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EXISTENCE, UNIQUENESS AND COMPACTNESS OF SOLUTIONS FOR RANDOM SEMILINEAR SYSTEM OF FUNCTIONAL DIFFERENTIAL EQUATIONS AND APPLICATION

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Abstract. Our results are used to prove the existence, uniqueness and compactness of solutions for systems of random semilinear functional differential equations with initial and boundary conditions. We assume that the linear part generates a strongly continuous semigroup on a general Banach space. Finally, some examples are given to illustrate the results.

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1. Introduction

In this paper, we give some existence results for functional differential equations with delay and random effects, we study the following systems

$$\begin{cases} x'(t,\omega) = A_1(\omega)x(t,\omega) + f(t, x_t(.,\omega), y_t(.,\omega), \omega), & t \in J := [0,T] \\ y'(t,\omega) = A_2(\omega)x(t,\omega) + g(t, x_t(.,\omega), y_t(.,\omega), \omega), & t \in J := [0,T] \\ x(\theta,\omega) = \varphi(\theta,\omega), & \theta \in [-r,0] \\ y(\theta,\omega) = \psi(\theta,\omega), & \theta \in [-r,0]. \end{cases} \tag{1}$$

where $f, g : J \times C([-r,0] \times \Omega, X) \times C([-r,0] \times \Omega, X) \times \Omega \to X$, $(\Omega, \mathcal{A})$ is a measurable space, $A_i : \Omega \times X \to X$, i = 1, 2 are random operators, φ, ψ are two random maps and X is a separable Banach space.

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For any function x defined on $[-r,T]\times\Omega$ and any $t\in J$ we denote by $x_t(.,\omega)$ the element of $C([-r,0]\times\Omega,X)$ defined by

$$x_t(\theta,\omega)=x(t+\theta,\omega),\ \theta\in[-r,0].$$

Here $x_t(.,\omega)$ represents the history of the state from time $t-r$, up to the present time t.

Functional differential equations arise in a variety of areas of biological, physical, and engineering applications, and such equations have received much attention in recent years. An important guide to investigations of functional differential equations of various types, see the books of Kolmanovskii and Myshkis [6] and the references therein.

Probabilistic functional analysis is an important to research due to its applications to probabilistic models. Random operator theory is needed for the study of random equations. The problem of fixed points for random mappings was initialed by the Prague school of probabilities. Several well-known fixed point theorems of single-valued mappings such as Banach's and Schauder's have been extended in generalized Banach spaces; see [5]. Some new fixed point theorems for single-valued operators on a set with two vectors-valued metric and a generalization of the Banach contraction principle for operators see for examples [9].

In recent years, much literature has dealt with the existence, uniqueness, and multiplicity of solutions to systems of difference equations [1,3] and the references therein.

This paper is organized as follows: In Section 2, we introduce the material needed in this paper such as generalized metric space, some fixed point theorems and preliminary facts which will be used and some random fixed point theorems. In Section 3, we shall use a random version of the Perov type theorem for the study of the semilinear initial value problems of random functional differential equations. In Section 4, we present an illustrative and comparative example.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. Let $(X,\|.\|)$ be a Banach space.

$C([-r,T]\times\Omega,X)$ is the Banach space of all continuous functions from $[-r,T]\times\Omega$ into X with the norm

$$\|x\|_\infty=\sup_{t\in[0,T]}\sup_{\theta\in[-r,0]}|x(t+\theta,\omega)|.$$

Definition 2.1. *A map N is said compact if the image is relatively compact. N is said completely continuous if is continuous and the image of every bounded is relatively compact.*

Definition 2.2. *The map $f:J\times E\times\Omega\to X$ is called random Carathéodory if*

(i) *the map $(t,\omega)\to f(t,x,\omega)$ is jointly measurable for each $x\in E$;*

(ii) *the map $x\to f(t,x,\omega)$ is continuous for all $t\in J$ and $\omega\in\Omega$.*

Definition 2.3. *A Carathéodory function $f : J \times E \times \Omega \to X$ is called random L^1-Carathéodory for each $q > 0$, there exists $h_q \in L^1(J, \mathbb{R}^+)$ such that*

$$\|f(t,x)\| \leq h_q(t,\omega), \quad a.e.\ t \in J$$

for all $\|x\| \leq q$ and $\omega \in \Omega$.

