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INTEGRAL STABILITY OF IMPULSIVE DYNAMIC SYSTEMS ON TIME SCALE

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Abstract. This work presents a comprehensive analysis of integral stability for impulsive dynamic equations on time scales using the comparison principle framework. We first establish a comparison theorem, which provides a rigorous basis for comparing the behavior of the main complex system to that of a simpler system of lower order known as the comparison system, whose qualitative properties are easier to ascertain. Building on this result, we derive an integral stability theorem, offering sufficient conditions for integral stability in terms of the properties of the comparison system. Our approach leverages the vector Lyapunov functions and comparison equations to ensure the cumulative effect of impulses and system dynamics remains bounded. The theoretical findings are validated through an illustrative example, demonstrating the applicability of the proposed framework to systems with mixed continuous-discrete dynamics and impulsive effects.

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1. INTRODUCTION

The study of dynamic equations on time scales has emerged as a unifying mathematical framework that seamlessly integrates continuous and discrete dynamical systems ([4,5,9]). This hybrid approach is particularly valuable for modeling systems that exhibit both continuous evolution and discrete jumps, such as population dynamics with harvesting, control systems with impulsive inputs, and neural networks with instantaneous state changes. Impulsive dynamic equations, which incorporate sudden perturbations or state jumps, are a natural extension of this framework and have garnered significant

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attention due to their ability to capture real-world phenomena with abrupt changes. Stability analysis of such systems is of paramount importance, as it provides critical insights into the long-term behavior and robustness of solutions under external disturbances.

Stability analysis of systems in both classical [28,29,32–34] and non-classical [31,35] systems has occupied the interest of researchers in recent times due to its importance [20]. Stability concepts, such as Lyapunov stability and asymptotic stability, have been extensively studied for both continuous and discrete systems [7,8,12]. However, the presence of impulses introduces additional complexity, as these sudden changes can drastically alter the system's trajectory [11]. Furthermore, when such systems are analyzed on time scales [13,14,17–19], the interplay between continuous and discrete dynamics necessitates a more sophisticated approach to stability analysis. Integral stability, a generalization of classical stability concepts, offers a robust framework for analyzing systems where the cumulative effect of perturbations over time plays a critical role. This concept is particularly relevant for impulsive systems, as it accounts for the aggregated impact of impulses and system dynamics, providing a more comprehensive measure of stability than pointwise criteria.

In this work, we present a comprehensive study of integral stability for impulsive dynamic equations on time scales using the comparison principle framework. The comparison principle is a powerful tool in stability analysis, allowing us to relate the behavior of a complex system to that of a simpler, well-understood comparison system. By leveraging this principle, we establish a unified framework for analyzing integral stability that is applicable to a wide range of impulsive dynamic systems on time scales.

The foundation of our stability analysis lies in the formulation and proof of a comparison theorem for impulsive dynamic equations on time scales. This theorem provides a rigorous basis for comparing the behavior of the original system to that of a scalar comparison system, enabling a systematic approach to stability analysis. Building on this result, we proceed to state and prove an integral stability theorem, which establishes sufficient conditions for integral stability in terms of the properties of the comparison system. Our approach integrates Lyapunov functions and vector comparison equations to derive explicit criteria for integral stability, ensuring that the cumulative effect of impulses and system dynamics remains bounded. By bridging the gap between time scale calculus, impulsive systems, and integral stability theory, this work advances the understanding of dynamic systems with complex temporal behaviors.

This article is organized as follows: In Section 2, we provide the necessary preliminaries on time scale calculus, impulsive dynamic equations, and stability concepts. In Section 3, we present the comparison theorem, including its statement and proof, which was then applied to develop the integral stability theorem, and providing a detailed proof. In Section 4, we illustrate the theoretical findings through a practical example, and Section 5 concludes the article with a discussion of the results and highlighting the importance of the work.

2. Preliminaries Notes

In this section, we lay the groundwork by introducing key notations and definitions that will be instrumental in developing the main results.

Definition 2.1. ([2]) *For $t \in \mathbb{T}$, we define the forward jump operator as $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and the backward jump operator as $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$. $t \in \mathbb{T}$ is said to be right scattered (r-s) if $\sigma(t) > t$, left scattered (l-s) if $\rho(t) < t$, right dense (r-d) if $\sigma(t) = t$ and left dense (l-d) if $\rho(t) = t$. A function $\mu(t) = \sigma(t) - t$ is called the graininess function.*

Definition 2.2 ([2]). *A function $\mathfrak{G} : \mathbb{T} \to \mathbb{R}$ is said to be r-d continuous (C_{rd}) if it is continuous at all right-dense points of $\mathbb{T}$, and its left-sided limits exist and are finite at all left-dense points of $\mathbb{T}$.*

Definition 2.3 ([2]). *If a function $\mathfrak{K} \in C[[0, \mathrm{j}], [0, \infty)]$ is strictly increasing on $[0, \mathrm{j}]$ with $\mathfrak{K}(0) = 0$, then it is called a class $\mathcal{K}$ function.*

Definition 2.4. *A function $g \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n]$ is said to be quasimonotone non-decreasing in u if $u < v$ and $u_i = v_i$ for $1 \le i \le n$ implies $g_i(u) \le g_i(v)$, $\forall i \in \mathbb{N}$, $u, v \in \mathbb{R}^n$.*

Definition 2.5. *A function $b \in C_{rd}[\mathbb{T} \times \mathbb{R}_+, \mathbb{R}^n_+]$ is said to belong to a class $\mathcal{OK}$ if $b(t, \cdot) \in \mathcal{K}$ and for any $t \in \mathbb{T}^k$, and $b(t, \cdot) \to \infty$ as $t \to \infty$.*

