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APPLICATIONS OF LOCAL FRACTIONAL VARIATIONAL ITERATION METHOD FOR SOLVING PARTIAL DIFFERENTIAL EQUATIONS WITHIN LOCAL FRACTIONAL OPERATORS

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Abstract. In this paper, the local fractional variational iteration method (LFVIM) is first proposed in order to solve linear and nonlinear partial differential equations (PDEs) with local fractional derivative operators (LFDOs). Some examples are given to illustrate that this method provides us with a convenient way to find non-differentiable solutions to local fractional differential equations.
Key words and phrases. local fractional variational iteration method; Laplace transform; Local fractional operator.

1. Introduction

Many problems of physics and engineering are expressed by ODEs and PDEs, which are termed boundary value problems. We can mention, for example, the wave, the diffusion, the Laplace, the Klein Gordon, the Schrodinger, the Advection, the Burgers, the Boussinesq, and the Fisher equations, and others [1,2].

The history of research on PDEs goes back to the 18th century in the work of Euler, Alembert, Lagrange and Laplace as a central tool in the descripton of mechanics of continua and more generally, as the principal mode of analytical study of models in the physical science. The analysis of physical models has remaned to the present day one of the fundamntal concerns of the development of PDEs. Beginning in the middle of the 19th century, particularly with the work of Riemann, PDEs also became an esential tool in other branches of mathematics [4,5].

Nowadays, PDEs have become a suitable tool for descrbing the natural phenomena of science and enginering models. Most of the phenomena arising in mathematcal physics and engineering fields

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can be expresed by PDEs. Many engineering applications are simulated mathmatically as PDEs with initial and boundary conditions. Most physical phenomena of fluid dynamics, gravitation, chemical reaction, dispersion, nonlinear optics, plasma physics, quntum mechanics, and many other models are controlled within their domain of valdity by PDEs. Therefore, it becomes increasingly imporant to be familiar with all tradtional and recently developed methods for solving partial differential equations and the implemenations of these methods [4,5].

The variational iteration method (VIM) was first proposed by He [6–10] who was succesfully applied to autonmous ODEs [9], to nonlinear PDEs with variable coefficients [10], Schrodinger-KdV, generalized KdV and shallowwater equations [11], nonlinear evolution equations [12], nonlinear systems of partial differential equations [12], nonlinear heat transfer equations [13], Burger's and coupled Burger'sequations [14], the epidemic model and the prey and predtor problem [15], linear Helmholtz equation [16] and recently to nonlinear fractional differential equations with Caputodifferential derivative [17], and other fields, [18,19].

The local fractional VIM [20–28] was utilized to solve the PDEs on Cantor sets such as, wave [20], Helmholtz [21], Laplace [22], diffusion [23], and Fokker-Planck [24] equations.

Recently, there are manyanalytical and numercal methods used to solve PDEs on Cantor sets such as, Adomian decomposition method [2,20,31,32], series expansion method [31,33], Fourier series method [34,35], Laplace transform method [36], Picard successive approximation method [37], homotopy perturbation method [38,39], similarity solution [40], reduce differential transform method [41], differential transform method [42,42,44], Laplace decomposition method [45,46], and another methods [47–69] involving the local fractional operators.

In this chapter, the LFVIM is first proposed in order to solve PDEs that arising in mathematical physics, nolinear gasdynamic equation, two-dimensional diffusion model in fractal heattransfer and system of coupled Korteweg-de Vries Equations involving LFDOs.

2. Analysis of the Local Fractional VIM

To illustrate the basic concept of the local fractional variational Iteration method, we consider the following general local fractional partial differential equation:

$$L_\vartheta \psi(\eta,\kappa) + R_\vartheta \psi(\eta,\kappa) + N_\vartheta \psi(\eta,\kappa) = \Lambda(\eta,\kappa),\ 0 < \vartheta \leq 1 \tag{2.1}$$

where $L_\vartheta = \dfrac{\partial^{m\vartheta}}{\partial \eta^{m\vartheta}}$ denotes the linear local fractional differential operator, R_ϑ is the remaining linear local fractional operators, N_ϑ denotes nonlinear local fractional operators and $\Lambda(\eta,\kappa)$ is a source term. The local fractional variational iteration algorithm for (2.1) is defined by [20–23,28]:

$$\psi_{n+1}(\eta) = \psi_n(\eta) + \frac{1}{\Gamma(1+\vartheta)} \int_0^\eta \sigma \left\{ L_\vartheta \psi_n(\xi) + R_\vartheta \widetilde{\psi_n}(\xi) + N_\vartheta \widetilde{\psi_n}(\xi) - \Lambda(\xi) \right\} (d\xi)^\vartheta, \tag{2.2}$$

where σ is a fractal Lagrange multiplier.

Making use of (2.2), the iteration formula can be reformulates as follows:

$$\delta^{\vartheta}\psi_{n+1}(\eta)=\delta^{\vartheta}\psi_n(\eta)+\frac{\delta^{\vartheta}}{\Gamma(1+\vartheta)}\int_0^{\eta}\sigma\left\{L_{\vartheta}\psi_n(\xi)+R_{\vartheta}\widetilde{\psi_n}(\xi)+N_{\vartheta}\widetilde{\psi_n}(\xi)-\Lambda(\xi)\right\}(d\xi)^{\vartheta}, \quad (2.3)$$

which leads to

$$\begin{aligned}&(\sigma(\eta))^{m\vartheta}\mid_{\eta=\xi}=0,\ 1+(-1)^{m-1}\sigma^{(m-1)\vartheta}(\eta)\mid_{\eta=\xi}=0,\\&\sigma^{(j\vartheta)}(\eta)\mid_{\eta=\xi}=0,\ j=0,1,\ldots,m-2.\end{aligned} \quad (2.4)$$

where $\delta^{\vartheta}\widetilde{\psi_n}$ denotes a restricted local fractional variation, that is, $\delta^{\vartheta}\widetilde{\psi_n}=0$.

Therefore, following (2.4) we have the fractal Lagrange multiplier, which can be identified as follows:

$$\sigma(\eta)=\frac{(-1)^{(m-1)}(\xi-\eta)^{(m-1)\vartheta}}{\Gamma(1+(m-1)\vartheta)}. \quad (2.5)$$

In view of (2.2) and (2.5), the local fractional variational iteration algorithm can be written as follows:

$$\begin{aligned}\psi_{n+1}(\eta)=\psi_n(\eta)+\frac{1}{\Gamma(1+\vartheta)}\times \quad (2.6)\\ \int_0^{\eta}\frac{(-1)^{(m-1)}(\xi-\eta)^{(m-1)\vartheta}}{\Gamma(1+(m-1)\vartheta)}\left\{L_{\vartheta}\psi_n(\xi)+R_{\vartheta}\psi_n(\xi)+N_{\vartheta}\psi_n(\xi)-\Lambda(\xi)\right\}(d\xi)^{\vartheta},\end{aligned}$$

for $n\geq 0$ the successive approximations $\psi_{m+1}(\eta,\kappa)$ of the solution $\psi(\eta,\kappa)$ can be given by using any selective local fractional function $\psi_0(\eta,\kappa)$.