Definition 2.4. *A square matrix of real numbers is said to be convergent to zero if and only if its spectral radius $\rho(M)$ is strictly less than 1. In other words, this means that all the eigenvalues of M are in the open unit disc i.e. $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with $det(M - \lambda I) = 0$, where I denote the unit matrix of $\mathcal{M}_{n\times n}(\mathbb{R})$.*

Theorem 2.5 ([12], pages 12,88)**.** *Let $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$. The following assertions are equivalent:*

(i) *M is convergent towards zero;*
(ii) *$M^k \to 0$ as $k \to \infty$;*
(iii) *The matrix $(I - M)$ is nonsingular and*

$$(I - M)^{-1} = I + M + M^2 + \ldots + M^k + \ldots,$$

(iv) *The matrix $(I - M)$ is nonsingular and $(I - M)^{-1}$ has nonnegative elements.*

Definition 2.6. *Let (X, d) be a generalized metric space. An operator $N : X \to X$ is said to be contractive if there exists a convergent to zero matrix M such that*

$$d(N(x), N(y)) \leq M d(x,y) \text{ for all } x, y \in X.$$

For $n = 1$ we recover the classical Banach's contraction fixed point result.

Definition 2.7. *We say that a non-singular matrix $A = (a_{ij})_{1\leq i,j\leq n} \in \mathcal{M}_{n\times n}(\mathbb{R})$ has the absolute value property if*

$$A^{-1}|A| \leq I,$$

where

$$|A| = (|a_{ij}|)_{1\leq i,j\leq n} \in \mathcal{M}_{n\times n}(\mathbb{R}_+).$$

Some examples of matrices convergent to zero $A \in \mathcal{M}_{n\times n}(\mathbb{R})$, which also satisfies the property $(I - A)^{-1}|I - A| \leq I$ are:

1) $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, where $a, b \in \mathbb{R}_+$ and $\max(a,b) < 1$
2) $A = \begin{pmatrix} a & -c \\ 0 & b \end{pmatrix}$, where $a, b, c \in \mathbb{R}_+$ and $a + b < 1,\ c < 1$
3) $A = \begin{pmatrix} a & -a \\ b & -b \end{pmatrix}$, where $a, b, c \in \mathbb{R}_+$ and $|a - b| < 1,\ a > 1, b > 0$.

Theorem 2.8. [3] *Let (X, d) be a complete generalized metric space and $N : X \to X$ a contractive operator with Lipschitz matrix M. Then N has a unique fixed point x_* and for each $x_0 \in X$ we have*

$$d(N^k(x_0), x_*) \leq M^k(I - M)^{-1}d(x_0, N(x_0)) \text{ for all } k \in \mathbb{N}.$$

Theorem 2.9. [11] *Let $(\Omega, \mathcal{F})$ be a measurable space, X be a real separable generalized Banach space and $F : \Omega \times X \to X$ be a continuous random operator, and let $M(\omega) \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ be a random variable matric such that for every $\omega \in \Omega$ the matrix, $M(\omega)$ converge to 0 and*

$$d(F(\omega, x_1), F(\omega, x_2)) \leq M(\omega)d(x_1, x_2) \text{ for each } x_1, x_2 \in X,\ \omega \in \Omega.$$

then there exists any random variable $x : \Omega \to X$ which is the unique random fixed point of F.

Theorem 2.10. [11] *Let X be a separable generalized Banach space and let $F : \Omega \times X \to X$ be a completely continuous random operator. Then, either*

(i) *the random equation $F(\omega, x) = x$ has a random solution, i.e., there is a measurable function $x : \Omega \to X$ such that $F(\omega, x(\omega)) = x(\omega)$ for all $\omega \in \Omega$, or*
(ii) *the set $\mathcal{M} = \{x : \Omega \to X \text{ is measurable} | \lambda(\omega)F(\omega, x) = x\}$ is unbounded for some measurable $\lambda : \Omega \to X$ with $0 < \lambda(\omega) < 1$ on Ω.*

Theorem 2.11. (*Carathéodory*) [11] *Let X be a separable metric space and $G : \Omega \times X \to X$ be a mapping such that $G(., x)$ is measurable for all $x \in X$ and $G(\omega, .)$ is continuous for all $\omega \in \Omega$. Then the map $(\omega, x) \to G(\omega, x)$ is jointly measurable.*

As consequence of above theorem we can easily prove the following result.

Lemma 2.12. [11] *Let X be a separable generalized metric space and $G : \Omega \times X \to X$ be a mapping such that $G(., x)$ is measurable for all $x \in X$ and $G(\omega, .)$ is continuous for all $\omega \in \Omega$. Then the map $(\omega, x) \to G(\omega, x)$ is jointly measurable.*

Proposition 2.13. [7] *Let X be a separable Banach space, and D be a dense linear subspace of X. Let $L : \Omega \times D \to X$ be a closed linear random operator such that, for each $\omega \in \Omega$, $L(\omega)$ is one to one and onto. Then the operator $R : \omega \times X \to X$ defined by $R(\omega)x = L^{-1}(\omega)x$ is random.*

3. Existence and uniqueness of random solutions

In this section, we establish the existence, uniqueness, and compactness of solutions set of random functional differential equations (1).