Statement of the Problem

Consider the impulsive dynamic system on time scale

$$\begin{aligned} & x^{\Delta}(t) = f(t, x),\ t \in \mathbb{T},\ t \neq t_i \\ & \Delta x(t) = x(t_i^+) - x(t_i) = I_i(x(t)),\ t = t_i \\ & x(t_0) = x_0, \end{aligned} \tag{1}$$

where $x(t_i^+)$ denotes the right limit of $x(t_i)$ at $t = t_i$, $i \in \mathbb{N}$, $x : \mathbb{T} \to \mathbb{R}^N$, $f \in C_{rd}[\mathbb{T} \times \mathbb{R}^N, \mathbb{R}^N]$, $f(t_{10}) = 0$, $0 \le t_0 < t_1 < \cdots \infty$. $I_i \in C_{rd}[\mathbb{R}^n, \mathbb{R}^n]$, $I_i(0) = 0$ is the sequence of instantaneous impulse operators. $\mathbb{T}$ is a strictly monotone increasing function such that $\forall t_i \in \mathbb{T}$, $\lim_{i \to \infty} t_i = \infty$. The solution of the impulsive dynamic equation with impulse effect (1) depends not only on the initial condition (t_0, x_0) but also on the moments of impulses t_k for each $k \in \mathbb{N}$. Let $x(t) = x(t, t_0, x_0)$ be the unique solution of (1) satisfying the initial condition $x(t_0, t_0, x_0) = x_0$, for the purpose of this work, we assume that the solution $x(t) = x(t; t_0, x_0)$ of (1) exists and is unique (see [10,21–23]), this work aims to investigate the integral stability.

To achieve this aim, we consider the following comparison impulsive dynamic system on time scale systems

$$\begin{aligned} u^{\Delta} &= g_1(t,u)\ t \neq t_i \\ \Delta u &= \Omega_i(u(t_i))\ t = t_i \\ u(t_0) &= u_0 \end{aligned} \tag{2}$$

$$\begin{aligned} V^{\Delta} &= g_2(t,v)\ t \neq t_i \\ \Delta V &= \Psi_i(V(t_i))\ t = t_i \\ v(t_0) &= v_0 \end{aligned} \tag{3}$$

and the perturbed system of (3)

$$\begin{aligned} v^{\Delta} &= g_2(t,v) + h(t)\ t \neq t_i \\ \Delta v &= \Psi_i(v(t_i)) + \eta_i(t_i)\ t = t_i \\ v(t_0) &= v_0, \end{aligned} \tag{4}$$

where

(i) $g_1 \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n]$, and $\Omega_i \in C_{rd}[\mathbb{R}^n, \mathbb{R}^n]$ are quasimonotone non-decreasing functions, such that $g_1(t,0) = 0$, $\Omega_i(0) = 0$. Note that $g_1(t,u)$ is quasimonotone non-decreasing in u.

(ii) $g_2 \in C_{rd}[\mathbb{T} \times \mathbb{R}, \mathbb{R}^n]$, is quasimonotone non-decreasing with respect to its second argument, $\Psi_i \in [\mathbb{R}^n, \mathbb{R}^n]$, $h(t) \in [\mathbb{T}, \mathbb{R}^n]$, $\eta_i \in [\mathbb{T}, \mathbb{R}^n]$ and $g_2(t,0) = \Psi_i(0) = 0$, $\eta_i(0) = 0$.

Note that the solution $u(t) = u(t; t_0, u_0)$ of (2) exists and is unique (see [10,24–27,30]).

Definition 2.6. *A function $V : \mathbb{T}^k \times \mathbb{R}^N \to \mathbb{R}^N_+$ is said to belong to class $\mathcal{M}$ if*

(1) $V(t,x) \in C_{rd}[\mathbb{T}^k \times \mathbb{R}^N, \mathbb{R}^N_+]$

(2) *$V(t,x)$ is locally Lipschitzian with respect to its second argument*

(3) *$\lim\limits_{t \to t_i^-} V(t,x) = V(t_i - 0, x) = v(t_i, x)$ and $\lim\limits_{t \to t_i^+} V(t,x)$ exists $\forall i \in \mathbb{N}$, at each (t,x) with left dense t and $\mathbb{T}^k = T - \{m\}$ if $\mathbb{T}$ has a right scattered maximum m or else $\mathbb{T}^k = \mathbb{T}$.*

Definition 2.7. *Let $V(t,x) \in \mathcal{M}$ then, we define the Dini derivative of $V(t,x)$ relative to (1) as follows:*

$$D^+V^{\Delta}(t,x) = \limsup_{\mu(t) \to 0} \frac{1}{\mu(t)} \Big\{ V(t + \mu(t), x + \mu(t) f(t,x)) - V(t,x) \Big\}.$$

Definition 2.8. *A zero solution of (1) is said to be stable if for every $\epsilon > 0$, and any $t_0 \in \mathbb{T}^k$, there exists a positive function $\delta(t_0, \epsilon) \in K$ which is continuous in t_0 for each ϵ such that the inequality*

$$\|x_0\| < \delta,$$

implies

$$\|x(t)\| < \epsilon,$$

where $x(t) = x(t; t_0, x_0)$ is any solution of (1).

For the purpose of this research, we shall define the following sets;

(1) $S_\rho = \{(t,x) \in \mathbb{T} \times \mathbb{R}^N : \|x\| < \rho \text{ and } \rho > 0\}$;

(2) $S_\rho^* = \{(t,x) \in \mathbb{T} \times \mathbb{R}^N : \|x\| \geq \rho, \text{ and } \rho > 0\}$;

(3) $\Lambda(t, T, \rho) = \{x \in \mathbb{R}^N : (t,x) \in S_\rho, S \in [t, t+T]\}$.

