Asia Pac. J. Math. 2025 **12**:43

EXTENSIONS OF UNCERTAINTY MEASURES IN MULTI-CRITERIA DECISION-MAKING METHODS AND CLASSIFICATION

HANAN H. SAKR[1,2,*]

[1]Department of Management Information Systems, College of Business Administration in Hawtat Bani Tamim, Prince Sattam Bin Abdulaziz University, Saudi Arabia

[2]Mathematics Department, Faculty of Education, Ain Shams University, Cairo 11341, Egypt

[*]Corresponding author: h.sakr@psau.edu.sa; hananhassan@edu.asu.edu.eg

Received Feb. 9, 2025

Abstract. In multi-criteria decision-making, there are different approaches to determining the weights before using any techniques (e.g., analytic hierarchy process and fuzzy entropy methods). In this paper, we integrate machine learning techniques with multi-criteria decision-making to address two distinct challenges: the selection of optimal phase transition materials and the classification of diabetes data. For the multi-criteria decision-making problem, we utilize extended entropy functions—including entropy, fractional entropy, and Tsallis entropy—to calculate criteria weights based on non-probabilistic and probabilistic aspects. These weights guide the selection process using multi-objective optimization methods like ratio analysis and complex proportional assessment, aimed at identifying materials with superior thermal performance at minimal cost for latent heat thermal energy storage systems. Empirical results validate the effectiveness of our proposed strategies in phase transition material selection and highlight their advantages when compared to the technique for order performance by similarity to the ideal solution. Additionally, a classification problem for diabetes data is addressed using pattern recognition, demonstrating the synergy between machine learning and multi-criteria decision-making in tackling diverse decision-making challenges.

2020 Mathematics Subject Classification. 26-08; 58-08; 62C05; 62E99; 94-08; 94A17.

Key words and phrases. fuzzy set; entropy; multi-criteria decision-making; optimization; pattern recognition; ratio analysis.

1. Introduction

In order to give a framework for addressing a range of issues where an indefiniteness originating more from a type of inherent ambiguity than from a statistical fluctuation plays a vital role, Zadeh [27] created the fuzzy sets theory. Since users often select a fuzzy set's membership function subjectively,

DOI: 10.28924/APJM/12-43

different people may provide quite different membership functions for the same idea. Users are always presented with the knowledge grade of the items in the fuzzy sets via the forms of the membership functions. To put it another way, each membership function also helps users see the fuzziness of the associated fuzzy set. Therefore, in order to quantify the fuzziness of fuzzy collections, we need certain criteria. To express the fuzziness of a fuzzy set, De Luca and Termini [6] used the idea of entropy. This note proposes the introduction of a measure for a generalized set's degree of entropy or fuzziness. Since no probabilistic idea is required to define this number, its meaning differs significantly from that of classical entropy. A global measure of the situation of interest's indefiniteness is provided by this function. A notion from fuzzy set theory and information theory, fuzzy entropy quantifies the degree of fuzziness or uncertainty in a fuzzy set or system. Entropy is generally used to quantify information content, randomness, or uncertainty. Fuzzy entropy measures the degree of ambiguity or uncertainty around the items of a fuzzy set in the context of fuzzy systems. Entropy is unquestionably an appropriate measurement of a fuzzy collection. The entropy of fuzzy sets has been the subject of several works (see, for example, [7], [16] and [26]). Fuzzy entropy is used in fields like pattern recognition, image processing, time-series analysis, and machine learning. For example, it can help assess the complexity of physiological signals or determine the uncertainty in classification problems. If a fuzzy set represents the concept of "young age," the fuzzy entropy would indicate the uncertainty in determining who qualifies as young. If many people have membership values close to 0.5, the set has high entropy. If most values are close to 0 or 1, the entropy is lower, indicating clearer distinctions. Fuzzy entropy is a useful metric when assessing or quantifying uncertainty in systems where information is not binary or absolute but rather exists on a continuum. A fuzzy set allows elements to have degrees of membership between 0 and 1, rather than being simply in or out of the set. Each element in a fuzzy set has a membership value, often denoted by $S(x)$, where $0 \leq S(x) \leq 1$.

In classical systems, entropy (like Shannon entropy [21]) measures uncertainty based on a probability distribution, where each event has a certain probability. In fuzzy systems, entropy measures the fuzziness or the uncertainty of membership values. The closer the membership values are to 0.5, the higher the fuzziness (because they are neither close to full membership 1 nor full non-membership 0).

A universal set Y, where Y is real and finite, defines a fuzzy set S. Let $S(y) : y \to [0, 1]$ be the fuzzy set S's membership function for $y \in Y$. The following characteristics apply to the measure of fuzziness $En(S)$ [6]:

(1) If S is a crisp set in Y, then $En(S) = 0$.
(2) If $S(y) = \frac{1}{2}$, then $En(S)$ is a unique greatest for every $y \in Y$.
(3) If $S_2(y) \leq S_1(y)$ for $A_1 \leq \frac{1}{2}$ and $S_2(y) \geq S_1(y)$ for $A_1 \geq \frac{1}{2}$, then $En(S_1) \geq En(S_2)$ for two fuzzy sets S_1 and S_2.
(4) As S^c is the traditional complement of S, then $En(S^c) = 1 - En(S)$.

While specific formulas for fuzzy entropy vary based on the approach (e.g., De Luca and Termini's fuzzy entropy [6]), a common expression could look like this:

$$En(S) = -M\left[\sum_{y\in Y} S(y)\log(S(y)) + (1-S(y))\log(1-S(y))\right], \tag{1}$$

where M is constant.

