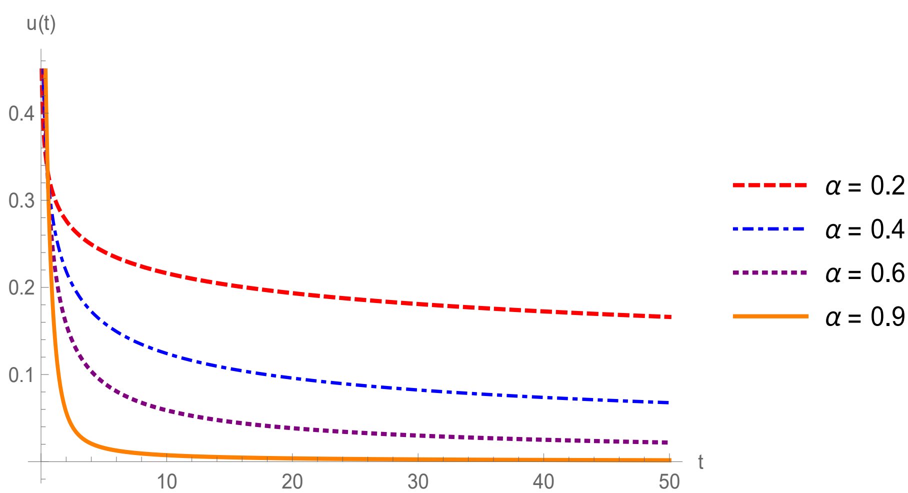

FIGURE 1. Graph of $u(t)=E_{\alpha,1}(-2t^{\alpha})$ against t for various values of α

Figure 1 above is the graphical representation of $u(t)=E_{\alpha,1}(-2t^{\alpha})$. The behaviour of the curves further buttresses the stability of the solution $u(t)$ of over time for different values of $\alpha\in(0,1)$.

5. Conclusion

This manuscript has presented a comprehensive framework for the total stability analysis of Caputo fractional dynamic equations on time scales. By leveraging the generalized Caputo fractional delta Dini derivative, we have developed new criteria for T_1-total stability and T_2-total stability, demonstrating their practical implications through illustrative examples. The unification of stability theories for fractional differential and difference equations within a hybrid system context marks a significant advancement in the field. Beyond theoretical insights, this work highlights the importance of robust stability guarantees for real-world systems subject to perturbations. Future research directions include exploring extensions to more complex systems, refining stability criteria, and broadening the scope of applications in multidisciplinary domains.

Authors' contributions. All authors contributed equally to the manuscript.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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