Definition 2.9. *The zero solution of (1) is said to be integrally stable if for every $\epsilon \geq 0$, $t_0 \in \mathbb{T}^k$, there exists $\delta(t_0, \epsilon) > 0 \in K$ which is $rd-$continuous in t_0 for each ϵ such that for any solution $x^*(t) = x^*(t, t_0, x_0$ of (),*

$$\|x_0\| \leq \epsilon \textit{ implies } \|x^*(t)\| < \delta \textit{ and for } T > 0, \tag{5}$$

the perturbations $f^*(t,x)$ and $I_i^*(x)$, $i \in \mathbb{N}$ satisfy

$$\int_{t_0}^{t_0+T} \sup_{\|x(t)\|<\delta} \|f^*(s,x^*)\|\Delta s + \sum_{t_0 < t_i \leq t_0+T} \sup_{\|x^*(t_i)\|<\delta} \|I_i^*(x^*)\| \leq \epsilon \tag{6}$$

3. Main Results

Theorem 3.1 (Comparison Theorem). *Assume that*

(1) $g_1(t,u) \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n]$ *is quasimonotone nondecreasing in u*

(2) $V \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n_+]$ *is locally Lipschitzian in x and satisfy*

$$D^+V^\Delta(t,x) \leq g_1(t, V(t,x)) \; t \neq t_i, \;\; i \in \mathbb{N}$$

(3) $\Omega_i \in [\mathbb{R}^n, \mathbb{R}^n]$, $V(t+0, x+I_i(x)) \leq \Omega_i(V(t,x))$ *where $\Omega_i \in K$.*

Let $m(t) = m(t, t_0, u_0)$ be existing on $\mathbb{T}^k$. Then

$$V(t_0+0, x_0) = V(t_0^+, x_0) \leq u_0$$

implies

$$V(t, x(t)) \leq m(t)$$

where $x(t) = x(t, t_0, x_0)$ is any solution of (1) existing on $\mathbb{T}^k$.

Proof. Let $x(t)$ be the solution of (1) defined for $t \geq t_0$, $t, t_0 \in \mathbb{T}^k$ such that $V(t_0^+, x_0) \leq u_0$. Also let $r(t) = V(t, x(t))$ for $t \neq t_i$. Then

$$\begin{aligned} r(t+\mu(t)) - r(t) &= V(t+\mu(t), x(t+\mu(t))) - V(t, x(t)) \\ &= V(t+\mu(t), x(t+\mu(t))) - V(t+\mu(t), x+\mu()f(t,x)) \\ &\quad + V(t+\mu(t), x+\mu(t)f(t,x)) - V(t, x(t)) \end{aligned}$$

Since V is Lipschitzian with respect to x, we have

$$r(t+\mu(t)) - r(t) \leq L\|x(t+\mu(t)) - (x+\mu(t)f(t,x))\|_e + V(t+\mu(t), x+\mu(t)f(t,x)) - V(t, x(t))$$

where L is the Lipschitz constant and $e = (1, 1, \cdots, 1)^T$. Taking $\limsup_{\mu(t)\to 0} \frac{1}{\mu(t)}$ of both sides.

$$\begin{aligned}
\limsup_{\mu(t)\to 0} \frac{1}{\mu(t)}\{r(t+\mu(t)) - r(t)\} &= \limsup_{\mu(t)\to 0} \frac{1}{\mu(t)}\{L\|x(t+\mu(t)) - (x+\mu(t)f(t,x))\|_e \\
&\quad + V(t+\mu(t), x+\mu(t)f(t,x)) - V(t,x(t))\} \\
D^+r^\Delta(t) &\leq \frac{1}{\mu(t)}[\limsup_{\mu(t)\to 0} L\|x(t+\mu(t)) - \limsup_{\mu(t)\to 0}(x+\mu(t)f(t,x))\|_e] \\
&\quad + \limsup_{\mu(t) to 0} \frac{1}{\mu(t)}\{V(t+\mu(t), x+\mu(t)f(t,x)) - V(t,x(t))\} \\
D^+r^\Delta(t) &= \frac{1}{\mu(t)}[L\|x(t) - x(t)\|_e] \\
&\quad + \limsup_{\mu(t)\to 0} \frac{1}{\mu(t)}\{V(t+\mu(t), x+\mu(t)f(t,x)) - V(t,x(t))\}
\end{aligned}$$

Applying definition 3 and from condition 2 of the theorem

$$D^+r^\Delta(t) \leq D^+V(t,x(t)) \leq g_1(t, V(t,x(t))).$$

From condition (3) of the theorem and equation (1), we have $r(t_)^+ \leq u_0$ and from equation (1) and condition (3) we also have

$$m(t_i^+) = V(t_i^+, x(t_i^+)) = V(t_i^+, x(t_i) + I(x(t_i))) \leq \Omega_i(V(t_i, x))$$

Then the inequality

$$V(t,x(t)) \leq m(t) \text{ for } t \geq t_0$$

completing the proof. □

Theorem 3.2 (**Integral Stability Theorem**). *Assume that:*

(1) $f \in C_{rd}[\mathbb{T}\times\mathbb{R}^n, \mathbb{R}^n]$, $I_i \in C_{rd}[\mathbb{R}^n, \mathbb{R}^n]$ *and* $g_1(t,u) \in C_{rd}[\mathbb{T}\times\mathbb{R}^n, \mathbb{R}^n]$ *is quasimonotone nondecreasing in* u.