Consequently, we can obtain the solution in the following form:

$$\psi(\eta,\kappa)=\lim_{m\to\infty}\psi_m(\eta,\kappa) \quad (2.7)$$

3. Applications of LFVIM to Partial Differential Equations That Arising in Mathematical Physics

Example 3.1. . Let us start with local fractional Laplace equation given in the following form:

$$\frac{\partial^{2\vartheta}\psi(\eta,\kappa)}{\partial\kappa^{2\vartheta}}+\frac{\partial^{2\vartheta}\psi(\eta,\kappa)}{\partial\eta^{2\vartheta}}=0, \quad (3.1)$$

and subject to the fractal value conditions

$$\begin{aligned}&\psi(\eta,0)=-E_{\vartheta}(\eta^{\vartheta}),\\&\frac{\partial^{\vartheta}\psi(\eta,0)}{\partial\kappa^{\vartheta}}=0,\end{aligned} \quad (3.2)$$

Following (2.6), we arrive at the iteration relation given by

$$\psi_{n+1}(\eta,\kappa)=\psi_n(\eta,\kappa)+\frac{1}{\Gamma(1+\vartheta)}\int_0^{\kappa}\frac{(\xi-\kappa)^{\vartheta}}{\Gamma(1+\vartheta)}\left\{\frac{\partial^{2\vartheta}\psi_n(\eta,\xi)}{\partial\xi^{2\vartheta}}+\frac{\partial^{2\vartheta}\psi_n(\eta,\xi)}{\partial\eta^{2\vartheta}}\right\}(d\xi)^{\vartheta}. \quad (3.3)$$

From(3.2) we take the initial value, which reads as

$$\psi_0(\eta,\kappa)=-E_{\vartheta}(\eta^{\vartheta}). \quad (3.4)$$

Hence, we can derive the fisrt approximation term as

$$\begin{aligned}\psi_1(\eta,\kappa) &= \psi_0(\eta,\kappa)+\frac{1}{\Gamma(1+\vartheta)}\int_0^{\kappa}\frac{(\xi-\kappa)^{\vartheta}}{\Gamma(1+\vartheta)}\left\{\frac{\partial^{2\vartheta}\psi_0(\eta,\xi)}{\partial\xi^{2\vartheta}}+\frac{\partial^{2\vartheta}\psi_0(\eta,\xi)}{\partial\eta^{2\vartheta}}\right\}(d\xi)^{\vartheta}\\ &= E_{\vartheta}(\eta^{\vartheta})\left[-1+\frac{\kappa^{2\vartheta}}{\Gamma(1+2\vartheta)}\right],\end{aligned}$$

In the same manner, the second approximation is given by

$$\begin{aligned}\psi_2(\eta,\kappa) &= \psi_1(\eta,\kappa)+\frac{1}{\Gamma(1+\vartheta)}\int_0^{\kappa}\frac{(\xi-\kappa)^{\vartheta}}{\Gamma(1+\vartheta)}\left\{\frac{\partial^{2\vartheta}\psi_1(\eta,\xi)}{\partial\xi^{2\vartheta}}+\frac{\partial^{2\vartheta}\psi_1(\eta,\xi)}{\partial\eta^{2\vartheta}}\right\}(d\xi)^{\vartheta}\\ &= E_{\vartheta}(\eta^{\vartheta})\left[-1+\frac{\kappa^{2\vartheta}}{\Gamma(1+2\vartheta}-\frac{\kappa^{4\vartheta}}{\Gamma(1+4\vartheta}\right].\end{aligned}$$

The third approximation can be calculated in the similar way, which is

$$\begin{aligned}\psi_3(\eta,\kappa) &= \psi_2(\eta,\kappa)+\frac{1}{\Gamma(1+\vartheta)}\int_0^{\kappa}\frac{(\xi-\kappa)^{\vartheta}}{\Gamma(1+\vartheta)}\left\{\frac{\partial^{2\vartheta}\psi_2(\eta,\xi)}{\partial\xi^{2\vartheta}}+\frac{\partial^{2\vartheta}\psi_2(\eta,\xi)}{\partial\eta^{2\vartheta}}\right\}(d\xi)^{\vartheta}\\ &= E_{\vartheta}(\eta^{\vartheta})\left[-1+\frac{\kappa^{2\vartheta}}{\Gamma(1+2\vartheta}-\frac{\kappa^{4\vartheta}}{\Gamma(1+4\vartheta}+\frac{\kappa^{6\vartheta}}{\Gamma(1+6\vartheta}\right].\end{aligned}$$

Finally, we can obtain the local fractional series solution as follows:

$$\psi(\eta,\kappa)=-E_{\vartheta}(\eta^{\vartheta})\left[\sum_{\rho=0}^{n}(-1)^{\rho}\frac{\kappa^{2\rho\vartheta}}{\Gamma(1+2\rho\vartheta)}\right]. \tag{3.5}$$

Thus, the expression of the final solution is given by

$$\begin{aligned}\psi(\eta,\kappa) &= \lim_{m\to\infty}\psi_m(\eta,\kappa)\\ &= E_{\vartheta}(\eta^{\vartheta})\left[\sum_{\rho=0}^{\infty}(-1)^{\rho}\frac{\kappa^{2\rho\vartheta}}{\Gamma(1+2\rho\vartheta)}\right]\\ &= -E_{\vartheta}(\eta^{\vartheta})cos_{\vartheta}(\eta^{\vartheta}).\end{aligned} \tag{3.6}$$

Example 3.2. [20]. Let us consider wave equation within local fractional operators:

$$\frac{\partial^{2\vartheta}\psi(\eta,\kappa)}{\partial\kappa^{2\vartheta}}-\frac{\eta^{2\vartheta}}{\Gamma(1+2\vartheta)}\frac{\partial^{2\vartheta}\psi(\eta,\kappa)}{\partial\eta^{2\vartheta}}=0, 0<\vartheta\leqslant 1 \tag{3.7}$$

is presented and its initial valuses are defined as follows:

$$\psi(\eta,0)=\frac{\eta^{2\vartheta}}{\Gamma(1+2\vartheta)},\ \frac{\partial^{\vartheta}\psi(\eta,0)}{\partial\kappa^{\vartheta}}=0. \tag{3.8}$$

Using (2.6), we have the iterative formula:

$$\begin{aligned}&\psi_{n+1}(\eta,\kappa)=\psi_n(\eta,\kappa)+\frac{1}{\Gamma(1+\vartheta)}\times\\ &\int_0^{\kappa}\frac{(\xi-\kappa)^{\vartheta}}{\Gamma(1+\vartheta)}\left\{\frac{\partial^{2\vartheta}\psi_n(\eta,\xi)}{\partial\xi^{2\vartheta}}-\frac{\eta^{2\vartheta}}{\Gamma(1+2\vartheta)}\frac{\partial^{2\vartheta}\psi_n(\eta,\xi)}{\partial\eta^{2\vartheta}}\right\}(d\xi)^{\vartheta}.\end{aligned} \tag{3.9}$$

From (3.8)we take the initial value, which reads as

$$\psi_0(\eta,\kappa) = \frac{\eta^{2\vartheta}}{\Gamma(1+2\vartheta)}. \tag{3.10}$$