Set $C_r := C([-r, 0] \times \Omega, X)$ and $C := C([-r, T] \times \Omega, X)$.

Definition 3.1. *We say that a function* $x : [-r, T] \times \Omega \to X$ *is a mild solution of problem* (1) *if* $(x(t,\omega), y(t,\omega)) = (\varphi(t,\omega), \psi(t,\omega))$, $t \in [-r, 0]$ *and*

$$\begin{cases} x(t,\omega) = S_1(\omega,t)\varphi(0,\omega) + \displaystyle\int_0^t S_1(\omega,t-s)f(s,x_s(.,\omega),y_s(.,\omega),\omega)ds, & t \in J \\ y(t,\omega) = S_2(\omega,t)\psi(0,\omega) + \displaystyle\int_0^t S_2(\omega,t-s)g(s,x_s(.,\omega),y_s(.,\omega),\omega)ds, & t \in J. \end{cases}$$

where $\{S_1(\omega,t)\}_{t\geq 0}$, $\{S_2(\omega,t)\}_{t\geq 0}$ *are random* C_0*-semigroups of bounded linear operators on* X *with infinitesimal generators* A_1, A_2*, respectively and* $\omega \in \Omega$.

There exist random variables $M_1, M_2 : \Omega \to (0; +\infty)$ such that $\|S_i(\omega,t)\| \leq M_i(\omega)$ for each $i = 1, 2$ and $\omega \in \Omega$.

Theorem 3.2. $f, g : J \times C_r \times C_r \times \Omega \to X$ *are two Carathéodory functions. Assume that the following condition hold:*

(H_1) *There exist* $p_1, p_2, p_3, p_4 : \Omega \to L^1(J, \mathbb{R}_+)$ *are random variable such that*

$$\|f(t,x,y,\omega) - f(t,\widetilde{x},\widetilde{y},\omega)\| \leq p_1(\omega)\|x - \widetilde{x}| + p_2(\omega)\|y - \widetilde{y}\|$$

and

$$\|g(t,x,y,\omega) - g(t,\widetilde{x},\widetilde{y},\omega)\| \leq p_3(\omega)\|x - \widetilde{x}\| + p_4(\omega)\|y - \widetilde{y}\|,$$

for each $t \in J$, $x, y, \widetilde{x}, \widetilde{y} \in C_r$ *and* $\omega \in \Omega$.

If $M(\omega)$ *converges to* 0*, then the problem* (1) *has a unique random solution.*

Proof: Consider the operator $N : C \times C \times \Omega \to C \times C$, $(x, y, \omega) \to (L_1(x, y, \omega), L_2(x, y, \omega))$ where

$$L_1(x(t,\omega), y(t,\omega), \omega) = \begin{cases} \varphi(t,\omega), & \text{t} \in [-r, 0] \\ S_1(\omega,t)\varphi(0,\omega) & \\ + \displaystyle\int_0^t S_1(\omega,t-s)f(s,x_s(.,\omega),y_s(.,\omega),\omega)ds\ , & \text{t} \in J \end{cases}$$

and

$$L_2(x(t,\omega), y(t,\omega), \omega) = \begin{cases} \psi(\omega,t), & \text{t} \in [-r, 0] \\ S_2(\omega,t)\psi(0,\omega) & \\ + \displaystyle\int_0^t S_2(\omega,t-s)g(s,x_s(.,\omega),y_s(.,\omega),\omega)ds\ , & \text{t} \in J. \end{cases}$$

First we show that N is a random operator on $C \times C \times \Omega$. Since f and g are Carathéodory functions, then $\omega \to f(t,x,y,\omega)$ and $\omega \to g(t,x,y,\omega)$ are measurable maps in view of lemma 2.12. By the Crandall-Liggett formula, we have

$$S_i(\omega,t)=\lim_{n\to\infty}(I-\frac{t}{n}A_i(\omega))^{-n}x,\ \ i=1,2.$$

From Proposition 2.13, we know that $\omega\to(I-\frac{t}{n}A_i(\omega))^{-n}x$ are measurable operators, thus $\omega\to S_i(\omega,t)$ are measurable. Using the continuity properties of the semigroups $S_1(\omega,t)$, $S_2(\omega,t)$, we get