(2) *For* $V_1(t,x) \in \mathcal{M}$, $V_1(t,0) = 0$, V_1 *is decrescent and*
   (i) $D^+V_1^\Delta(t,x) \leq g_1(t, V_1(t,x))$ *holds for* $(t,x) \in S_\rho$, $t \neq t_i$
   (ii) $V_1(t, x+I_i(x)) \leq \Omega_i(V_1(t,x))$ *for all* $(t, x \in S_\rho)$ *where* Ω_i *is quasimonotone non decreasing such that* $\Omega_i(x) \geq x$

(3) *For* $V_2(t,x) \in \mathcal{M}$, $V_2(t,0) = 0$,
   (i) $b(\|x\|) \leq V_2(t,x) \leq a(\|x\|)$ *for* $(t,x) \in S_\rho \cap S_\rho^*$ *and* $a, b \in K$
   (ii) $D^+V_1^\Delta(t,x) + D^+V_2^\Delta(t,x) \leq g_2(t, V_1(t,x) + V_2(t,x))$ *holds for* $(t,x) \in S_\rho \cap S_{\rho^*}$ *and* $t \neq t_i$, $\forall i \in \mathbb{N}$ *where* g_2 *is quasimonotone with respect to the second argument*
   (iii) $V_1(t_i^+, x+I_i(x)) + V_2(t_i^+, x+I_i(x)) \leq \Psi_i(V_1(t_i, x(t_i))) + V_2(t_i, x(t_i))$ *for* $\Psi_i \in K$ *and* $\Psi_i \geq x$

(4) *The zero solution of systems* (1),(2), (3) *exists. Then if the zero solution of* (2) *is stable and* (3) *is integrally stable, then it implies that* (1) *will be integrally stable.*

Proof. By the decrescent property of $V_1(t,x)$, then there exists $\rho_1 < \rho$ which is positive and $d_1 \in \mathcal{OK}$ such that

$$\|x\| < \rho_1$$

implies

$$V_1(t,x) < d_1(t,x) \tag{7}$$

Choose $\epsilon > 0$ such that $\epsilon < \rho_1$.

By the Lipschitzian property on $V_1(t,x)$ and $V_2(t,x)$, then there exists Lipschitz constants L_1 and L_2 of V_1 and V_2. Set $(L_1+L_2)\epsilon = \epsilon_1$.

Byb stability of (2), given $\epsilon_1 > 0$, and $t_0 \in \mathbb{T}$, there exists $\delta_1(t_0,\epsilon) > 0$ such that

$$\|u_0\| < \delta_1$$

implies

$$\|m(t,t_0,u_0)\| < \frac{\epsilon_1}{2},\ t \geq t_0 \tag{8}$$

where $m(t,t_0,u_0) = m(t)$ is the maximal solution of (2) which of course is any solution of (2).

Since $d_1 \in \mathcal{DK}$, exists $\delta_2(\delta_1) > 0$, such that

$$\|u\| < \delta_2 \text{ implies } d_1(t,u) < \delta_1 \tag{9}$$

Also, by the integral stability of (3), we can find $\lambda_1(t_0,\epsilon_1) \in \mathcal{K}$ for each ϵ_1 such that for every solution $V(t) = V(t,t_0,v_0)$ of (4),

$$\|V(t)\| < \lambda_1 \tag{10}$$

holds, where $\|v_0\| \leq \epsilon_1$ and for every $T > 0$, $h(t)$ and η_i satisfy

$$\int_{t_0}^{t_0+T} |h(s)|\Delta s + \sum_{t_0<t_i<t_0+T} |\eta_i(t_i)| \leq \epsilon_1. \tag{11}$$

Moreso, for $b \in \mathcal{K}$, $\lim_{s\to\infty} b(s) = \infty$. Take $\lambda(\lambda_1) > 0$ such that $b(\lambda) \geq \lambda_1$ and $\lambda > d_2(\epsilon)$ for $d_2 \in \mathcal{K}$ satisfying $d_2(\epsilon) < \rho_1$. Choose $\delta_3 = \delta_3(\epsilon_1,\lambda)$, $\epsilon < \delta_3 < \min\{\delta_2,\rho_1\}$ such that

$$a(\delta_3) < \frac{\epsilon_1}{2} \text{ and } d_2(\delta_3) < \lambda \tag{12}$$

hold.

For any solution $x^*(t) = x^*(t,t_0,x_0)$ of (2), we show that if (5) and (6) hold, then

$$\|x^*(t)\| < \lambda \text{ for } t \geq t_0,\ t,t_0 \in \mathbb{T} \tag{13}$$

Suppose this claim is false, then we can find a point $t_1 > t_0$ such that

$$\|x^*(t_1)\| \geq \lambda \text{ and } \|x^*(t)\| < \lambda \text{ at } t \in [t_0,t_1) \subset \mathbb{T} \tag{14}$$

For $t_1 \neq t_i$, the solution $x^*(r)$ is continuous at t_1 and $\|x^*(t_1)\| = \lambda$ but $\|x^*(t_1)\| > \delta_3$ because if $\|x^*(t_1)\| \leq \delta_3$, then $d_2(x^*(t_1)) < \lambda$ which contradicts (14) by the choice of δ_3.
Take the interval $(t_0, t_1) \subset \mathbb{T}$, choose $t_{10} \in (t_0, t_1$ such that $t_{10} \neq t_k$, $\|x^*(t_{10})\| = \delta_3$ and for $t \in [t_{10}, t_1)$,

$$(t, x^*(t)) \in S_\lambda \cap S^*_\rho$$

Set $u_0 = V_1(t_{10}, x^*(t_{10}))$, and let $m_1(t, t_{10}, u_0)$ be the maximal solution of (2) then from $(2i)$ and $(2ii)$ of our theorem and applying the comparison theory, we have that

$$V_1(t, x(t, t_{10}, x_0)) \leq m_1(t_1, t_{10}, u_0) \text{ for } t \in [t_{10}, t_1] \in \mathbb{T} \tag{15}$$

where $x(t, t_{10}, x_0)$ is a solution of (1) with respect to t_{10}.
By the choice of our $\delta_3 = \|x^*(t_{10})\|$, $\delta < \delta_2$ and $u_0 = V_1(t_{10}, x^*(t_{10}))$, applying (7) and (9) we have that

$$u_0 = V_1(t_{10}, x^*(t_{10})) \leq d_1(t_{10}, x^*(t_{10})) < \delta_1$$

Applying (8) to (15) we have that

$$V_1(t, x(t, t_{10}, x_0)) \leq m_1(t, t_{10}, u_0) < \frac{\epsilon_1}{2} \text{ for } t \in [t_{10}, t_1] \in \mathbb{T} \tag{16}$$