Making decisions is a fundamental part of almost all human activity, whether it be carrying out everyday tasks or any kind of professional work. Making the best choices in industries may improve product quality while reducing various hazards. Numerous professionals and scholars have researched and developed various techniques to provide the best forecasts and crucial choices. Multi-criteria decision-making (MCDM) is one method for resolving multi-objective optimization issues; see Jaimes et al. [10]. When choosing from the given options, MCDM is utilized. It offers qualitative and quantitative evaluations to determine the relative relevance of the requirements in relation to the overall goal of the problems, as well as the value of each alternative with respect to each criterion (Ma et al. [15], Dalalah et al. [5]). There are several approaches for MCDM. The labels Multi-Attribute Decision-Making (MADM), Multi-Criteria Decision Assistance (MCDA), or Multi-Objective Decision-Making (MODM) have been substituted by researchers for MCDM throughout time, see Yalcin et al. [25]. The "Multiobjective Optimization on the basis of Ratio Analysis" (MOORA) approach is one of the many MCDM techniques that have been created and evaluated over the past 20 years for resolving non-trivial issues. It has garnered a lot of interest in a variety of application fields. Brauers and Zavadskas [3] first used the MOORA technique to rank the alternatives using dimensionless square root ratios for privatization issues in a transition economy. Additionally, Zavadskas et al. [29] presented the COPRAS (Complex Proportional Assessment) approach as an MCDM tool. It is also used to determine which option, out of all the options, is the best. To find a solution that has a ratio to the optimal solution, the COPRAS approach compares the proportional ratios of the best and worst ideal solutions.

In information theory, many extensions of uncertainty were presented. Fractional entropy (Ubriaco [23]) and Tsallis entropy (Tsallis [22]) are obtained, respectively, as

$$\sum_{s} pt(y)(-\ln pt(y))^{\theta_1}, \tag{2}$$

$$\frac{1}{\theta_2 - 1}\left[1 - \sum_{s}(pt(y))^{\theta_2}\right], \tag{3}$$

where $0 \le \theta_1 \le 1$, $1 \ne \theta_2 \ge 0$, $pt(.)$ is the probability mass function.

Researchers have recently shown a great deal of interest in the latent heat of phase transition material (PTM), which stores thermal energy. An MCDM challenge is choosing the best material for an engineering application among two or more alternative materials based on two or more characteristics or criteria (see, for example, [11], [13], and [14]).

This paper aims to apply MOORA and COPRAS methods of MCDM technique by obtaining the criteria weights based on non-probabilistic and probabilistic entropy, fractional entropy, and Tsallis entropy to suggest a methodical assessment technique for choosing the optimal PTM for a latent heat energy thermal unit storage. The rest of the paper is formed as: In Section 2, the steps of getting the weights of non-probabilistic and probabilistic entropy, fractional entropy, and Tsallis entropy are presented. In Section 3, the MOORA and COPRAS techniques are illustrated. A systematic assessment model for the selection process utilizing the provided methodologies, as well as a problem of classification for data of diabetes using recognition of patterns, are provided in Section 4.

2. Utilizing the entropy extensions, calculate the weights

In this section, we will use the entropy extensions in their probabilistic and non-probabilistic to calculate the weights. MCDM was used to choose the best solution from the set of solutions $S = \{S_1, S_2, ..., S_r\}$ based on the set of criteria $T = \{T_1, T_2, ..., T_t\}$. In which, each crirerion T_j is assigned with a weight w_j, $j = 1, 2, ..., t$, so that $\sum(w_j) = 1$. A MCDM problem was presented by the matrix $[\eta_{ij}]_{r\times t}$:

Weight (w)	w_1	w_2	$\cdots$	w_t
Alternatives/Criteria	T_1	T_2	$\cdots$	T_t
Δ_1	η_{11}	η_{12}	$\cdots$	η_{1t}
Δ_2	η_{21}	η_{22}	$\cdots$	η_{2t}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
Δ_r	η_{r1}	η_{r2}	$\cdots$	η_{rt}
Total Weighted Score	$\sum_{j=1}^{t} w_j\eta_{1j}$	$\sum_{j=1}^{t} w_j\eta_{2j}$	$\cdots$	$\sum_{j=1}^{t} w_j\eta_{rj}$

where $\eta_{ij} \in \mathbb{R}^+$, $i = 1, 2, ..., r$, $j = 1, 2, ..., t$. Using the provided value, the entropy extension techniques determine the weight for each criterion based on how important the criteria are both within and between. The weights acquired will be used for the COPRAS and MOORA MCDM techniques.

2.1. Non-probabilistic entropy extensions.

In this subsection, we will use the fuzzy entropy to obtain the weights; see [1], [6], and [9]. The process below should be used to calculate objective weight using entropy extensions.

Step 1: Determining the amounts of pt_{ij} with $i = 1, 2, ..., r$ and $j = 1, 2, ..., t$ utilizing

$$pt_{ij} = \frac{\eta_{ij}}{r + \sum_{i=1}^{r} \eta_{ij}}, \tag{4}$$

with noting that this choice makes $pt_{ij} \in [0,1]$, but prevent it from the probability attribute.

Step 2: Using $j = 1, 2, ..., t$, the measurement entropy en_j of each criteria T_j is computed under the same conditions in (1). Therefore, we can use the entropy or its extensions, which are given as:

(1) We can use the entropy function in (1) (With $M = \frac{1}{r}$) as

$$en_j = -\frac{1}{r}\left[\sum_{i=1}^{r} pt_{ij}\ln pt_{ij} + (1-pt_{ij})\ln(1-pt_{ij})\right]. \tag{5}$$

(2) For the fractional entropy function given in (2), we can obtain a familiar form that satisfies the same conditions in (1). Then, we have

$$\begin{aligned} en_j &= \frac{1}{r}\left[\sum_{i=1}^{r} pt_{ij}(-\ln pt_{ij})^{\theta_1} + (1-pt_{ij})(-\ln(1-pt_{ij}))^{\theta_1}\right] \\ &= \frac{1}{r}\left[\sum_{i=1}^{r} Fr(\theta_1)\right], \end{aligned} \tag{6}$$

where $Fr(\theta_1) = pt_{ij}(-\ln pt_{ij})^{\theta_1} + (1-pt_{ij})(-\ln(1-pt_{ij}))^{\theta_1}$.