Hence, we can derive the fisrt approximation term as

$$\begin{aligned}
\psi_1(\eta,\kappa) &= \psi_0(\eta,\kappa) + \frac{1}{\Gamma(1+\vartheta)} \times \\
&\int_0^\kappa \frac{(\xi-\kappa)^\vartheta}{\Gamma(1+\vartheta)} \left\{ \frac{\partial^{2\vartheta}\psi_0(\eta,\xi)}{\partial\xi^{2\vartheta}} - \frac{\eta^{2\vartheta}}{\Gamma(1+2\vartheta)} \frac{\partial^{2\vartheta}\psi_0(\eta,\xi)}{\partial\eta^{2\vartheta}} \right\} (d\xi)^\vartheta \\
&= \frac{\eta^{2\vartheta}}{\Gamma(1+2\vartheta)} \left[1 + \frac{\kappa^{2\vartheta}}{\Gamma(1+2\vartheta} \right],
\end{aligned}$$

The second approximation can be calculated in the similar way, which is

$$\begin{aligned}
\psi_2(\eta,\kappa) &= \psi_1(\eta,\kappa) + \frac{1}{\Gamma(1+\vartheta)} \times \\
&\int_0^\kappa \frac{(\xi-\kappa)^\vartheta}{\Gamma(1+\vartheta)} \left\{ \frac{\partial^{2\vartheta}\psi_1(\eta,\xi)}{\partial\xi^{2\vartheta}} - \frac{\eta^{2\vartheta}}{\Gamma(1+2\vartheta)} \frac{\partial^{2\vartheta}\psi_1(\eta,\xi)}{\partial\eta^{2\vartheta}} \right\} (d\xi)^\vartheta \\
&= \frac{\eta^{2\vartheta}}{\Gamma(1+2\vartheta)} \left[1 + \frac{\kappa^{2\vartheta}}{\Gamma(1+2\vartheta} + \frac{\kappa^{4\vartheta}}{\Gamma(1+4\vartheta} \right].
\end{aligned}$$

Proceeding in this manner, we get the third approximation as

$$\begin{aligned}
\psi_3(\eta,\kappa) &= \psi_2(\eta,\kappa) + \frac{1}{\Gamma(1+\vartheta)} \times \\
&\int_0^\kappa \frac{(\xi-\kappa)^\vartheta}{\Gamma(1+\vartheta)} \left\{ \frac{\partial^{2\vartheta}\psi_2(\eta,\xi)}{\partial\xi^{2\vartheta}} - \frac{\eta^{2\vartheta}}{\Gamma(1+2\vartheta)} \frac{\partial^{2\vartheta}\psi_2(\eta,\xi)}{\partial\eta^{2\vartheta}} \right\} (d\xi)^\vartheta \\
&= \frac{\eta^{2\vartheta}}{\Gamma(1+2\vartheta)} \left[1 + \frac{\kappa^{2\vartheta}}{\Gamma(1+2\vartheta} + \frac{\kappa^{4\vartheta}}{\Gamma(1+4\vartheta} + \frac{\kappa^{6\vartheta}}{\Gamma(1+6\vartheta} \right],
\end{aligned}$$

and so on. Therefore, the expression of the final solution is given by

$$\begin{aligned}
\psi(\eta,\kappa) &= \lim_{m\to\infty} \psi_m(\eta,\kappa) \\
&= \frac{\eta^{2\vartheta}}{\Gamma(1+2\vartheta)} \left[\sum_{\rho=0}^{\infty} \frac{\kappa^{2\rho\vartheta}}{\Gamma(1+2\rho\vartheta)} \right] \\
&= \frac{\eta^{2\vartheta}}{\Gamma(1+2\vartheta)} cosh_\vartheta(\kappa^\vartheta).
\end{aligned} \tag{3.11}$$

Example 3.3. . We present the following local fractional inhomogeneous Helmholtz equation:

$$\frac{\partial^{2\vartheta}\varphi(\eta,\kappa)}{\partial\kappa^{2\vartheta}} + \frac{\partial^{2\vartheta}\varphi(\eta,\kappa)}{\partial\eta^{2\vartheta}} + \varphi(\eta,\kappa) = \frac{\kappa^\vartheta}{\Gamma(1+\vartheta)} E_\vartheta(\eta^\vartheta),\ 0 < \vartheta \leqslant 1 \tag{3.12}$$

with the initial value conditions as follows:

$$\varphi(\eta,0) = E_\vartheta(-\eta^\vartheta),\ \frac{\partial^\vartheta\varphi(\eta,0)}{\partial\kappa^\vartheta} = 0.$$

Making use of (2.6), we structure the local fractional variational iteration algorithm as follows:

$$\varphi_{n+1}(\eta,\kappa) = \varphi_n(\eta,\kappa) + \frac{1}{\Gamma(1+\vartheta)} \times \int_0^{\kappa} \frac{(\xi-\kappa)^{\vartheta}}{\Gamma(1+\vartheta)} \left\{ \frac{\partial^{2\vartheta}\varphi_n(\eta,\xi)}{\partial \xi^{2\vartheta}} + \frac{\partial^{2\vartheta}\varphi_n(\eta,\xi)}{\partial \eta^{2\vartheta}} + \varphi_n(\eta,\xi) - \frac{\xi^{\vartheta}}{\Gamma(1+\vartheta)} E_{\vartheta}(\eta^{\vartheta}) \right\} (d\xi)^{\vartheta}, \tag{3.13}$$

where the initial value is presented as

$$\varphi_0(\eta,\kappa) = \frac{\kappa^{\vartheta}}{\Gamma(1+\vartheta)} E_{\vartheta}(-\eta^{\vartheta}). \tag{3.14}$$

In view of (3.13) and (3.14), we obtain the following approximations:

$$\begin{aligned} \varphi_1(\eta,\kappa) &= \varphi_0(\eta,\kappa) + \frac{1}{\Gamma(1+\vartheta)} \times \\ &\quad \int_0^{\kappa} \frac{(\xi-\kappa)^{\vartheta}}{\Gamma(1+\vartheta)} \left\{ \frac{\partial^{2\vartheta}\varphi_0(\eta,\xi)}{\partial \xi^{2\vartheta}} + \frac{\partial^{2\vartheta}\varphi_0(\eta,\xi)}{\partial \eta^{2\vartheta}} + \varphi_0(\eta,\xi) - \frac{\xi^{\vartheta}}{\Gamma(1+\vartheta)} E_{\vartheta}(\eta^{\vartheta}) \right\} (d\xi)^{\vartheta} \\ &= \frac{\kappa^{\vartheta}}{\Gamma(1+\vartheta)} E_{\vartheta}(-\eta^{\vartheta}) + \frac{1}{\Gamma(1+\vartheta)} \int_0^{\kappa} \frac{(\xi-\kappa)^{\vartheta}}{\Gamma(1+\vartheta)} \left\{ \frac{\kappa^{\vartheta}}{\Gamma(1+\vartheta)} E_{\vartheta}(-\eta^{\vartheta}) \right\} (d\xi)^{\vartheta} \\ &= \frac{\kappa^{\vartheta}}{\Gamma(1+\vartheta)} E_{\vartheta}(-\eta^{\vartheta}) - \frac{\kappa^{3\vartheta}}{\Gamma(1+3\vartheta)} E_{\vartheta}(-\eta^{\vartheta}), \\ &= E_{\vartheta}(-\eta^{\vartheta}) \left[\frac{\kappa^{\vartheta}}{\Gamma(1+\vartheta)} - \frac{\kappa^{3\vartheta}}{\Gamma(1+3\vartheta)} \right], \end{aligned} \tag{3.15}$$