Combining (12) and (3i) of our theorem, we have

$$V_2(t_{10}, x^*(t_{10})) \leq a\|x^*(t_{10})\| < a(\delta_1) < \frac{\epsilon_1}{2} \tag{17}$$

For $V(t, x) \in \mathcal{M}$, let $V_1(t, x) + V_2(t, x)$, from (3ii) of our theorem and by the Lipschitzian of V_1 and V_2, we get that

$$\begin{aligned}
D^+V^\Delta(t, x) &= D^+V_1^\Delta(t, x) + D^+V_2^\Delta(t, x) \\
&= \limsup_{\mu(t)\to 0} \frac{1}{\mu(t)}\{V_1(t + \mu(t), x + \mu(t)(f(t, x) + f^*(t, x))) - V_1(t, x)\} \\
&\quad + \limsup_{\mu^*(t)\to 0} \frac{1}{\mu^*(t)}\Big\{V_2(t + \mu^*(t), x + \mu^*(t)(f(t, x) + f^*(t, x))) - V_2(t, x)\Big\} \\
&= \limsup_{\mu(t)\to 0} \frac{1}{\mu(t)}\Big\{V_1(t + \mu(t), x + \mu(t)(f(t, x) + f^*(t, x))) \\
&\quad - V_1(t + \mu(t), x + \mu(t)f(t, x)) \\
&\quad + V_1(t + \mu(t), x + \mu(t)f(t, x)) - V_1(t, x)\Big\} \\
&\quad + \limsup_{\mu^*(t)\to 0} \frac{1}{\mu^*(t)}\Big\{V_2(t + \mu^*(t), x + \mu^*(t)(f(t, x) + f^*(t, x))) \\
&\quad - V_2(t + \mu^*(t), x + \mu^*(t)(f(t, x))) \\
&\quad + V_2(t + \mu^*(t), x + \mu^*(t)f(t, x)) - V_2(t, x)\Big\} \\
&= \limsup_{\mu(t)\to 0} \frac{1}{\mu(t)}\{V_1(t + \mu(t), x + \mu(t)(f(t, x) + f^*(t, x))) \\
&\quad - V_1(t + \mu(t), x + \mu(t)f(t, x))\}
\end{aligned}$$

$$+\limsup_{\mu^*(t)\to 0} \frac{1}{\mu^*(t)} \{V_2(t+\mu^*(t), x+\mu^*(t)(f(t,x)+f^*(t,x)))$$
$$-V_2(t+\mu^*(t), x+\mu^*(t)f(t,x))\}$$
$$+\limsup_{\mu(t)\to 0} \frac{1}{\mu(t)} \{V_1(t+\mu(t), x+\mu(t)f(t,x)) - V_1(t,x)\}$$
$$+\limsup_{\mu^*(t)\to 0} \frac{1}{\mu^*(t)} \{V_2(t+\mu^*(t), x+\mu^*(t)f(t,x)) - V_2(t,x)\}$$

Since V_1 and V_2 are Lipschitzian, we get

$$\leq L_1 \sup_{x\in\Lambda(t_{10},T^*,\lambda)} \|f^*(t,x)\| + L_2 \sup_{x\in L(t_{10},T^*,\lambda)} \|f^*(t,x)\| + D^+V_1^{\Delta}(t,x) + D^+V_2^{\Delta}(t,x)$$

from (3ii) we have

$$\leq (L_1+L_2) \sup_{x\in L(t_{10},T^*,\lambda)} \|f^*(t,x)\| + g_2(V_1(t,x)+V_2(t,x)), \tag{18}$$

where $T^* = t_1 - t_{10} \in \mathbb{T}$

Also for $t_i \in (t_{10}, t_1)$, and $(t_i, x) \in S_\lambda \cap S^*_\epsilon$ and applying (3iii) we get

$$\begin{aligned}
&V(t_i^+, x+I_i(x)+I_i^*(x))\\
&\quad= V(t_i, x+I_i(x)) + V(t_i^*, x+I_i(x)) - V(t_i^+, x+I_i(x))\\
&\quad= V(t_i^+, x+I_i(x)) + V(t_i^+, x+I_i(x)+I_i^*(x)) - V(t_i^+, x+I_i(x))\\
&\quad= V(t_i^+, x+I_i(x)) + V_1(t_i^+, x+I_i(x)+I_i^*(x))\\
&\qquad + V_2(t_i^+, x+I_i(x)+I_i^*(x)) - V_1(t_i^+, x+I_i(x)) - V_2(t_i^+, x+I_i(x))\\
&\quad= V(t_i^+, x+I_i(x)) + V_1(t_i^+, x+I_i(x)+I_i^*(x)) - V_1(t_i^+, x+I_i(x))\\
&\qquad + V_2(t_i^+, x+I_i(x)+I_i^*(x)) - V_2(t_i^+, x+I_i(x)).
\end{aligned} \tag{19}$$

Applying the Lipschitz property on V_1 and V_2, we get

$$\begin{aligned}
&\leq \Psi_i(V(t_i, x(t_i))) + L_1\|I_i^*(x)\| + L_2\|I_+^*(x)\|\\
&= \Psi_i(V(t_i, x(t_i))) + (L_i+L_2)\|I_i^*(x)\|\\
&\leq \Psi_i(V(t_i, x(t_i))) + (L_1+L_2) \sum_{t_0<t_i<t_0+T} \sup_{x(t_i)<\lambda} \|I_i^*(x)\|.
\end{aligned}$$