(3) For the Tsallis function given in (3), we can obtain a familiar form that satisfies the same conditions in (1). Then, we have

$$\begin{aligned} en_j &= \frac{1}{r(\theta_2-1)}\left[\sum_{i=1}^{r} 1-(pt_{ij})^{\theta_2} - (1-pt_{ij})^{\theta_2}\right] \\ &= \frac{1}{r}\left[\sum_{i=1}^{r} Ts(\theta_2)\right], \end{aligned} \tag{7}$$

where $Ts(\theta_2) = \frac{1}{(\theta_2-1)}1-(pt_{ij})^{\theta_2} - (1-pt_{ij})^{\theta_2}$.

The average quantity of information resulting from fuzziness may be understood as those entropies. Figures 1 and 2 show the plot of $Fr(\theta_1)$ and $Ts(\theta_2)$ for 1000 simulated data set over $[0, 0.5]$ and $[0.5, 1]$ with different values of θ_1 and θ_2, and we can see the satisfaction of the conditions of the measure of fuzziness (The plots over $[0, 0.5]$ increases and over $[0.5, 1]$ decreases).

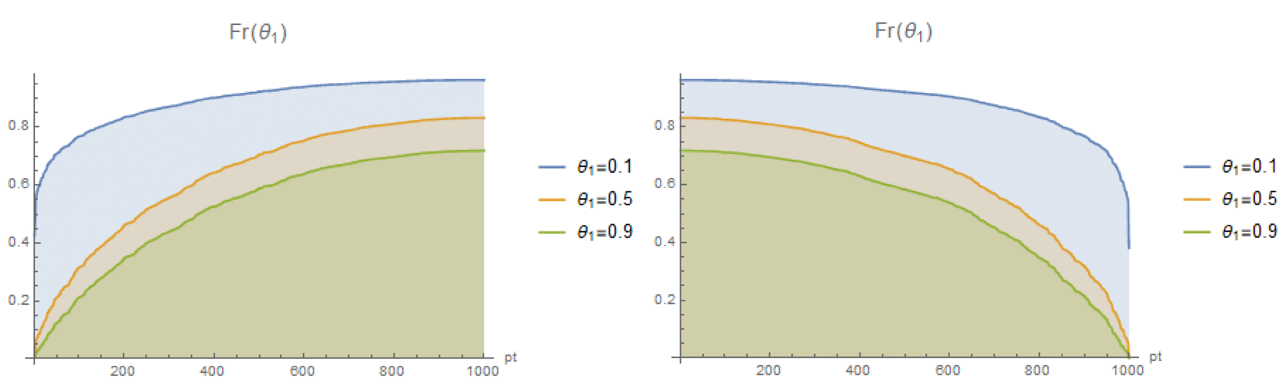

FIGURE 1. Plot of $Fr(\theta_1)$ for 1000 simulated data set over $[0, 0.5]$ (Left) and $[0.5, 1]$ (Right).

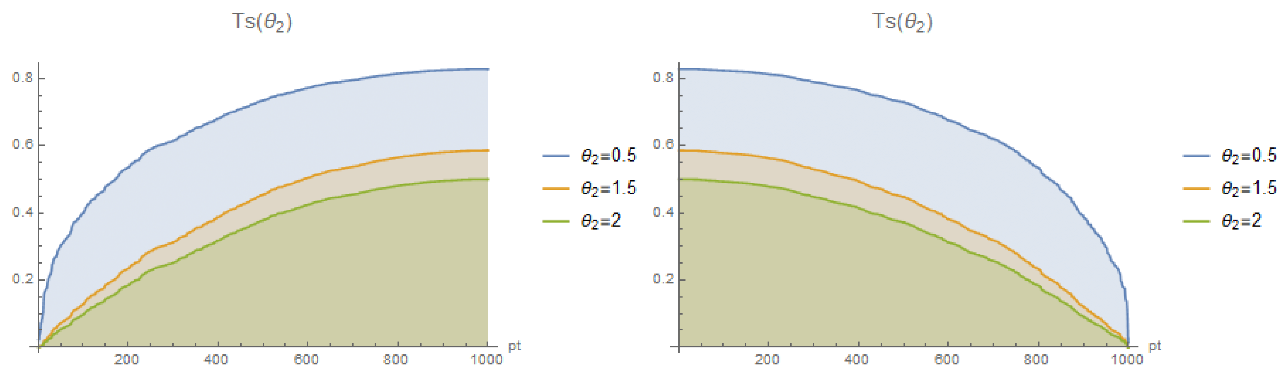

FIGURE 2. Plot of $Ts(\theta_2)$ for 1000 simulated data set over $[0, 0.5]$ (Left) and $[0.5, 1]$ (Right).

Step 3: Since the normalized maximal entropy is assigned to 1, then we can use the expression $1 - en_j$ to determine the weights. Then, using $j = 1, 2, ..., t$, the weight w_j for every criteria T_j is determined by

$$w_j = \frac{1 - en_j}{\sum_{j=1}^{t} 1 - en_j}. \tag{8}$$

2.2. **Probabilistic entropy extensions.** In this subsection, we will use the entropy to obtain the weights; see [8] and [1]. The process below should be used to calculate objective weight using entropy extensions.
Step 1: Determining the amounts of pt_{ij} with $i = 1, 2, ..., r$ and $j = 1, 2, ..., t$ utilizing the softmax function:

$$pt_{ij} = \frac{e^{\eta_{ij}}}{\sum_{i=1}^{r} e^{\eta_{ij}}}, \tag{9}$$

with noting that this choice make $pt_{ij} \in [0, 1]$, but keep the probability attribute.
Step 2: Since pt_{ij} can serve as a probability distribution, then we can use the original entropy function and their extensions to compute en_j of each criterion T_j, $j = 1, 2, ..., t$.