The second approximation is

$$\begin{aligned} \varphi_2(\eta,\kappa) &= \varphi_1(\eta,\kappa) + \frac{1}{\Gamma(1+\vartheta)} \times \\ &\quad \int_0^{\kappa} \frac{(\xi-\kappa)^{\vartheta}}{\Gamma(1+\vartheta)} \left\{ \frac{\partial^{2\vartheta}\varphi_1(\eta,\xi)}{\partial \xi^{2\vartheta}} + \frac{\partial^{2\vartheta}\varphi_1(\eta,\xi)}{\partial \eta^{2\vartheta}} + \varphi_1(\eta,\xi) - \frac{\xi^{\vartheta}}{\Gamma(1+\vartheta)} E_{\vartheta}(\eta^{\vartheta}) \right\} (d\xi)^{\vartheta} \\ &= E_{\vartheta}(-\eta^{\vartheta}) \left[\frac{\kappa^{\vartheta}}{\Gamma(1+\vartheta)} - \frac{\kappa^{3\vartheta}}{\Gamma(1+3\vartheta)} \right] + \\ &\quad \frac{1}{\Gamma(1+\vartheta)} \int_0^{\kappa} \frac{(\xi-\kappa)^{\vartheta}}{\Gamma(1+\vartheta)} \left\{ -\frac{2\xi^{3\vartheta}}{\Gamma(1+3\vartheta)} E_{\vartheta}(-\eta^{\vartheta}) \right\} (d\xi)^{\vartheta} \\ &= \frac{\kappa^{\vartheta}}{\Gamma(1+\vartheta)} E_{\vartheta}(-\eta^{\vartheta}) - \frac{\kappa^{3\vartheta}}{\Gamma(1+3\vartheta)} E_{\vartheta}(-\eta^{\vartheta}) + \frac{2\kappa^{5\vartheta}}{\Gamma(1+5\vartheta)} E_{\vartheta}(-\eta^{\vartheta}), \\ &= E_{\vartheta}(-\eta^{\vartheta}) \left[\frac{\kappa^{\vartheta}}{\Gamma(1+\vartheta)} - \frac{\kappa^{3\vartheta}}{\Gamma(1+3\vartheta)} + \frac{2\kappa^{5\vartheta}}{\Gamma(1+5\vartheta)} \right]. \end{aligned}$$

The second approximation is

$$\begin{aligned} \varphi_3(\eta,\kappa) &= \varphi_2(\eta,\kappa) + \frac{1}{\Gamma(1+\vartheta)} \times \\ &\quad \int_0^{\kappa} \frac{(\xi-\kappa)^{\vartheta}}{\Gamma(1+\vartheta)} \left\{ \frac{\partial^{2\vartheta}\varphi_2(\eta,\xi)}{\partial \xi^{2\vartheta}} + \frac{\partial^{2\vartheta}\varphi_2(\eta,\xi)}{\partial \eta^{2\vartheta}} + \varphi_2(\eta,\xi) - \frac{\xi^{\vartheta}}{\Gamma(1+\vartheta)} E_{\vartheta}(\eta^{\vartheta}) \right\} (d\xi)^{\vartheta} \\ &= E_{\vartheta}(-\eta^{\vartheta}) \left[\frac{\kappa^{\vartheta}}{\Gamma(1+\vartheta)} - \frac{\kappa^{3\vartheta}}{\Gamma(1+3\vartheta)} + \frac{2\kappa^{5\vartheta}}{\Gamma(1+5\vartheta)} \right] + \end{aligned}$$

$$\begin{aligned}
&\frac{1}{\Gamma(1+\vartheta)}\int_0^{\kappa}\frac{(\xi-\kappa)^{\vartheta}}{\Gamma(1+\vartheta)}\left\{\frac{4\kappa^{5\vartheta}}{\Gamma(1+5\vartheta)}E_{\vartheta}(-\eta^{\vartheta})\right\}(d\xi)^{\vartheta}\\
=\;&\frac{\kappa^{\vartheta}}{\Gamma(1+\vartheta)}E_{\vartheta}(-\eta^{\vartheta})-\frac{\kappa^{3\vartheta}}{\Gamma(1+3\vartheta)}E_{\vartheta}(-\eta^{\vartheta})+\frac{2\kappa^{5\vartheta}}{\Gamma(1+5\vartheta)}E_{\vartheta}(-\eta^{\vartheta})-\\
&\frac{4\kappa^{7\vartheta}}{\Gamma(1+7\vartheta)}E_{\vartheta}(-\eta^{\vartheta}),\\
=\;&E_{\vartheta}(-\eta^{\vartheta})\left[\frac{\kappa^{\vartheta}}{\Gamma(1+\vartheta)}-\frac{\kappa^{3\vartheta}}{\Gamma(1+3\vartheta)}+\frac{2\kappa^{5\vartheta}}{\Gamma(1+5\vartheta)}-\frac{4\kappa^{7\vartheta}}{\Gamma(1+7\vartheta)}\right].
\end{aligned}$$

and so on.

Hence, we have the local fractional series solution of (3.13):

$$\begin{aligned}
\varphi_n(\eta,\kappa) &= E_{\vartheta}(-\eta^{\vartheta})\left[\frac{\kappa^{\vartheta}}{\Gamma(1+\vartheta)}-\frac{\kappa^{3\vartheta}}{\Gamma(1+3\vartheta)}+\frac{2\kappa^{5\vartheta}}{\Gamma(1+5\vartheta)}-\frac{4\kappa^{7\vartheta}}{\Gamma(1+7\vartheta)}+\cdots\right]\\
&= E_{\vartheta}(-\eta^{\vartheta})\left[\frac{1}{2}\frac{\kappa^{\vartheta}}{\Gamma(1+\vartheta)}+\frac{1}{2}\sum_{i=0}^{n}(-1)^i\frac{2^i\kappa^{(2i+1)\vartheta}}{\Gamma(1+(2i+1)\vartheta}\right].
\end{aligned}$$

From (2.7), we get the exact solution of (3.13) given as

$$\begin{aligned}
\varphi(\eta,\kappa) &= \lim_{n\to\infty}\varphi_n(\eta,\kappa)\\
&= \lim_{n\to\infty}\left(E_{\vartheta}(-\eta^{\vartheta})\left[\frac{1}{2}\frac{\kappa^{\vartheta}}{\Gamma(1+\vartheta)}+\frac{1}{2}\sum_{i=0}^{n}(-1)^i\frac{2^i\kappa^{(2i+1)\vartheta}}{\Gamma(1+(2i+1)\vartheta}\right]\right)\\
&= E_{\vartheta}(-\eta^{\vartheta})\left[\frac{1}{2}\frac{\kappa^{\vartheta}}{\Gamma(1+\vartheta)}+\frac{1}{2}\sin_{\vartheta}(2\kappa^{\vartheta})\right]. \qquad (3.16)
\end{aligned}$$