From (4), let $h(t) = (L_1+L_2) \sup_{x\in\Lambda(t_{10},T^*,\lambda)} \|f^*(t,x)\|$ and $\eta_i = (L_1+L_2) \sup_{x(t_i)<\lambda} \|I_i^*(x)\|$ Then from (18), we have

$$D^+V(t,x) \leq g_2(t, V(t,x)) + h(t),$$

and from (19), we have

$$V(t_i^*, I_i(x)+I_i^*(x)) \leq \Psi_i(V(t_i, x(t_i))) + \eta_i(t_i).$$

Take $m^*(t, t_{10}, v_{10})$ to be the maximal solution of (4) and set $v_{10} = V(t_{10}, x^*(t_{10}))$ applying (18) and (19) and from the comparison theorem, we have

$$V(t, x^*(t, t_0, x_0)) \le m^*(t, t_{10}, v_{10}) \text{ for } t \in \mathbb{T}^k$$

From (7) and by the choice of ϵ_1.

$$\begin{aligned}\int_{t_{10}}^{t_1} h(s)\Delta s + \sum_{t_0<t_i\le t_1} \eta_i = (L_1+L_2) &\int_{t_{10}}^{t_1} \sup_{x^*\in\Lambda(t_{10},T,\lambda)} \|f^*(s,x)\|\Delta s \\ &+ (M_1+M_2) \sum_{t_0<t_i\le t^*} \sup_{x(t_i)<\lambda} \|I_i^*(x)\| \\ =&(L_1+L_2)\Big[\int_{t_{10}}^{t_1} \sup_{x\in\Lambda(t_{10},T,\lambda)} \|f^*(s,x)\|\Delta s \\ &+ \sum_{t_{10}<t_i<t_1} \sup_{x(t_i)<\lambda} \|I_i^*(x)\|\Big] < \epsilon(L_1+L_2) < \epsilon_1,\end{aligned} \tag{20}$$

Take $T^* > t^*\ \forall t^*, T^* \in \mathbb{T}^k$ such that $\int_{t_{10}}^{t_1} h(s)\Delta s + \frac{1}{2}(T^* - t^*)h(t_1) < \epsilon_1$
let $h^*(t)$ be such that $h^*(t) \in [\mathbb{T}, \mathbb{R}]$ and
$h^*(t) = h(t)$ for $t \in [t_{10}, t_1] \in \mathbb{T}^k$
$h^*(t) = \frac{h(t_1)}{t_1 - T^*}(t - T^*)$ for $t \in (t_1, T^*] \in \mathbb{T}^k$
$h^*(t) = 0$ when $t \ge T^* \forall t, T^* \in \mathbb{T}^k$.
Also define constants η_i^* such that $\eta_i^* = \eta_i$ when $t_i \in (t_{10}, t_1] \in \mathbb{T}^k$
$\eta_i^* = 0$ when $t_i > t_1$
And if (6) holds then by (20), we get that

$$\int_{t_{10}}^{t_{10}+T} |h^*(s)|\Delta s + \sum_{t_0<t_i<t_0+T} |\eta_i^*| < \epsilon_1, \tag{21}$$

take $m^*(t, t_{10}, v_{10})$ to be the maximal solution of (4) and $h(s) = h^*(s)$, $\eta_i = \eta_i^*$, then

$$m_1^*(t, t_{10}, v_{10}) = m^*(t, t_{10}, v_{10}) \text{ for } t \in \mathbb{T}^k,$$

from (16) and (17), we deduce that

$$V(t_{10}, x^*(t_{10})) = V_1(t_{10}, x^*(t_{10})) + V_2(t_{10}, x^*(t_{10})) < \epsilon_1,$$

can be rewritten as

$$\|v_{10}\|\epsilon_1, \tag{22}$$

and applying (10), we have

$$\|m_1^*(t, t_{10}, w_{10})\| < \lambda_1 \text{ for } t \ge t_{10} \in \mathbb{T}^k \tag{23}$$

Since ϵ_1 was chosen arbitrarily, then from (23) and from (3i) in our theorem, we have that

$$b(\lambda) \ge \lambda_1 > \|m_1^*(t, t_{10}, v_{10})\| = \|m^*(t, t_{10}, v_{10})\| \ge V(t_1, x^*(t_i, t_{10}, v_{10}))$$

$$
\begin{aligned}
&= V_1(t_1, x^*(t_1, t_{10}, v_{10})) + V_2(t_1, x^*(t_1, t_{10}, v_{10})) \geq V_2(t_1, x^*(t_1, t_{10}, v_{10})) \\
&\geq b(\|x^*(t_1, t_{10}, v_{10})\|) = b(\lambda).
\end{aligned}
$$

this is a contradiction since $b(\lambda) > b(\lambda)$, so (13) is true for $t \geq t_0 \in \mathbb{T}^k$

Choose $t_i \in (t_0, t_1)$ such that $\delta_3 = \|x^*(t_i)\|$ and $(t, x^*(t)) \in S_\lambda \cap S^*_\rho$ and $t \in [t_i, t_1) \in \mathbb{T}^k$.

Select δ^*_3 such that $\delta_3 < \delta^*_3 < \lambda$ and $\delta^*_3 = \|x^*(t_{10}, t_0, x_0)\|$, $t_{10} \neq t_i \in (t_0, t_1) \in \mathbb{T}^k$. If we repeat the previous staeps with $\delta_3 = \delta^*_3$ we will still get a contradiction and then show that (13) is true.