$$\omega\to S_1(\omega,t)\varphi(0,\omega),\ (s,\omega)\to S_1(\omega,t-s)f(s,x_s(.,\omega),y_s(.,\omega),\omega)$$

and

$$\omega\to S_2(\omega,t)\psi(0,\omega),\ (s,\omega)\to S_2(\omega,t-s)g(s,x_s(.,\omega),y_s(.,\omega),\omega)$$

are measurable. Further, the integral is a limit of a finite sum of measurable functions, therefore, the maps

$$\omega\to L_1(x(t,\omega),y(t,\omega),\omega),\ \ \omega\to L_2(x(t,\omega),y(t,\omega),\omega)$$

are measurable. As a result, N is a random operator on $N: C\times C\times\Omega$ into $C\times C$.

We show that N satisfies all the conditions of Theorem 2.9 on $C\times C\times\Omega$.

Let $(x,y),(\widetilde{x},\widetilde{y})\in C\times C$ then

$$\begin{aligned}
&\|L_1(x(t,\omega),y(t,\omega),\omega)-L_1(\widetilde{x}(t,\omega),\widetilde{y}(t,\omega),\omega)\|=\\
&\qquad\|\int_0^t S_1(\omega,t-s)(f(s,x_s(.,\omega),y_s(.,\omega),\omega)-f(s,\widetilde{x}_s(.,\omega),\widetilde{y}_s(.,\omega),\omega))ds\|\\
&\leq\int_0^t\|S_1(\omega,t-s)\|\|f(s,x_s(.,\omega),y_s(.,\omega),\omega)-f(s,\widetilde{x}_s(.,\omega),\widetilde{y}_s(.,\omega),\omega)\|ds\\
&\leq M_1(\omega)\int_0^t p_1(s,\omega)\|x_s(.,\omega)-\widetilde{x}_s(.,\omega)\|ds\\
&\quad+M_1(\omega)\int_0^t p_2(s,\omega)\|y_s(.,\omega)-\widetilde{y}_s(.,\omega)\|ds
\end{aligned}$$

Then

$$\|L_1(x,y,\omega)-L_1(\widetilde{x},\widetilde{y},\omega)\|_H\ \leq\ \frac{M_1(\omega)}{\tau}(\|x-\widetilde{x}\|_H+\|y-\widetilde{y}\|_H)$$

where

$$\|x\|_H=\sup_{t\in J}e^{-\tau K(t,\omega)},\quad K(t,\omega)=\int_0^t p(r,\omega)dr,\ \tau\geq M_1(\omega)+M_2(\omega)$$

and

$$p(t,\omega)=\sum_{i=1}^{i=4}p_i(t,\omega).$$

Similarly, we obtains

$$\|L_2(x,y,\omega)-L_2(\widetilde{x},\widetilde{y},\omega)\|_H\ \leq\ \frac{M_2(\omega)}{\tau}(\|x-\widetilde{x}\|_H+\|y-\widetilde{y}\|_H).$$

Hence

$$d_0(N(x,y,\omega),N(\widetilde{x},\widetilde{y},\omega))\ \leq\ M(\omega)d_0((x,y),(\widetilde{x},\widetilde{y}))$$

where

$$M(\omega) = \begin{pmatrix} \frac{M_1(\omega)}{\tau} & \frac{M_1(\omega)}{\tau} \\ \frac{M_2(\omega)}{\tau} & \frac{M_2(\omega)}{\tau} \end{pmatrix}$$

and

$$d_0(x,y) = \begin{pmatrix} \|x-y\|_H \\ \|\widetilde{x}-\widetilde{y}\|_H \end{pmatrix}.$$

It is clear that the spectral $\rho(M(\omega)) = \frac{M_1(\omega)+M_2(\omega)}{\tau} < 1$, then the matrix $M(\omega)$ has converge to 0. From theorem 2.9 there exists unique random solution of problem (1). We denote by $(x(t,\omega), y(t,\omega))$ the mild solution of (1).

Lemma 3.3. (*Grönwall-Bihari*) [2] *Let $I = [a,b]$ and let $u, g : I \to \mathbb{R}$ be positive continuous functions. Assume there exist $c > 0$ and a continuous nondecreasing function $h : [0,\infty) \to (0,+\infty)$ such that*

$$u(t) \leq c + \int_a^t g(s)h(u(s))ds, \quad \forall t \in I.$$

Then

$$u(t) \leq H^{-1}\Big(\int_a^t g(s)ds\Big), \quad \forall t \in I,$$

provided

$$\int_c^{+\infty} \frac{dy}{h(y)} > \int_a^b g(s)ds,$$

where H^{-1} refers to inverse of the function $H(u) = \int_c^u \frac{dy}{h(y)}$ for $u \geq c$.