4. Applications of VIM for Solving Nonlinear Gas Dynamic Equation with Local Fractional Operators

Example 4.1. *Let us consider the following nonlinear gas dynamics equation involving local fractional derivative operators:*

$$\frac{\partial^{\vartheta}\varphi(\eta,\kappa)}{\partial\kappa^{\vartheta}}+\frac{1}{2}\frac{\partial^{\vartheta}\varphi^2(\eta,\kappa)}{\partial\eta^{\vartheta}}+\varphi^2(\eta,\kappa)-\varphi(\eta,\kappa)=0,\ 0<\vartheta\leqslant 1 \qquad (4.1)$$

with the initial value conditions as follows:

$$\varphi(\eta,0)=E_{\vartheta}(-\eta^{\vartheta}). \qquad (4.2)$$

By using (2.6), we have the local fractional iteration procedure as:

$$\begin{aligned}
\varphi_{n+1}(\eta,\kappa) &= \varphi_n(\eta,\kappa)-\frac{1}{\Gamma(1+\vartheta)} \qquad (4.3)\\
&\int_0^{\kappa}\left\{\frac{\partial^{\vartheta}\varphi_n(\eta,\xi)}{\partial\xi^{\vartheta}}+\frac{1}{2}\frac{\partial^{\vartheta}\varphi_n^2(\eta,\xi)}{\partial\eta^{\vartheta}}+\varphi_n^2(\eta,\xi)-\varphi_n(\eta,\xi)\right\}(d\xi)^{\vartheta}.
\end{aligned}$$

From (4.2) we take the initial value, which reads as

$$\varphi_0(\eta,\kappa)=E_{\vartheta}(-\eta^{\vartheta}). \qquad (4.4)$$

In view of (4.3) and (4.4), we obtain the following approximations:

$$\begin{aligned}
\varphi_1(\eta,\kappa) &= \varphi_0(\eta,\kappa) - \frac{1}{\Gamma(1+\vartheta)} \\
&\quad \int_0^\kappa \left\{ \frac{\partial^\vartheta \varphi_0(\eta,\xi)}{\partial \xi^\vartheta} + \frac{1}{2}\frac{\partial^\vartheta \varphi_0^2(\eta,\xi)}{\partial \eta^\vartheta} + \varphi_0^2(\eta,\xi) - \varphi_n(\eta,\xi) \right\} (d\xi)^\vartheta \\
&= E_\vartheta(-\eta^\vartheta)\left[1 + \frac{\kappa^\vartheta}{\Gamma(1+\vartheta}\right], \\
\varphi_2(\eta,\kappa) &= \varphi_1(\eta,\kappa) - \frac{1}{\Gamma(1+\vartheta)} \\
&\quad \int_0^\kappa \left\{ \frac{\partial^\vartheta \varphi_1(\eta,\xi)}{\partial \xi^\vartheta} + \frac{1}{2}\frac{\partial^\vartheta \varphi_1^2(\eta,\xi)}{\partial \eta^\vartheta} + \varphi_1^2(\eta,\xi) - \varphi_1(\eta,\xi) \right\} (d\xi)^\vartheta \\
&= E_\vartheta(-\eta^\vartheta)\left[1 + \frac{\kappa^\vartheta}{\Gamma(1+\vartheta} + \frac{\kappa^{2\vartheta}}{\Gamma(1+2\vartheta}\right], \\
\varphi_3(\eta,\kappa) &= \varphi_2(\eta,\kappa) - \frac{1}{\Gamma(1+\vartheta)} \\
&\quad \int_0^\kappa \left\{ \frac{\partial^\vartheta \varphi_2(\eta,\xi)}{\partial \xi^\vartheta} + \frac{1}{2}\frac{\partial^\vartheta \varphi_2^2(\eta,\xi)}{\partial \eta^\vartheta} + \varphi_2^2(\eta,\xi) - \varphi_2(\eta,\xi) \right\} (d\xi)^\vartheta \\
&= E_\vartheta(-\eta^\vartheta)\left[1 + \frac{\kappa^\vartheta}{\Gamma(1+\vartheta} + \frac{\kappa^{2\vartheta}}{\Gamma(1+2\vartheta} + \frac{\kappa^{3\vartheta}}{\Gamma(1+3\vartheta}\right], \\
\varphi_4(\eta,\kappa) &= \varphi_3(\eta,\kappa) - \frac{1}{\Gamma(1+\vartheta)} \\
&\quad \int_0^\kappa \left\{ \frac{\partial^\vartheta \varphi_3(\eta,\xi)}{\partial \xi^\vartheta} + \frac{1}{2}\frac{\partial^\vartheta \varphi_3^2(\eta,\xi)}{\partial \eta^\vartheta} + \varphi_3^2(\eta,\xi) - \varphi_3(\eta,\xi) \right\} (d\xi)^\vartheta \\
&= E_\vartheta(-\eta^\vartheta)\left[1 + \frac{\kappa^\vartheta}{\Gamma(1+\vartheta} + \frac{\kappa^{2\vartheta}}{\Gamma(1+2\vartheta} + \frac{\kappa^{3\vartheta}}{\Gamma(1+3\vartheta} + \frac{\kappa^{4\vartheta}}{\Gamma(1+4\vartheta}\right], \\
&\vdots
\end{aligned}$$

$$\begin{aligned}
\varphi_n(\eta,\kappa) &= \varphi_{n-1}(\eta,\kappa) - \frac{1}{\Gamma(1+\vartheta)} \\
&\quad \int_0^\kappa \left\{ \frac{\partial^\vartheta \varphi_{n-1}(\eta,\xi)}{\partial \xi^\vartheta} + \frac{1}{2}\frac{\partial^\vartheta \varphi_{n-1}^2(\eta,\xi)}{\partial \eta^\vartheta} + \varphi_{n-1}^2(\eta,\xi) - \varphi_{n-1}(\eta,\xi) \right\} (d\xi)^\vartheta \\
&= E_\vartheta(-\eta^\vartheta)\left[1 + \frac{\kappa^\vartheta}{\Gamma(1+\vartheta} + \frac{\kappa^{2\vartheta}}{\Gamma(1+2\vartheta} + \frac{\kappa^{3\vartheta}}{\Gamma(1+3\vartheta} + \cdots + \frac{\kappa^{n\vartheta}}{\Gamma(1+n\vartheta}\right],
\end{aligned}$$

Consequently, the analytical solution is given by

$$\begin{aligned}
\varphi(\eta,\kappa) &= \lim_{n\to\infty} \varphi_n(\eta,\kappa) \\
&= E_\vartheta(-\eta^\vartheta)\left[\sum_{\rho=0}^{\infty} \frac{\kappa^{\rho\vartheta}}{\Gamma(1+\rho\vartheta)}\right] \\
&= E_\vartheta(\kappa^\vartheta - \eta^\vartheta). \qquad (4.5)
\end{aligned}$$

5. LFVIM for 2D Diffusion Problem in Fractal Heat Transfer

The two-dimensional diffusion model in fractal heat transfer involving local fractional derivatives was presented as :

$$\nabla^{2\vartheta}\psi(\eta_1,\eta_2,\kappa)-\frac{1}{D^{\vartheta}}\frac{\partial^{\vartheta}\psi(\eta_1,\eta_2,\kappa)}{\partial\kappa^{\vartheta}}=0, \tag{5.1}$$

subject to the initial and boundary conditions

$$\begin{aligned}&\psi(\eta_1,\eta_2,0)=f(\eta_1,\eta_2),\\&\psi(0,\eta_2,\kappa)=\psi(a,\eta_2,\kappa)=g_1(\eta_2,\kappa),\\&\psi(\eta_1,0,\kappa)=\psi(\eta_1,b,\kappa)=g_2(\eta_1,\kappa),\end{aligned} \tag{5.2}$$

where the local fractional Laplace operator reads as (see [26])

$$\nabla^{2\vartheta}=\frac{\partial^{2\vartheta}}{\partial\eta_1^{2\vartheta}}+\frac{\partial^{2\vartheta}}{\partial\eta_2^{2\vartheta}}, \tag{5.3}$$

and D^{ϑ} denotes the fractal diffusion constant which is, in essence, a measure for the efficiency of the spreading of the underlying substance.