At $t_1 = t_k$ from (14) we have $\|x^*(t_k)\| \geq \lambda$ and $\|x^*(t)\| < \lambda$, for $t \in [t_0, t_i]$

$\|x^*(t_i^+, t_0, x_0)\| = x^*(t_i) + (I_i + I_i^*)(x^*(t_i))$

Choose $\lambda(\lambda_1) > 0$ such that $b(\lambda) \geq \sup_i \Psi_i(m^*_i(t_i, t_0, v_{10}))$ and the following same pattern as above, we get (23) and applying (3i) and (3iii) to (23) we get

$$
\begin{aligned}
b(\lambda) &\geq \sup_i(\Psi_i(m^*_1(t_i, t_0, v_{10}))) > \Psi_i(m^*_1(t_i, t_0, v_{10})) \geq \Psi_i(V(t_i, x^*(t_i))) \\
&\geq V(t_i, x^*(t_i)) \geq V_2(t_i, x^*(t_i)) \geq b(x^*(t_i)) \geq b(\lambda),
\end{aligned}
$$

which is a contradiction since $b(\lambda) > b(\lambda)$.

This implies that (13) is true for all $t \in \mathbb{T}^k$ and therefore (1) is integrally stable. □

4. Illustration

Consider the system

$$
\begin{cases}
x_1^\Delta = x_2 \cos t + e^{-t} x_1 - (x_1^3 + x_1 x_2^2)\cos^2 t, & t \neq t_i \\
x_2^\Delta = x_1 \cos t + e^{-t} x_2 - (x_1^2 x_2 + x_2^2)\cos^2 t, & t \neq t_i \\
\Delta x_1 = y x_1 + z x_2, \ \ \Delta x_2 = y x_1 + z x_2, & t = t_i,
\end{cases}
$$

for $2y = \sqrt{1+\alpha_1} + \sqrt{1+\alpha_2} - 2$, $2z = \sqrt{1+\alpha_1} - \sqrt{1+\alpha_2}$ and $-1 < \alpha_1 \leq 0$, $-1 < \alpha_2 \leq 0$. Applying the concept of vector Lyapunov function, we can choose $V = (V_1, V_2)^T$ where $V_1 = \frac{1}{2}(x_1 + x_2)^2$, $V_2 = \frac{1}{2}(x_1 - x_2)^2$ so that the associated norm $\|x\| = \sqrt{x_1^2 + x_2^2}$.

$$
\begin{aligned}
V_0(t, x) = \sum_{i=1}^{2} V_1(x_1, x_2) &= \frac{1}{2}(x_1 + x_2)^2 + \frac{1}{2}(x_1 - x_2)^2 \\
&= \frac{1}{2}(x_1^2 + 2x_1x_2 + x_2^2) + \frac{1}{2}(x_1^2 - 2x_1x_2 + x_2^2) \\
&= \frac{1}{2}x_1^2 + x_1 + x_2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_1^2 + x_1x_2 + \frac{1}{2}x_2^2 \\
&= x_1^2 + x_2^2 \\
V_0(t, x) &= x_1^2 + x_2^2.
\end{aligned}
$$

Choose $b(r) = r$, $a(r) = 2r^2$,

So the assumption

$$b(\|x\|) \leq V_0(t,x) \leq a(\|x\|)$$

becomes

$$\sqrt{x_1^2 + x_2^2} \leq x_1^2 + x_2^2 \leq 2(\sqrt{x_1^2 + x_2^2}).$$

Computing for $D^+V_1^{\Delta}(t,x)$ for $V_1(t,x) = V_1(x_1,x_2) = \frac{1}{2}(x_1 - x_2)^2$.

We have

$$\begin{aligned}
D^+V_1^{\Delta}(x_1,x_2) &= \limsup_{\mu(t)\to 0} \frac{1}{\mu(t)}\{V_1(x_1 + \mu(t)f_1(t,x), x_2 + \mu(t)f_2(t,x)) - V(x_1,x_2)\} \\
&= \limsup_{\mu(t)\to 0} \frac{1}{\mu(t)}\{\frac{1}{2}[x_1 + \mu(t)f_1(t,x) - (x_2 + \mu(t)f_2(t,x))]^2 - \frac{1}{2}(x_1 - x_2)^2\} \\
&= \limsup_{\mu(t)\to 0} \frac{1}{\mu(t)}\{\frac{1}{2}[(x_1 + \mu(t)f_1(t,x))^2 + (x_2 + \mu(t)f_2(t,x))^2 \\
&\quad -2((x_1 + \mu(t)f_1(t,x))(x_2 + \mu(t)f_2(t,x)))] - \frac{1}{2}(x_1 - x_2)^2\} \\
&= \limsup_{\mu(t)\to 0} \frac{1}{\mu(t)}\{\frac{1}{2}[x_1^2 + 2x_1\mu(t)f_1(t,x) + \mu^2(t)f_1^2(t,x) + x_2^2 \\
&\quad +2x_2\mu(t)f_2(t,x) + \mu^2(t)f_2^2(t,x) - 2x_1x_2 - 2x_1\mu(t)f_2(t,x) \\
&\quad -2x_2\mu(t)f_(t,x) - 2\mu^2(t)f_1(t,x)f_2(t,x)] - \frac{1}{2}(x_1 - x_2)^2\} \\
&= \limsup_{\mu(t)\to 0} \frac{1}{\mu(t)}\{\frac{1}{2}[(x_1 - x_2)^2 + 2\mu(t)f_1(t,x)(x_1 - x_2) \\
&\quad -2\mu(t)f_2(t,x)(x_1 - x_2)] - \frac{1}{2}(x_1 - x_2)^2\} \\
&= \frac{1}{2}[(x_1 - x_2)^2 + 2f_1(t,x)(x_1 - x_2) - 2f_2(t,x)(x_1 - x_2)] - \frac{1}{2}(x_1 - x_2)^2 \\
&= \frac{1}{2}(x_1 - x_2)^2 + (x_1 - x_2)(f_1(t,x) - f_2(t,x)) - \frac{1}{2}(x_1 - x_2)^2 \\
&= (x_1 - x_2)(f_1(t,x) - f_2(t,x)) \\
&= (x_1 - x_2)(x_2\cos t + e^{-t}x_1 \\
&\quad -(x_1^3 + x_1x_2^2)\cos^2 t - x_1\cos t - e^{-t}x_2 + (x_1^2x_2 + x_2^3)\cos^2 t) \\
&= (x_1 - x_2)(e^{-t}(x_1 - x_2) - \cos t(x_1 - x_2) - (x_1^3 + x_1x_2^2)\cos^2 t \\
&\quad +(x_1^2x_2 + x_2^3)\cos^2 t) \\
&\leq (x_1 - x_2)[(x_1 - x_2)(e^{-t} - \cos t)] \\
&= (x_1 - x_2)^2(e^{-t} - \cos t) \\
D^+V^{\Delta}(x_1,x_2) &\leq (e^{-t} - \cos t)V_1(x_1,x_2).
\end{aligned}$$