We consider the following set of hypotheses in what follows:

(H_2) The functions f and g are random Carathéodory on $[0,T] \times C_r \times C_r \times \Omega$.

(H_3) There exist a measurable and bounded functions $\gamma_1, \gamma_2 : \Omega \to L^1([0,T], \mathbb{R}_+)$ and a continuous and nondecreasing function $\psi_1, \psi_2 : \mathbb{R}_+ \to (0,\infty)$ such that

$$\|f(t,x,y,\omega)\| \leq \gamma_1(t,\omega)\psi_1(\|x\|+\|y\|), \quad \|g(t,x,y,\omega)\| \leq \gamma_2(t,\omega)\psi_2(\|x\|+\|y\|)$$

for all $\omega \in \Omega$, $t \in [0,T]$ and $x, y \in C_r$.

(H_4) A_1, A_2 are the generators of a strongly continuous semigroup $S_1(\omega,t)$, $S_2(\omega,t)$ respectively for $t \in J$ and $\omega \in \Omega$ which are compact for $t > 0$ in the Banach space X.

Now, we give prove of the existence result of problem (1) by using Schaefer's random fixed point theorem type in generalized Banach space.

Theorem 3.4. *Assume that the hypotheses (H_2), (H_3) and (H_4) hold. If*

$$\int_0^T \phi(s,\omega)ds < \int_c^{\infty} \frac{du}{\Gamma(u)} \quad \text{for all } \omega \in \Omega,$$

where

$$c = M_1(\omega)\|\varphi(0,\omega)\| + M_2(\omega)\|\psi(0,\omega)\|,\ \phi = \max\{M_1(\omega)\gamma_1, M_2(\omega)\gamma_2\} \text{ and } \Gamma = \psi_1 + \psi_2.$$

Then the problem (1) has a random solution.

moreover the set

$$S = \{(x,y) \in C \times C : (x,y) \textit{ is solution of the problem } (1)\ \}$$

is compact.

Proof: Let $N : C \times C \times \Omega \to C \times C$ a random operator defined in Theorem 3.2.

Clearly, the random fixe point of N are solutions to (1), where N is defined in Theorem 3.2. In order to apply Theorem 2.10 , we first show that N is completely continuous. The proof will be given in several steps.

Step 1: $N(.,.,\omega) = (L_1(.,.,\omega), L_2(.,.,\omega)$ is continuous.

Let (x_n, y_n) be a sequence such that $(x_n, y_n) \to (x,y)$ in $C \times C$ as $n \to \infty$. Then

$$\begin{aligned} & \|L_1(x_n(t,\omega), y_n(t,\omega), \omega) - L_1(x(t,\omega), y(t,\omega), \omega)\| \\ \leq \quad & M_1(\omega) \int_0^t \|f(s, x_{ns}(.,\omega), y_{ns}(.,\omega), \omega) - f(s, x_s(.,\omega), y_s(.,\omega), \omega)\| ds \end{aligned}$$

and so

$$\begin{aligned} & \|L_1(x_n(.,\omega), y_n(.,\omega), \omega) - L_1(x(.,\omega), y(.,\omega), \omega)\|_\infty \\ \leq \quad & M_1(\omega) \int_0^T \|f(s, x_{ns}(.,\omega), y_{ns}(.,\omega), \omega) - f(s, x_s(.,\omega), y_s(.,\omega), \omega)\| ds. \end{aligned}$$

Since f is an L^1-Carathéodory function, we have by the Lebesgue dominated convergence theorem, we have

$$\|L_1(x_n(.,\omega), y_n(.,\omega), \omega) - L_1(x(.,\omega), y(.,\omega), \omega)\|_\infty \to 0 \quad \text{as} \quad n \to \infty.$$

Similarly

$$\|L_2(x_n(.,\omega), y_n(.,\omega), \omega) - L_2(x(.,\omega), y(.,\omega), \omega)\|_\infty \to 0 \quad \text{as} \quad n \to \infty.$$

Thus N is continuous.