According to the theory of local fractional variational iteration method in chapter IV, we write the iteration formula of (5.1) as follows:

$$\psi_{n+1}(\eta_1,\eta_2,\kappa)=\psi_n(\eta_1,\eta_2,\kappa)+\\ \frac{1}{\Gamma(1+\vartheta)}\int_0^{\kappa}\frac{\gamma^{\vartheta}}{\Gamma(1+\vartheta)}\left[\frac{\partial^{\vartheta}\psi_n(\eta_1,\eta_2,\xi)}{\partial\xi^{\vartheta}}-D^{\vartheta}\nabla^{2\vartheta}\widetilde{\psi}_n(\eta_1,\eta_2,\xi)\right](d\xi)^{\vartheta}, \tag{5.4}$$

where $\frac{\gamma^{\vartheta}}{\Gamma(1+\vartheta)}$ is a fractal Lagrange multiplier.

Taking local fractional variation of (5.4), we present

$$\delta^{\vartheta}\psi_{n+1}(\eta_1,\eta_2,\kappa)=\delta^{\vartheta}\psi_n(\eta_1,\eta_2,\kappa)+\\ \frac{\delta^{\vartheta}}{\Gamma(1+\vartheta)}\int_0^{\kappa}\frac{\gamma^{\vartheta}}{\Gamma(1+\vartheta)}\left[\frac{\partial^{\vartheta}\psi_n(\eta_1,\eta_2,\xi)}{\partial\xi^{\vartheta}}-D^{\vartheta}\nabla^{2\vartheta}\widetilde{\psi}_n(\eta_1,\eta_2,\xi)\right](d\xi)^{\vartheta}, \tag{5.5}$$

The extremum conditions of $\psi_{n+1}(\eta_1,\eta_2,\kappa)$ requires that $\delta^{\vartheta}\psi_{n+1}(\eta_1,\eta_2,\kappa)=0$.

This yields the stationary conditions:

$$1+\frac{\gamma^{\vartheta}}{\Gamma(1+\vartheta)}\mid_{\kappa=\xi}=0,\ \left(\frac{\gamma^{\vartheta}}{\Gamma(1+\vartheta)}\right)^{(\vartheta)}(\eta)\mid_{\kappa=\xi}=0, \tag{5.6}$$

where $\delta^{\vartheta}\widetilde{\psi_n}$ denotes a restricted local fractional variation, that is, $\delta^{\vartheta}\widetilde{\psi_n}=0$.

This in turn presents Lagrange multiplier:

$$\frac{\gamma^{\vartheta}}{\Gamma(1+\vartheta)}=-1. \tag{5.7}$$

Substituting (5.7) into (5.4), we obtain the iteration formula as:

$$\psi_{n+1}(\eta_1,\eta_2,\kappa) = \psi_n(\eta_1,\eta_2,\kappa) - \frac{1}{\Gamma(1+\vartheta)}\int_0^\kappa \left[\frac{\partial^\vartheta \psi_n(\eta_1,\eta_2,\xi)}{\partial \xi^\vartheta} - D^\vartheta \nabla^{2\vartheta}\psi_n(\eta_1,\eta_2,\xi)\right](d\xi)^\vartheta, \tag{5.8}$$

for $n \geq 0$ the successive approximations $\psi_{n+1}(\eta_1,\eta_2,\kappa)$ of the solution $\psi(\eta_1,\eta_2,\kappa)$ can be given by using any selective local fractional function $u_0(\eta_1,\eta_2,\kappa)$.

We now consider the initial condition of (5.2), namely

$$\psi(\eta_1,\eta_2,0) = E_\vartheta(\eta_1^\vartheta)E_\vartheta(\eta_2^\vartheta). \tag{5.9}$$

Starting with the zeroth approximation:

$$\psi_0(\eta_1,\eta_2,\kappa) = E_\vartheta(\eta_1^\vartheta)E_\vartheta(\eta_2^\vartheta). \tag{5.10}$$

Substituting (5.10) in (5.8) we obtain the following successive approximations

$$\begin{aligned}
\psi_1(\eta_1,\eta_2,\kappa) &= \psi_0(\eta_1,\eta_2,\kappa) - \frac{1}{\Gamma(1+\vartheta)}\int_0^\kappa \left[\frac{\partial^\vartheta \psi_0(\eta_1,\eta_2,\xi)}{\partial \xi^\vartheta} - D^\vartheta \nabla^{2\vartheta}\psi_0(\eta_1,\eta_2,\xi)\right](d\xi)^\vartheta \\
&= E_\vartheta(\eta_1^\vartheta)E_\vartheta(\eta_2^\vartheta)\left[1+\frac{2D^\vartheta\kappa^\vartheta}{\Gamma(1+\vartheta)}\right], \\
\psi_2(\eta_1,\eta_2,\kappa) &= \psi_1(\eta_1,\eta_2,\kappa) - \frac{1}{\Gamma(1+\vartheta)}\int_0^\kappa \left[\frac{\partial^\vartheta \psi_1(\eta_1,\eta_2,\xi)}{\partial \xi^\vartheta} - D^\vartheta \nabla^{2\vartheta}\psi_1(\eta_1,\eta_2,\xi)\right](d\xi)^\vartheta \\
&= E_\vartheta(\eta_1^\vartheta)E_\vartheta(\eta_2^\vartheta)\left[1+\frac{2D^\vartheta\kappa^\vartheta}{\Gamma(1+\vartheta)}+\frac{2^2D^{2\vartheta}\kappa^{2\vartheta}}{\Gamma(1+2\vartheta)}\right],
\end{aligned}$$