(1) Since the maximal entropy is achieved for the uniform distribution of the set of r alternatives (i.e., $pt_{ij} = \frac{1}{r}$). Then, for the Shannon entropy, we have

$$Max_{en} = -\sum_{i=1}^{r} \frac{1}{r} \ln \frac{1}{r} = \ln r.$$

The entropy function can then be normalized by comparing it to its maximum (to guarantee that each value falls between 0 and 1) as

$$en_j = \frac{-\sum_{i=1}^{r} pt_{ij} \ln pt_{ij}}{Max_{en}} = \frac{-1}{\ln r} \sum_{i=1}^{r} pt_{ij} \ln pt_{ij}. \tag{10}$$

(2) For the fractional entropy function given in (2), its maximum attained for the uniform distribution of the set of r alternatives (i.e., $pt_{ij} = \frac{1}{r}$). Then, we have

$$Max_{fen} = \sum_{i=1}^{r} \frac{1}{r}\left(-\ln\frac{1}{r}\right)^{\theta_1} = (\ln r)^{\theta_1}.$$

The fractional entropy function can then be normalized by comparing it to its maximum as

$$en_j = \frac{\sum_{i=1}^{r} pt_{ij}(-\ln pt_{ij})^{\theta_1}}{Max_{fen}} = \frac{1}{(\ln r)^{\theta_1}} \sum_{i=1}^{r} pt_{ij}(-\ln pt_{ij})^{\theta_1}, \tag{11}$$

where $0 \le \theta_1 \le 1$.

(3) For the Tsallis function given in (3), its maximum attained for the uniform distribution of the set of r alternatives (i.e., $pt_{ij} = \frac{1}{r}$). Then, we have

$$Max_{Ts} = \frac{1}{\theta_2 - 1}\left(1 - \sum_{j=1}^{r}\left(\frac{1}{r}\right)^{\theta_2}\right) = \frac{1}{\theta_2 - 1}\left(1 - (r)^{1-\theta_2}\right).$$

The Tsallis entropy function can then be normalized by comparing it to its maximum as

$$en_j = \frac{\frac{1}{\theta_2 - 1}\left(1 - \sum_{i=1}^{r}(pt_{ij})^{\theta_2}\right)}{Max_{Ts}} = \frac{\left(1 - \sum_{i=1}^{r}(pt_{ij})^{\theta_2}\right)}{\left(1 - (r)^{1-\theta_2}\right)}, \tag{12}$$

where $1 \neq \theta_2 \ge 0, t \neq 1$.

Step 3: Using $j = 1, 2, ..., t$, the weight w_j for every criteria T_j is determined by

$$w_j = \frac{1 - en_j}{\sum_{j=1}^{t} 1 - en_j}. \tag{13}$$

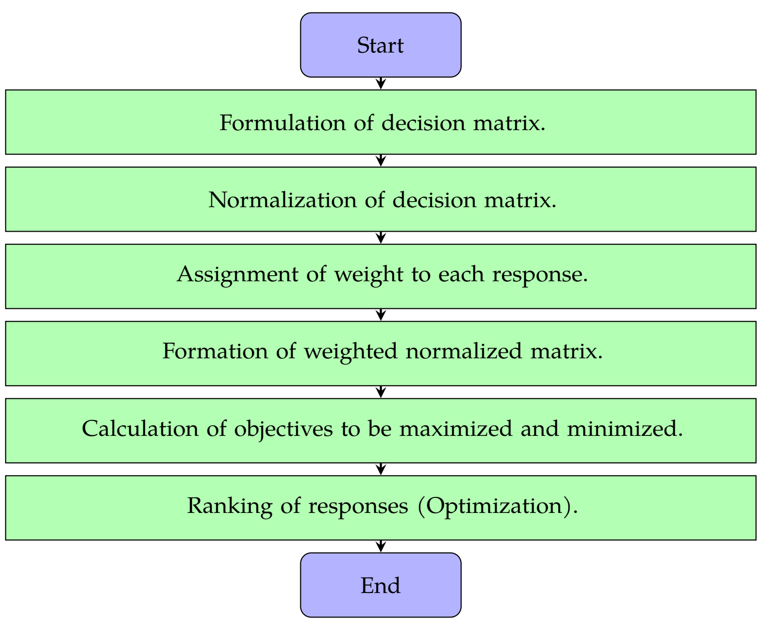

Figure 3. Steps in MOORA and COPRAS methods.

3. The MOORA and COPRAS techniques

In this section, we will show the steps to implement the techniques of MOORA and COPRAS procedures. The flow of the experiment is depicted in Figure 3.

3.1. **MOORA technique.** Brauers and Zavadskas originally presented the MOORA approach in 2004 [3]. In a production setting, complicated decision issues with competing objectives can be effectively resolved by using this multi-objective optimization approach. The following stages are part of the MOORA technique:

Step 1: Using $i = 1, 2, ..., r$ and $j = 1, 2, ..., t$, the standardized matrix is calculated by

$$Y = [y_{ij}]_{r\times t} \text{ where } y_{ij} = \frac{\eta_{ij}}{\sqrt{\sum_{i=1}^{r}(\eta_{ij})^2}}. \tag{14}$$

Note that The square root of the sum of each alternative performance index is the most robust option among the several options for the denominator of the normalization ratio, according to Brauers and Zavadskas [3].

Step 2: After standardizing with the weight $W = [W_{ij}]_{r\times t}$ for $i = 1, 2, ..., r$ and $j = 1, 2, ..., t$, the decision matrix is calculated by

$$W_{ij} = w_j \times y_{ij}, \tag{15}$$

where w_j is the weights obtained from (8) or (13).