Step 2: N maps bounded sets into bounded sets in $C \times C$. Indeed, it is enough to show that for any $q > 0$ there exists a positive constant K such that for each $(x,y) \in B_q = \{(x,y) \in C \times C : \|x\|_\infty \leq q, \|y\|_\infty \leq q\}$, we have

$$\|N(x,y,\omega)\|_\infty \leq K = (K_1, K_2).$$

Then for each $t \in [0,T]$, we get

$$\begin{aligned} \|L_1(x(t,\omega), y(t,\omega), \omega)\| \quad &= \quad \|\varphi(0,\omega) S_1(\omega, t) + \int_0^t S_1(\omega, t-s) f(s, x_s(.,\omega), y_s(.,\omega), \omega) ds\| \\ &\leq \quad M_1(\omega)(\|\varphi(0,\omega)\| + \int_0^t \|f(s, x_s(.,\omega), y_s(.,\omega), \omega)\| ds). \end{aligned}$$

From (H_3), we have

$$\|L_1(x(.,\omega),y(.,\omega),\omega)\|_\infty \leq M_1(\omega)(q+\psi_1(2q)\int_0^T \gamma_1(s,\omega)ds) := K_1.$$

Similarly, we have

$$\|L_2(x(.,\omega),y(.,\omega),\omega)\|_\infty \leq M_2(\omega)(q+\psi_2(2q)\int_0^T \gamma_2(s,\omega)ds) := K_2.$$

Step 3: N maps bounded sets into equicontinuous sets of $C \times C$.

Let $0 < \tau_1, \tau_2 \in J$, $\tau_1 < \tau_2$ and B_q be a bounded set of $C \times C$ as in Step 2. Let $(x,y) \in B_q$ then for each $t \in J$ we have

$$\begin{aligned}\|h(\tau_2)-h(\tau_1)\| \leq\ & \|S_1(\omega,\tau_2)-S_1(\omega,\tau_2)\|\|\varphi(0,\omega)\| \\ & +\psi_1(2q)\int_0^{\tau_1-\epsilon}\|S_1(\omega,\tau_2-s)-S_1(\omega,\tau_2-s)\|\gamma_1(s,\omega)ds \\ & +\psi_1(2q)\int_{\tau_1}^{\tau_1-\epsilon}\|S_1(\omega,\tau_2-s)-S_1(\omega,\tau_1-s)\|\gamma_1(s,\omega)ds \\ & +\psi_1(2q)\int_{\tau_1}^{\tau_2}\|S_1(\omega,\tau_2-s)\|\gamma_1(s,\omega)ds,\end{aligned}$$

where

$$h_1(\tau_i) = L_1(\omega, x(\tau_i,\omega), y(\tau_i,\omega)),\ \ i = 1,2.$$

The right-hand side tends to zero as $\tau_2 - \tau_1 \to 0$, and ϵ sufficiently small, since $\{S_1(\omega,t)\}_{t\geq 0}$ is a strongly continuous operator and the compactness of $\{S_1(\omega,t)\}_{t\geq 0}$ for $t > 0$ implies the continuity in the uniform operator topology. By a similar way we can prove the equicontinuity for $L_2(B_g \times B_g)$.

As a consequence of Steps 2, 3 and the Arzel$\acute{a}$ -Ascoli theorem we can conclude that we conclude that N maps B_q into a precompact set in $C \times C$.

Step 4: (*A priori bounds on solutions.*)

Now, it remains to show that the set

$$\Sigma = \left\{ (x,y) \in C \times C : (x,y) = \lambda(\omega)N(x,y), \lambda(\omega) \in (0,1) \right\} \text{is bounded.}$$

Let $(x,y) \in \Sigma$. Then $x = \lambda(\omega)L_1(x,y)$ and $y = \lambda(\omega)L_2(x,y)$ for some $0 < \lambda(\omega) < 1$.

Thus, for $t \in [0,T]$, we have

$$\begin{aligned}\|x(t,\omega)\| &\leq M_1(\omega)(\|\varphi(0,\omega)\| + \int_0^t \|f(s,x_s(.,\omega),y_s(.,\omega),\omega)\|ds) \\ &\leq M_1(\omega)(\|\varphi(0,\omega)\| + \int_0^t \gamma_1(s,\omega)\psi_1(\|x_s(.,\omega)\|+\|y_s(.,\omega)\|))ds\end{aligned}$$

and

$$\|y(t,\omega)\| \leq M_2(\omega)(\|\psi(0,\omega)\| + \int_0^t \gamma_2(s,\omega)\psi_2(\|x_s(.,\omega)\|+\|y_s(.,\omega)\|))ds.$$

Therefore

$$\|x(t,\omega)\| + \|y(t,\omega)\| \quad \leq \quad c + \int_0^t \phi(s,\omega)\Gamma(\|x_s(.,\omega)\| + \|y_s(.,\omega)\|)ds,$$

By Lemma 3.3, we have

$$\|x(t,\omega)\| + \|y(t,\omega)\| \leq H^{-1}\Big(\int_0^t \phi(s)ds\Big) := K_*, \text{ for each } t \in [0,T],$$

where

$$H(z) = \int_c^z \frac{du}{\Gamma(u)}.$$

Consequently

$$\|x\|_\infty \leq K_* \text{ and } \|y\|_\infty \leq K_*.$$

This shows that Σ is bounded. As a consequence of Theorem 2.10 we deduce that N has at least one fixed point which is a random mild solution of problem(1).