$$\begin{aligned}
\psi_3(\eta_1,\eta_2,\kappa) &= \psi_2(\eta_1,\eta_2,\kappa) - \frac{1}{\Gamma(1+\vartheta)}\int_0^\kappa \left[\frac{\partial^\vartheta \psi_2(\eta_1,\eta_2,\xi)}{\partial \xi^\vartheta} - D^\vartheta \nabla^{2\vartheta}\psi_2(\eta_1,\eta_2,\xi)\right](d\xi)^\vartheta \\
&= E_\vartheta(\eta_1^\vartheta)E_\vartheta(\eta_2^\vartheta)\left[1+\frac{2D^\vartheta\kappa^\vartheta}{\Gamma(1+\vartheta)}+\frac{2^2D^{2\vartheta}\kappa^{2\vartheta}}{\Gamma(1+2\vartheta)}+\frac{2^3D^{3\vartheta}\kappa^{3\vartheta}}{\Gamma(1+3\vartheta)}\right], \\
&\vdots \\
\psi_n(\eta_1,\eta_2,\kappa) &= \psi_{n-1}(\eta_1,\eta_2,\kappa) - \frac{1}{\Gamma(1+\vartheta)}\int_0^\kappa \left[\frac{\partial^\vartheta \psi_{n-1}(\eta_1,\eta_2,\xi)}{\partial \xi^\vartheta} - D^\vartheta \nabla^{2\vartheta}\psi_{n-1}(\eta_1,\eta_2,\xi)\right](d\xi)^\vartheta \\
&= E_\vartheta(\eta_1^\vartheta)E_\vartheta(\eta_2^\vartheta)\left[1+\frac{2D^\vartheta\kappa^\vartheta}{\Gamma(1+\vartheta)}+\frac{2^2D^{2\vartheta}\kappa^{2\vartheta}}{\Gamma(1+2\vartheta)}+\cdots+\frac{2^nD^{n\vartheta}\kappa^{n\vartheta}}{\Gamma(1+n\vartheta)}\right].
\end{aligned}$$

Hence, we estimate the analytical solution for the two-dimensional diffusion problem (5.1) in fractal heat transfer

$$\psi(\eta_1,\eta_2,\kappa) = E_\vartheta(\eta_1^\vartheta)E_\vartheta(\eta_2^\vartheta)E_\vartheta(2D^\vartheta\kappa^\vartheta). \tag{5.11}$$

6. Applications of VIM for System of coupled Korteweg-de Vries Equations with Local Fractional Operators

Example 6.1. . Let us consider the following nonlinear coupled KdV equations involving local fractional derivative operator:

$$\frac{\partial^{\vartheta}\varphi(\eta,\kappa)}{\partial\kappa^{\vartheta}}+\frac{\partial^{3\vartheta}\varphi(\eta,\kappa)}{\partial\eta^{3\vartheta}}+2\varphi(\eta,\kappa)\frac{\partial^{\vartheta}\varphi(\eta,\kappa)}{\partial\eta^{\vartheta}}+2\psi(\eta,\kappa)\frac{\partial^{\vartheta}\varphi(\eta,\kappa)}{\partial\eta^{\vartheta}}=0, \tag{6.1}$$

$$\frac{\partial^{\vartheta}\psi(\eta,\kappa)}{\partial\kappa^{\vartheta}}+\frac{\partial^{3\vartheta}\psi(\eta,\kappa)}{\partial\eta^{3\vartheta}}+2\psi(\eta,\kappa)\frac{\partial^{\vartheta}\psi(\eta,\kappa)}{\partial\eta^{\vartheta}}+2\varphi(\eta,\kappa)\frac{\partial^{\vartheta}\psi(\eta,\kappa)}{\partial\eta^{\vartheta}}=0, \tag{6.2}$$

with initial values

$$\varphi(\eta,0)=E_{\vartheta}(-\eta^{\vartheta}), \tag{6.3}$$

$$\psi(\eta,0)=-E_{\vartheta}(-\eta^{\vartheta}). \tag{6.4}$$

To solve the system of equations (6.1) and (6.2) by means of the local fractional variational iteration method, we construct the following correction functionals:

$$\begin{aligned}\varphi_{n+1}(\eta,\kappa) &= \varphi_n(\eta,\kappa)-\frac{1}{\Gamma(1+\vartheta)} \\ &\quad \int_0^{\kappa}\left\{\frac{\partial^{\vartheta}\varphi_n}{\partial\kappa^{\vartheta}}+\frac{\partial^{3\vartheta}\varphi_n}{\partial\eta^{3\vartheta}}+2\varphi_n\frac{\partial^{\vartheta}\varphi_n}{\partial\eta^{\vartheta}}+2\psi_n\frac{\partial^{\vartheta}\varphi_n}{\partial\eta^{\vartheta}}\right\}(d\xi)^{\vartheta},\end{aligned} \tag{6.5}$$

$$\begin{aligned}\psi_{n+1}(\eta,\kappa) &= \psi_n(\eta,\kappa)-\frac{1}{\Gamma(1+\vartheta)} \\ &\quad \int_0^{\kappa}\left\{\frac{\partial^{\vartheta}\psi_n}{\partial\kappa^{\vartheta}}+\frac{\partial^{3\vartheta}\psi_n}{\partial\eta^{3\vartheta}}+2\psi_n\frac{\partial^{\vartheta}\psi_n}{\partial\eta^{\vartheta}}+2\varphi_n\frac{\partial^{\vartheta}\psi_n}{\partial\eta^{\vartheta}}\right\}(d\xi)^{\vartheta}.\end{aligned} \tag{6.6}$$

From (6.3) and (6.4), we take the initial value, which reads as

$$\varphi_0(\eta,\kappa) = E_{\vartheta}(-\eta^{\vartheta}), \tag{6.7}$$

$$\varphi_0(\eta,\kappa) = -E_{\vartheta}(-\eta^{\vartheta}). \tag{6.8}$$

From (6.5)-(6.8), we obtain the following approximations:

$$\begin{aligned}\varphi_1(\eta,\kappa) &= \varphi_0(\eta,\kappa)-\frac{1}{\Gamma(1+\vartheta)}\int_0^{\kappa}\left\{\frac{\partial^{\vartheta}\varphi_0}{\partial\kappa^{\vartheta}}+\frac{\partial^{3\vartheta}\varphi_0}{\partial\eta^{3\vartheta}}+2\varphi_0\frac{\partial^{\vartheta}\varphi_0}{\partial\eta^{\vartheta}}+2\psi_0\frac{\partial^{\vartheta}\varphi_0}{\partial\eta^{\vartheta}}\right\}(d\xi)^{\vartheta} \\ \psi_1(\eta,\kappa) &= \psi_0(\eta,\kappa)-\frac{1}{\Gamma(1+\vartheta)}\int_0^{\kappa}\left\{\frac{\partial^{\vartheta}\psi_0}{\partial\kappa^{\vartheta}}+\frac{\partial^{3\vartheta}\psi_0}{\partial\eta^{3\vartheta}}+2\psi_0\frac{\partial^{\vartheta}\psi_0}{\partial\eta^{\vartheta}}+2\varphi_0\frac{\partial^{\vartheta}\psi_0}{\partial\eta^{\vartheta}}\right\}(d\xi)^{\vartheta} \\ &= E_{\vartheta}(-\eta^{\vartheta})\left[1+\frac{\kappa^{\vartheta}}{\Gamma(1+\vartheta}\right], \\ &= -E_{\vartheta}(-\eta^{\vartheta})\left[1+\frac{\kappa^{\vartheta}}{\Gamma(1+\vartheta}\right], \\ \varphi_2(\eta,\kappa) &= \varphi_1(\eta,\kappa)-\frac{1}{\Gamma(1+\vartheta)}\int_0^{\kappa}\left\{\frac{\partial^{\vartheta}\varphi_1}{\partial\kappa^{\vartheta}}+\frac{\partial^{3\vartheta}\varphi_1}{\partial\eta^{3\vartheta}}+2\varphi_1\frac{\partial^{\vartheta}\varphi_1}{\partial\eta^{\vartheta}}+2\psi_1\frac{\partial^{\vartheta}\varphi_1}{\partial\eta^{\vartheta}}\right\}(d\xi)^{\vartheta}\end{aligned}$$