for $V_2(x_1, x_2) = \frac{1}{2}(x_1 + x_2)^2$,

$$\begin{aligned}
D^+V_2^{\Delta}(x_1,x_2) &= \limsup_{\mu(t)\to 0} \frac{1}{\mu(t)}\{V_2(x_1+\mu(t)f_1(t,x), x_2+\mu(t)f_2(t,s)) - V(x_1,x_2)\} \\
&= \limsup_{\mu(t)\to 0} \frac{1}{\mu(t)}\{\frac{1}{2}(x_1+\mu(t)+x_2+\mu(t)f_2(t,x))^2 - \frac{1}{2}(x_+x_2)^2\} \\
&= \limsup_{\mu(t)\to 0} \frac{1}{\mu(t)}\{\frac{1}{2}((x_1+\mu(t)f_1(t,x))^2 + (x_2+\mu(t)f_2(t,x))^2 \\
&\quad +2(x_1+\mu(t)f_1(t,x))(x_2+\mu(t)f_2(t,x))) - \frac{1}{2}(x_1+x_2)^2\} \\
&= \limsup_{\mu(t)\to 0} \frac{1}{\mu(t)}\{\frac{1}{2}[x_1^2 + 2x_1\mu(t)f_1(t,x) + \mu^2(t)f_1^2(t,x) + x_2^2 + 2x_2\mu(t)f_2(t,x) \\
&\quad +\mu^2(t)f_2^2(t,x) + 2x_1x_2 + 2x_1\mu(t)f_2(t,x) \\
&\quad +2x_2\mu(t)f_1(t,x) + 2\mu^2(t)f_1(t,x)f_2(t,x)] - \frac{1}{2}(x_1+x_2)^2\} \\
&= \limsup_{\mu(t)\to 0} \frac{1}{\mu(t)}\{\frac{1}{2}[(x_1+x_2)^2 \\
&\quad +2\mu(t)f_1(t,x)(x_1+x_2) + 2\mu(t)f_2(t,x)(x_1+x_2)] - \frac{1}{2}(x_1+x_2)^2\} \\
&= \frac{1}{2}[(x_1+x_2)^2 + 2f_1(t,x)(x_1+x_2) + 2f_2(t,x)(x_1+x_2)] - \frac{1}{2}(x_1+x_2)^2 \\
&= (x_1+x_2)(2f_1(t,x) + 2f_2(t,x)) \\
&= (x_1+x_2)(2[(x_2\cos t + e^{-t}x_1 - (x_1^3 + x_1x_2^2)\cos^2 t) \\
&\quad +(x_1\cos t) + e^{-t}x_2 - (x_1^2x_2 + x_2^2)\cos^2 t]) \\
&\leq (x_1+x_2)(x_1+x_2)(e^{-t} + \cos t) \\
D^+V_2^{\Delta}(x_1,x_2) &\leq (e^{-t} + \cos t)V_2(x_1,x_2).
\end{aligned}$$

This follows that

$$\begin{pmatrix} D^+V_1^{\Delta}(x_1,x_2) \\ D^+V_2^{\Delta}(x_1,x_2) \end{pmatrix} = \begin{pmatrix} e^{-t} & \cos t \\ e^{-t} & -\cos t \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}. \tag{24}$$

From system (24), we obtain the comparison result

$$g_1(t,u_1,u_2) = (e^{-t} - \cos t)u_1$$

$$g_2(t,u_1,u_2) = (e^{-t} + \cos t)u_2$$

$$u_1(t_i^+) = (1+c_1)u_1(t_i),\ \ u_2(t_i^+) = (1+c_2)u_2(t_i)$$

It is clear that $u_1 = u_2 = 0$ of (24) is integrally stable and hence by Theorem 3.2, $x = y = 0$ is also integrally stable.

5. Conclusion

In this work, we developed a comprehensive framework for analyzing the integral stability of impulsive dynamic equations on time scales using the comparison principle. By establishing a comparison theorem, we provided a rigorous foundation for comparing the behavior of the original system to that of a comparison system of lower order whose qualitative properties can easily be ascertained. This enabled us to derive an integral stability theorem, which offers explicit sufficient conditions for ensuring the cumulative effect of impulses and system dynamics remains bounded. Our approach, which integrates Lyapunov functions and scalar comparison equations, not only unifies and extends existing stability results but also provides a computationally tractable method for stability analysis.The applicability of the proposed framework was demonstrated through an illustrative example, highlighting its effectiveness in analyzing systems with mixed continuous-discrete dynamics and impulsive effects. This example underscores the versatility of our approach and its potential for addressing real-world problems in fields such as engineering, biology, and economics.

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