Step 5: It remains to show that the set S is compact.

Let the sequence $(x_n, y_n)_{n\in\mathbb{N}} \subset S$, then

$$x_n(t,\Omega) = \begin{cases} \varphi(t,\omega), & \text{t} \in [-r,0] \\ S_1(\omega,t)\varphi(0,\omega) & \\ + \displaystyle\int_0^t S_1(\omega,t-s)f(s,x_{ns}(.,\omega),y_{ns}(.,\omega),\omega)ds\ , & \text{t} \in J \end{cases}$$

and

$$y_n(t,\omega) = \begin{cases} \psi(t,\omega), & \text{t} \in [-r,0] \\ S_2(\omega,t)\psi(0,\omega) & \\ + \displaystyle\int_0^t S_2(\omega,t-s)g(s,x_{ns}(.,\omega),y_{ns}(.,\omega),\omega)ds\ , & \text{t} \in J. \end{cases}$$

Let $B = \{(x_n, y_n) : n \in \mathbb{N}\} \subseteq C \times C$.

Then from earlier parts of the proof of this theorem, we conclude that B is bounded and equicontinuous. Then from the Ascoli-Arzel$\grave{a}$ theorem we can conclude that B is compact, then there exists a subsequence $(x_{nm}, y_{nm}) \subset S$; $(x_{nm}, y_{nm}) \to (x,y)$ as $n_m \to \infty$. Consider

$$z(t,\Omega) = \begin{cases} \varphi(t,\omega), & \text{t} \in [-r,0] \\ S_1(\omega,t)\varphi(0,\omega) & \\ + \displaystyle\int_0^t S_1(\omega,t-s)f(s,z_s(.,\omega),j_s(.,\omega),\omega)ds\ , & \text{t} \in J \end{cases}$$

and

$$j(t,\omega) = \begin{cases} \psi(t,\omega), & \text{t} \in [-r,0] \\ S_2(\omega,t)\psi(0,\omega) & \\ + \displaystyle\int_0^t S_2(\omega,t-s)g(s,z_s(.,\omega),j_s(.,\omega),\omega)ds\ , & \text{t} \in J, \end{cases}$$

then

$$\|x_{nm}(t,\omega)-z(t,\omega)\| \leq M_1(\omega)\int_0^t \|f(s,x_{ns}(.,\omega),y_{ns}(.,\omega),\omega)-f(s,z_s(.,\omega),j_s(.,\omega),\omega)\|ds$$

and so

$$\|x_{nm}(.,\omega)-z(.,\omega)\|_\infty \leq M_1(\omega)\int_0^T \|f(s,x_{ns}(.,\omega),y_{ns}(.,\omega),\omega)-f(s,z_s(.,\omega),j_s(.,\omega),\omega)\|_\infty ds$$

Since f is an L^1-Carath*é*odory function, we have by the Lebesgue dominated convergence theorem, we have

$$\|x_{nm}(.,\omega)-z(.,\omega)\|_\infty \to 0 \quad \text{as} \quad n\to\infty.$$

Similarly

$$\|y_{nm}(.,\omega)-j(.,\omega)\|_\infty \to 0 \quad \text{as} \quad n\to\infty.$$

Thus

$$x(t,\Omega)=\begin{cases}\varphi(t,\omega), & \text{t} \in [-r,0]\\ S_1(\omega,t)\varphi(0,\omega) & \\ +\displaystyle\int_0^t S_1(\omega,t-s)f(s,x_s(.,\omega),y_s(.,\omega),\omega)ds\ , & \text{t} \in J\end{cases}$$

and

$$y(t,\omega)=\begin{cases}\psi(t,\omega), & \text{t} \in [-r,0]\\ S_2(\omega,t)\psi(0,\omega) & \\ +\displaystyle\int_0^t S_2(\omega,t-s)g(s,x_s(.,\omega),y_s(.,\omega),\omega)ds\ , & \text{t} \in J,\end{cases}$$

4. An example

Let $\Omega=\mathbb{R}$ be equipped with the usual $\sigma-$ algebra consisting of Lebesgue measurable subsets of $(-\infty,0)$ and $J:=[0,1]$.