$$\begin{aligned}
\psi_2(\eta,\kappa) &= \psi_1(\eta,\kappa) - \frac{1}{\Gamma(1+\vartheta)}\int_0^{\kappa}\left\{\frac{\partial^{\vartheta}\psi_1}{\partial\kappa^{\vartheta}} + \frac{\partial^{3\vartheta}\psi_1}{\partial\eta^{3\vartheta}} + 2\psi_1\frac{\partial^{\vartheta}\psi_1}{\partial\eta^{\vartheta}} + 2\varphi_1\frac{\partial^{\vartheta}\psi_1}{\partial\eta^{\vartheta}}\right\}(d\xi)^{\vartheta} \\
&= E_{\vartheta}(-\eta^{\vartheta})\left[1 + \frac{\kappa^{\vartheta}}{\Gamma(1+\vartheta} + \frac{\kappa^{2\vartheta}}{\Gamma(1+2\vartheta}\right], \\
&= -E_{\vartheta}(-\eta^{\vartheta})\left[1 + \frac{\kappa^{\vartheta}}{\Gamma(1+\vartheta} + \frac{\kappa^{2\vartheta}}{\Gamma(1+2\vartheta}\right], \\
\varphi_3(\eta,\kappa) &= \varphi_2(\eta,\kappa) - \frac{1}{\Gamma(1+\vartheta)}\int_0^{\kappa}\left\{\frac{\partial^{\vartheta}\varphi_2}{\partial\kappa^{\vartheta}} + \frac{\partial^{3\vartheta}\varphi_2}{\partial\eta^{3\vartheta}} + 2\varphi_2\frac{\partial^{\vartheta}\varphi_2}{\partial\eta^{\vartheta}} + 2\psi_2\frac{\partial^{\vartheta}\varphi_2}{\partial\eta^{\vartheta}}\right\}(d\xi)^{\vartheta} \\
\psi_3(\eta,\kappa) &= \psi_2(\eta,\kappa) - \frac{1}{\Gamma(1+\vartheta)}\int_0^{\kappa}\left\{\frac{\partial^{\vartheta}\psi_2}{\partial\kappa^{\vartheta}} + \frac{\partial^{3\vartheta}\psi_2}{\partial\eta^{3\vartheta}} + 2\psi_2\frac{\partial^{\vartheta}\psi_2}{\partial\eta^{\vartheta}} + 2\varphi_2\frac{\partial^{\vartheta}\psi_2}{\partial\eta^{\vartheta}}\right\}(d\xi)^{\vartheta} \\
&= E_{\vartheta}(-\eta^{\vartheta})\left[1 + \frac{\kappa^{\vartheta}}{\Gamma(1+\vartheta} + \frac{\kappa^{2\vartheta}}{\Gamma(1+2\vartheta} + \frac{\kappa^{3\vartheta}}{\Gamma(1+3\vartheta}\right], \\
&= -E_{\vartheta}(-\eta^{\vartheta})\left[1 + \frac{\kappa^{\vartheta}}{\Gamma(1+\vartheta} + \frac{\kappa^{2\vartheta}}{\Gamma(1+2\vartheta} + \frac{\kappa^{3\vartheta}}{\Gamma(1+3\vartheta}\right], \\
&\vdots \\
\varphi_n(\eta,\kappa) &= \varphi_{n-1}(\eta,\kappa) - \frac{1}{\Gamma(1+\vartheta)} \\
&\quad \int_0^{\kappa}\left\{\frac{\partial^{\vartheta}\varphi_{n-1}}{\partial\kappa^{\vartheta}} + \frac{\partial^{3\vartheta}\varphi_{n-1}}{\partial\eta^{3\vartheta}} + 2\varphi_{n-1}\frac{\partial^{\vartheta}\varphi_{n-1}}{\partial\eta^{\vartheta}} + 2\psi_n\frac{\partial^{\vartheta}\varphi_n}{\partial\eta^{\vartheta}}\right\}(d\xi)^{\vartheta}, \\
\psi_n(\eta,\kappa) &= \psi_{n-1}(\eta,\kappa) - \frac{1}{\Gamma(1+\vartheta)} \\
&\quad \int_0^{\kappa}\left\{\frac{\partial^{\vartheta}\psi_{n-1}}{\partial\kappa^{\vartheta}} + \frac{\partial^{3\vartheta}\psi_{n-1}}{\partial\eta^{3\vartheta}} + 2\psi_{n-1}\frac{\partial^{\vartheta}\psi_{n-1}}{\partial\eta^{\vartheta}} + 2\varphi_{n-1}\frac{\partial^{\vartheta}\psi_{n-1}}{\partial\eta^{\vartheta}}\right\}(d\xi)^{\vartheta}. \\
&= E_{\vartheta}(-\eta^{\vartheta})\left[1 + \frac{\kappa^{\vartheta}}{\Gamma(1+\vartheta} + \frac{\kappa^{2\vartheta}}{\Gamma(1+2\vartheta} + \frac{\kappa^{3\vartheta}}{\Gamma(1+3\vartheta} + \cdots + \frac{\kappa^{n\vartheta}}{\Gamma(1+n\vartheta}\right], \\
&= -E_{\vartheta}(-\eta^{\vartheta})\left[1 + \frac{\kappa^{\vartheta}}{\Gamma(1+\vartheta} + \frac{\kappa^{2\vartheta}}{\Gamma(1+2\vartheta} + \frac{\kappa^{3\vartheta}}{\Gamma(1+3\vartheta} + \cdots + \frac{\kappa^{n\vartheta}}{\Gamma(1+n\vartheta}\right].
\end{aligned}$$

Therefore, the series solutions can be written in the form

$$\begin{aligned}
\varphi(\eta,\kappa) &= E_{\vartheta}(-\eta^{\vartheta})\left[1 + \frac{\kappa^{\vartheta}}{\Gamma(1+\vartheta} + \frac{\kappa^{2\vartheta}}{\Gamma(1+2\vartheta} + \frac{\kappa^{3\vartheta}}{\Gamma(1+3\vartheta} + \cdots + \frac{\kappa^{n\vartheta}}{\Gamma(1+n\vartheta}\right], \\
\psi(\eta,\kappa) &= -E_{\vartheta}(-\eta^{\vartheta})\left[1 + \frac{\kappa^{\vartheta}}{\Gamma(1+\vartheta} + \frac{\kappa^{2\vartheta}}{\Gamma(1+2\vartheta} + \frac{\kappa^{3\vartheta}}{\Gamma(1+3\vartheta} + \cdots + \frac{\kappa^{n\vartheta}}{\Gamma(1+n\vartheta}\right],
\end{aligned}$$

and finally in its closed form gives

$$\varphi(\eta,\kappa) = E_{\vartheta}(\kappa^{\vartheta} - \eta^{\vartheta}) \tag{6.9}$$

$$\psi(\eta,\kappa) = -E_{\vartheta}(\kappa^{\vartheta} - \eta^{\vartheta}). \tag{6.10}$$

Authors' Contributions

All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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