Step 3: For every $i = 1, 2, ..., r$, let β be the set of all non-benefit criteria (objectives to be minimized) and α be the set of benefit criteria (objectives to be maximized). Consequently, we identify the two expressions:

$$B_i = \frac{1}{\mid \alpha \mid}\sum_{j\in\alpha} W_{ij}, \tag{16}$$

$$NB_i = \frac{1}{\mid \beta \mid}\sum_{j\in\beta} W_{ij}. \tag{17}$$

Step 4: Using $i = 1, 2, ..., r$ to determine each alternative's priority value

$$D_i = B_i - NB_i. \tag{18}$$

Step 5: Ordering the alternatives $\Delta_k > \Delta_i$ if $D_k > D_i$ for all $i, k = 1, 2, ..., r$. The best option may then be identified by looking for the greatest of all D_i assessment values:

$$\Delta^* = \left\{\Delta_i \mid \max_i D_i\right\}. \tag{19}$$

As well as, we can calculate the performance index value for each alternative

$$P_i = \frac{D_i}{D_{Max}}. \tag{20}$$

The finest option is the one that has 1 degree. The alternatives are ranked from big to small.

3.2. **COPRAS technique.** In 1994, the COPRAS approach was originally presented [29]. The following is the presentation of the steps.

Step 1: Using $i = 1, 2, ..., r$ and $j = 1, 2, ..., t$, the standardized matrix is calculated by

$$Y^* = \left[y^*_{ij}\right]_{r\times t} \text{ where } y^*_{ij} = \frac{\eta_{ij}}{\sum_{i=1}^{r}\eta_{ij}}, \tag{21}$$

which is the Voogd ratio (Voogd [24]).

Step 2: After standardizing with the weight $W^* = \left[W^*_{ij}\right]_{r\times t}$ for $i = 1, 2, ..., r$ and $j = 1, 2, ..., t$, the decision matrix is calculated by

$$W^*_{ij} = w_j \times y^*_{ij}, \tag{22}$$

where w_j is the weights obtained from (8) or (13).

Step 3: For every $i = 1, 2, ..., r$, let β be the set of all non-benefit criteria (objectives to be minimized) and α be the set of benefit criteria (objectives to be maximized). Consequently, we identify the two expressions:

$$B^*_i = \sum_{j\in\alpha} W^*_{ij}, \tag{23}$$

$$NB^*_i = \sum_{j\in\beta} W^*_{ij}. \tag{24}$$

Step 4: Using $i = 1, 2, ..., r$ to determine each alternative's priority value

$$D^*_i = B^*_i + \frac{\sum_{k=1}^{r} NB^*_k}{NB^*_i \sum_{k=1}^{r}\frac{1}{NB^*_k}}. \tag{25}$$

Step 5: Ordering the alternatives $\Delta_k > \Delta_i$ if $D^*_k > D^*_i$ for all $i, k = 1, 2, ..., r$. The best option may then be identified by looking for the greatest of all D^*_i assessment values:

$$\Delta^* = \left\{\Delta_i \mid \max_i D^*_i\right\}. \tag{26}$$

As well as, we can calculate the performance index value for each alternative

$$P^*_i = \frac{D^*_i}{D^*_{Max}}. \tag{27}$$

The finest option is the one that has 1 degree. The alternatives are ranked from big to small.

4. Application

This part includes a case study, details on the materials, their characteristics, and the relative weights of the assessment criteria. This case study was utilized by Rathod and Kanzaria [17] and Zakeri et al. [28] to determine the optimal PTM for solar energy storage. Nine substitutes, including calcium hexahydrate chloride, acid stearic, p116, RT 60, wax paraffin RT 30, n-docosane, n-octadecane, n-nonadecane, and n-eicosane, are thought to be PTM, see Table 1.

The cost's objective value, in this instance, is qualitative in nature; that is, a numeric value is not accessible. A fuzzy conversion scale with ranked value judgment is used to translate them into quantitative value. Using fuzzy logic, an eleven-point scale is taken into consideration to reflect the material selection requirements on a qualitative scale. Table 2 displays the same information. The intended data of the criterion with fuzzy score is displayed in Table 3, which is a decision matrix. We have used the equations from (4) to (13), to obtain non-probabilistic and probabilistic entropy and its extensions decision matrix and their weights in Table 4. Moreover, in Tables 5, 6 and 7, we have calculated the weighted normalized non-probabilistic and probabilistic entropy and its extensions decision matrix using the MOORA and COPRAS techniques, which described in the equations (14), (15) and (21), (22) respectively.

Rathod and Kanzaria [17] used the TOPSIS (Technique for Order Performance by Similarity to Ideal Solution) method in this case study, where the weights of the criterion are determined using an analytical hierarchy process approach. In the TOPSIS method, B_i^+ denotes the separation of each option from the positive-ideal solution, and NB_i^- denotes the separation from the negative ideal solution, where $i = 1, 2, ..., r$. The following is an expression for the relative proximity to the ideal solution:

$$R_i = \frac{NB_i^-}{B_i^+ + NB_i^-},$$

We compared the different procedures with the TOPSIS method. The analysis results and ranking based on entropy, fractional entropy ($\theta_1 = 0.1$), and Tsallis entropy ($\theta_2 = 1.5$) are shown in Tables 8, 9 and 10. Moreover, the comparison of ranks impacted by the weights assigned to the criterion by non-probabilistic and probabilistic entropy, fractional entropy, and Tsallis entropy in MOORA and COPRAS methods, besides TOPSIS method, are displayed in Figures 4 and 5. We can see that all the techniques show that after taking into account six criteria, it is determined that the material labeled as alternative Δ_1, or calcium hexahydrate chloride, is the first best option for the specified design application under the provided conditions. Moreover, the technique based on Tsallis entropy gives the same ranks for non-probabilistic and probabilistic MOORA or COPRAS procedures. Also, the ranks of both MOORA and COPRAS methods based on non-probabilistic entropy and Tsallis entropy are the same. Furthermore, we show the Spearman rank correlation coefficient between TOPSIS and the other methods in Figure 6. We can see that the probabilistic entropy MOORA and COPRAS are the highly correlated with TOPSIS method.