Consider the following random differential equation system.

$$\begin{cases} x'(t,\omega)=\omega^2 x(t,\omega)+\frac{t\omega^2}{(2+\omega^2)(1+x_t^2(.,\omega)+y_t^2(.,\omega))}, & t\in J\\ y'(t,\omega)=\omega^4 x(t,\omega)+\frac{t^2\omega^2}{(2+\omega^2)(1+x_t^2(.,\omega)+y_t^2(.,\omega))}, & t\in J\\ x(\theta,\omega)=\varphi(\theta,\omega), & \theta\in[-r,0]\\ y(\theta,\omega)=\psi(\theta,\omega), & \theta\in[-r,0].\end{cases} \tag{2}$$

here

$$f(t,x,y,\omega)=\frac{t\omega^2}{(2+\omega^2)(1+x^2+y^2)}$$

$$g(t,x,y,\omega)=\frac{t^2\omega^2}{(2+\omega^2)(1+x^2+y^2)}$$

and

$$A_1(\omega)=\omega^2,\ \ A_2(\omega)=\omega^4.$$

Clearly, the map $(t,\omega)\mapsto f(t,x,y,\omega)$ is jointly continuous for all $x,y\in[1,\infty)$. The same for the map g. Also the maps $x\mapsto f(t,x,y,\omega)$ and $y\mapsto f(t,x,y,\omega)$ are continuous for all $t\in J$ and $\omega\in\Omega$. Similarly for the maps corresponding to function g. Thus the functions f and g are Carathéodory on $J\times[1,\infty)\times[1,\infty)\times\Omega$. Firstly, we show that f and g are Lipschitz functions. Indeed, let $x,y\in\mathbb{R}$, then

$$\begin{aligned}
|f(t,x,y,\omega)-f(t,\widetilde{x},\widetilde{y},\omega)| &= \left|\frac{t\omega^2}{(2+\omega^2)(1+x^2+y^2)}-\frac{t\omega^2}{(2+\omega^2)(1+\widetilde{x}^2+\widetilde{y}^2)}\right| \\
&= \left|\frac{t\omega^2[(1+\widetilde{x}^2+\widetilde{y}^2)-(1+x^2+y^2)]}{(2+\omega^2)(1+x^2+y^2)(1+\widetilde{x}^2+\widetilde{y}^2)}\right| \\
&= \frac{t\omega^2}{(2+\omega^2)(1+x^2+y^2)(1+\widetilde{x}^2+\widetilde{y}^2)}\left|\widetilde{x}^2+\widetilde{y}^2-x^2-y^2\right| \\
&\le \frac{\omega^2}{(2+\omega^2)}|x-\widetilde{x}|+\frac{\omega^2}{(2+\omega^2)}|y-\widetilde{y}|.
\end{aligned}$$

Then

$$A_1(\omega)=\omega^2, A_2(\omega)=\omega^4$$

and

$$\|f(t,x,y,\omega)-f(t,\widetilde{x},\widetilde{y},\omega)\|_\infty\le\frac{\omega^2}{(2+\omega^2)}\|x-\widetilde{x}\|_\infty+\frac{\omega^2}{(2+\omega^2)}\|y-\widetilde{y}\|_\infty.$$

Analogously for the function g, we get

$$\|g(t,x,y,\omega)-g(t,\widetilde{x},\widetilde{y},\omega)\|_\infty\le\frac{\omega^2}{(2+\omega^2)}\|x-\widetilde{x}\|_\infty+\frac{\omega^2}{(2+\omega^2)}\|y-\widetilde{y}\|_\infty.$$

We take

$$p_1(\omega)=p_2(\omega)=p_3(\omega)=p_4(\omega)=\frac{\omega^2}{(2+\omega^2)},$$

$$p(t,\omega)=\sum_{i=1}^{i=4}p_i(t,\omega)=\frac{4\omega^2}{(2+\omega^2)},$$

$$S_1(\omega,t)=e^{-w^2t},\ S_2(\omega,t)=e^{-w^4t},\ \|S_i(\omega,t)|\le 1=M_i(\omega), i=1,2 \text{ for all } \omega\in\Omega, t\in J$$

and

$$\tau>M_1(\omega)+M_2(\omega)=2.$$

We have

$$M(\omega)=\frac{1}{\tau}\begin{pmatrix}1&1\\1&1\end{pmatrix}.$$

It is clear that the spectral $\rho(M(\omega))=\frac{2}{\tau}<1$, then the matrix $M(\omega)$ has converge to 0. Therefore, all the conditions of Theorem 3.2 are satisfied. Hence the problem (2) has a unique random solution.

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