TABLE 1. Properties of PCMs for solar energy devices.

PTM	Material selection criteria					
	Heat Latent	**Density**	**Heat Specific (solid)**	**Heat Specific (liquid)**	**Conductivity Thermal**	**Cost**
	(T_1)	(T_2)	(T_3)	(T_4)	(T_5)	(T_6)
Calcium Hexahydrate Chloride (Δ_1)	169.98	1560.0	1.4600	2.1300	1.0900	Very low
Acid Stearic (Δ_2)	186.50	903.00	2.8300	2.3800	0.1800	Very high
p116 (Δ_3)	190.00	830.00	2.1000	2.1000	0.2100	Low
RT 60 (Δ_4)	214.40	850.00	0.9000	0.9000	0.2000	Very low
Wax Paraffin RT 30 (Δ_5)	206.00	789.00	1.8000	2.4000	0.1800	Low
n-Docosane (Δ_6)	194.60	785.00	1.9300	2.3800	0.2200	Low
n-Octadecane (Δ_7)	245.00	773.22	0.3767	2.2670	0.1400	Low
n-Nonadecane (Δ_8)	222.00	775.80	1.7189	1.9210	0.1420	High
n-Eicosane (Δ_9)	247.00	776.33	0.7467	2.3770	0.1380	Low

TABLE 2. Linguistic terms turn into fuzzy scores using an 11-point rating system. [18]

Term Linguistic	**Score Crisp**
Low Exceptionally	0.045
Low Extremely	0.135
Low Very	0.255
Low	0.335
Average Below	0.410
Average	0.500
Average Above	0.590
High	0.665
High Very	0.745
High Extremely	0.865
High Exceptionally	0.955

Table 3. Scores for the criteria's objective data (decision matrix).

	T_1	T_2	T_3	T_4	T_5	T_6
	Max	Max	Max	Max	Max	Min
Δ_1	169.98	1560.0	1.4600	2.1300	1.0900	0.255
Δ_2	186.50	903.00	2.8300	2.3800	0.1800	0.745
Δ_3	190.00	830.00	2.1000	2.1000	0.2100	0.335
Δ_4	214.40	850.00	0.9000	0.9000	0.2000	0.255
Δ_5	206.00	789.00	1.8000	2.4000	0.1800	0.335
Δ_6	194.60	785.00	1.9300	2.3800	0.2200	0.335
Δ_7	245.00	773.22	0.3767	2.2670	0.1400	0.335
Δ_8	222.00	775.80	1.7189	1.9210	0.1420	0.665
Δ_9	247.00	776.33	0.7467	2.3770	0.1380	0.335

Table 4. Non-probabilistic and probabilistic entropy and its extensions decision matrix and their weights w_j, $j = 1, 2, ..., t$.

	Non-probabilistic						Probabilistic					
	T_1	T_2	T_3	T_4	T_5	T_6	T_1	T_2	T_3	T_4	T_5	T_6
Δ_1	0.000428836	0.000202556	0.0417388	0.0423246	0.104364	0.0238576	0	1	0.0797958	0.106619	0.237549	0.0947367
Δ_2	0.000470514	0.000117249	0.0809046	0.0472923	0.0172344	0.0697016	0	0	0.314024	0.136902	0.0956191	0.15464
Δ_3	0.000479344	0.00010777	0.0600352	0.0417285	0.0201068	0.0313423	0	0	0.151331	0.103468	0.0985311	0.102627
Δ_4	0.000540902	0.000110367	0.0257294	0.0178836	0.0191494	0.0238576	0	0	0.0455801	0.0311641	0.0975507	0.0947367
Δ_5	0.00051971	0.000102446	0.0514587	0.0476897	0.0172344	0.0313423	0	0	0.112109	0.139668	0.0956191	0.102627
Δ_6	0.000490949	0.000101927	0.0551752	0.0472923	0.0210643	0.0313423	0	0	0.127673	0.136902	0.0995214	0.102627
Δ_7	0.000618102	0.000100397	0.0107692	0.0450469	0.0134046	0.0313423	0.119203	0	0.027009	0.122274	0.0918698	0.102627
Δ_8	0.000560076	0.000100732	0.0491402	0.0381716	0.0135961	0.0622168	0	0	0.103376	0.0865105	0.0920537	0.142751
Δ_9	0.000623147	0.000100801	0.0213468	0.0472326	0.0132131	0.0313423	0.880797	0	0.0391019	0.136492	0.0916863	0.102627
Entropy: w_j	0.185113	0.185732	0.153386	0.153986	0.164928	0.156854	0.416057	0.499032	0.0541038	0.0126834	0.0145073	0.00361651
Fractional: w_j, $\theta_1 = 0.1$	0.238553	0.268664	0.116832	0.115598	0.139414	0.120939	0.175887	0.803502	0.013387	0.0025728	0.00379848	0.000853029
w_j, $\theta_1 = 0.5$	0.195987	0.198655	0.146314	0.146684	0.161853	0.150506	0.377201	0.56213	0.0390371	0.00820388	0.0108717	0.0025557
w_j, $\theta_1 = 0.9$	0.185214	0.185815	0.153254	0.153915	0.164975	0.156826	0.412912	0.505669	0.0519716	0.0119208	0.0140629	0.00346388
Tsallis: w_j, $\theta_2 = 0.5$	0.207884	0.213104	0.139034	0.138743	0.157891	0.143344	0.186857	0.192315	0.157292	0.15478	0.154682	0.154074
w_j, $\theta_2 = 1.5$	0.177696	0.177911	0.158346	0.15907	0.166007	0.16097	0.159855	0.157024	0.169964	0.171033	0.170915	0.17121
w_j, $\theta_2 = 2$	0.174824	0.174967	0.160421	0.161072	0.166214	0.162502	0.157988	0.153456	0.171329	0.172423	0.172249	0.172556

Table 5. The weighed normalized non-probabilistic and probabilistic entropy decision matrix.

	Non-probabilistic (MOORA)						Probabilistic (MOORA)					
	T_1	T_2	T_3	T_4	T_5	T_6	T_1	T_2	T_3	T_4	T_5	T_6
Δ_1	0.0499789	0.104405	0.0439363	0.0510216	0.149591	0.0307819	0.112332	0.280519	0.0154976	0.00420249	0.0131583	0.000709724
Δ_2	0.0548362	0.0604344	0.0851643	0.05701	0.0247032	0.0899314	0.123249	0.162378	0.0300399	0.00469573	0.00217292	0.00207351
Δ_3	0.0558653	0.0555488	0.0631961	0.050303	0.0288204	0.040439	0.125562	0.149251	0.0222911	0.0041433	0.00253508	0.000932382
Δ_4	0.0630396	0.0568873	0.027084	0.0215584	0.027448	0.0307819	0.141687	0.152847	0.00955334	0.0017757	0.00241436	0.000709724
Δ_5	0.0605698	0.0528048	0.0541681	0.0574891	0.0247032	0.040439	0.136136	0.141878	0.0191067	0.00473519	0.00217292	0.000932382
Δ_6	0.0572179	0.0525371	0.0580802	0.05701	0.0301928	0.040439	0.128602	0.141159	0.0204866	0.00469573	0.0026558	0.000932382
Δ_7	0.0720369	0.0517487	0.0113362	0.0543032	0.0192136	0.040439	0.161909	0.13904	0.0039986	0.00447279	0.00169005	0.000932382
Δ_8	0.0652742	0.0519214	0.0517275	0.0460152	0.0194881	0.0802744	0.146709	0.139504	0.0182458	0.00379013	0.0017142	0.00185085
Δ_9	0.0726249	0.0519568	0.0224707	0.0569382	0.0189391	0.040439	0.163231	0.1396	0.00792609	0.00468982	0.00166591	0.000932382

	Non-probabilistic (COPRAS)						Probabilistic (COPRAS)					
	T_1	T_2	T_3	T_4	T_5	T_6	T_1	T_2	T_3	T_4	T_5	T_6
Δ_1	0.0167773	0.036027	0.0161549	0.0173954	0.0719087	0.011126	0.0377084	0.0967987	0.0056983	0.00143281	0.00632519	0.000256526
Δ_2	0.0184079	0.0208541	0.0313139	0.0194372	0.0118748	0.0325053	0.0413732	0.0560316	0.0110453	0.00160098	0.00104453	0.000749458
Δ_3	0.0187533	0.0191682	0.0232365	0.0171504	0.013854	0.0146165	0.0421497	0.0515019	0.00819619	0.00141263	0.00121861	0.000337005
Δ_4	0.0211616	0.0196301	0.00995849	0.00735019	0.0131943	0.011126	0.0475626	0.0527429	0.00351265	0.000605412	0.00116059	0.000256526
Δ_5	0.0203325	0.0182213	0.019917	0.0196005	0.0118748	0.0146165	0.0456991	0.0489578	0.0070253	0.00161443	0.00104453	0.000337005
Δ_6	0.0192073	0.018129	0.0213554	0.0194372	0.0145137	0.0146165	0.0431702	0.0487096	0.00753269	0.00160098	0.00127664	0.000337005
Δ_7	0.0241819	0.0178569	0.00416818	0.0185143	0.00923599	0.0146165	0.0543509	0.0479787	0.00147024	0.00152497	0.00081241	0.000337005
Δ_8	0.0219118	0.0179165	0.0190196	0.0156886	0.00936793	0.0290148	0.0492486	0.0481388	0.00670877	0.00129222	0.000824016	0.000668979
Δ_9	0.0243793	0.0179287	0.00826222	0.0194127	0.00910404	0.0146165	0.0547946	0.0481716	0.00291433	0.00159896	0.000800804	0.000337005

Table 6. The weighed normalized non-probabilistic and probabilistic fractional entropy decision matrix, $\theta_1 = 0.1$.

	Non-probabilistic (MOORA)						Probabilistic (MOORA)					
	T_1	T_2	T_3	T_4	T_5	T_6	T_1	T_2	T_3	T_4	T_5	T_6
Δ_1	0.0644072	0.151024	0.0334657	0.0383019	0.12645	0.0237338	0.0474879	0.45167	0.00383461	0.000852465	0.00344525	0.000167403
Δ_2	0.0706668	0.0874194	0.0648685	0.0427974	0.0208816	0.0693398	0.0521031	0.261448	0.00743285	0.00095252	0.000568941	0.000489079
Δ_3	0.071993	0.0803523	0.0481356	0.0377625	0.0243618	0.0311796	0.0530809	0.240312	0.00551554	0.000840459	0.000663764	0.000219921
Δ_4	0.0812384	0.0822885	0.0206295	0.0161839	0.0232017	0.0237338	0.0598976	0.246102	0.0023638	0.000360197	0.000632156	0.000167403
Δ_5	0.0780556	0.0763831	0.0412591	0.0431571	0.0208816	0.0311796	0.0575509	0.228441	0.00472761	0.000960524	0.000568941	0.000219921
Δ_6	0.073736	0.0759958	0.0442389	0.0427974	0.0255219	0.0311796	0.054366	0.227283	0.00506904	0.00095252	0.000695372	0.000219921
Δ_7	0.0928331	0.0748554	0.00863461	0.0407655	0.0162412	0.0311796	0.0684465	0.223872	0.000989383	0.000907295	0.000442509	0.000219921
Δ_8	0.0841181	0.0751052	0.0394001	0.0345437	0.0164732	0.0618939	0.0620209	0.224619	0.0045146	0.00076882	0.000448831	0.000436561
Δ_9	0.0935909	0.0751565	0.0171156	0.0427435	0.0160092	0.0311796	0.0690052	0.224773	0.00196117	0.000951319	0.000436188	0.000219921

	Non-probabilistic (COPRAS)						Probabilistic (COPRAS)					
	T_1	T_2	T_3	T_4	T_5	T_6	T_1	T_2	T_3	T_4	T_5	T_6
Δ_1	0.0216207	0.0521137	0.012305	0.0130588	0.0607844	0.00857844	0.0159411	0.155858	0.00140994	0.000290642	0.00165614	0.0000605069
Δ_2	0.023722	0.0301658	0.0238514	0.0145915	0.0100378	0.0250625	0.0174904	0.0902177	0.00273297	0.000324755	0.00027349	0.000176775
Δ_3	0.0241672	0.0277271	0.0176989	0.0128748	0.0117108	0.0112697	0.0178186	0.0829243	0.002028	0.000286548	0.000319072	0.0000794895
Δ_4	0.0272708	0.0283953	0.00758525	0.00551779	0.0111531	0.00857844	0.0201069	0.0849225	0.000869143	0.000122806	0.000303878	0.0000605069
Δ_5	0.0262023	0.0263575	0.0151705	0.0147141	0.0100378	0.0112697	0.0193191	0.0788281	0.00173829	0.000327484	0.00027349	0.0000794895
Δ_6	0.0247523	0.0262239	0.0162661	0.0145915	0.0122684	0.0112697	0.01825	0.0784284	0.00186383	0.000324755	0.000334266	0.0000794895
Δ_7	0.0311629	0.0258303	0.00317485	0.0138987	0.00780717	0.0112697	0.0229767	0.0772515	0.000363785	0.000309336	0.000212715	0.0000794895
Δ_8	0.0282374	0.0259165	0.014487	0.0117774	0.0079187	0.0223712	0.0208197	0.0775093	0.00165997	0.000262124	0.000215753	0.000157793
Δ_9	0.0314173	0.0259342	0.00629323	0.0145731	0.00769564	0.0112697	0.0231642	0.0775622	0.000721099	0.000324346	0.000209676	0.0000794895

Table 7. The weighed normalized non-probabilistic and probabilistic Tsallis entropy decision matrix, $\theta_2 = 1.5$.

	Non-probabilistic (MOORA)						Probabilistic (MOORA)					
	T_1	T_2	T_3	T_4	T_5	T_6	T_1	T_2	T_3	T_4	T_5	T_6
Δ_1	0.0479764	0.100009	0.0453569	0.052706	0.150569	0.0315897	0.0431593	0.0882673	0.0486848	0.0566697	0.155021	0.0335992
Δ_2	0.0526391	0.0578896	0.0879178	0.0588922	0.0248647	0.0922915	0.0473539	0.0510932	0.0943685	0.0633211	0.0255999	0.0981624
Δ_3	0.053627	0.0532097	0.0652394	0.0519637	0.0290088	0.0415002	0.0482426	0.0469627	0.0700261	0.0558716	0.0298665	0.0441402
Δ_4	0.0605138	0.0544919	0.0279597	0.0222702	0.0276274	0.0315897	0.0544379	0.0480943	0.0300112	0.023945	0.0284443	0.0335992
Δ_5	0.0581429	0.0505813	0.0559195	0.0593871	0.0248647	0.0415002	0.0523051	0.0446429	0.0600223	0.0638532	0.0255999	0.0441402
Δ_6	0.0549253	0.0503249	0.0599581	0.0588922	0.0303902	0.0415002	0.0494105	0.0444165	0.0643573	0.0633211	0.0312887	0.0441402
Δ_7	0.0691506	0.0495697	0.0117027	0.0560961	0.0193392	0.0415002	0.0622075	0.04375	0.0125613	0.0603147	0.019911	0.0441402
Δ_8	0.0626589	0.0497351	0.0534	0.0475344	0.0196155	0.082381	0.0563676	0.043896	0.057318	0.0511092	0.0201955	0.0876215
Δ_9	0.0697151	0.049769	0.0231973	0.058818	0.0190629	0.0415002	0.0627153	0.043926	0.0248993	0.0632413	0.0196266	0.0441402

	Non-probabilistic (COPRAS)						Probabilistic (COPRAS)					
	T_1	T_2	T_3	T_4	T_5	T_6	T_1	T_2	T_3	T_4	T_5	T_6
Δ_1	0.0161051	0.03451	0.0166772	0.0179698	0.0723789	0.0114179	0.0144881	0.0304584	0.0179008	0.0193212	0.0745189	0.0121443
Δ_2	0.0176703	0.019976	0.0323264	0.0200789	0.0119525	0.0333583	0.0158961	0.0176307	0.0346982	0.0215889	0.0123059	0.0354803
Δ_3	0.0180019	0.0183611	0.0239878	0.0177167	0.0139446	0.015	0.0161944	0.0162054	0.0257478	0.019049	0.0143569	0.0159542
Δ_4	0.0203138	0.0188035	0.0102805	0.00759285	0.0132805	0.0114179	0.0182742	0.0165959	0.0110348	0.00816387	0.0136732	0.0121443
Δ_5	0.0195179	0.0174541	0.0205609	0.0202476	0.0119525	0.015	0.0175582	0.0154049	0.0220695	0.0217703	0.0123059	0.0159542
Δ_6	0.0184378	0.0173656	0.0220459	0.0200789	0.0146086	0.015	0.0165865	0.0153268	0.0236634	0.0215889	0.0150405	0.0159542
Δ_7	0.023213	0.017105	0.00430295	0.0191256	0.00929637	0.015	0.0208823	0.0150968	0.00461866	0.0205639	0.00957124	0.0159542
Δ_8	0.0210338	0.0171621	0.0196346	0.0162065	0.00942918	0.0297762	0.0189219	0.0151472	0.0210752	0.0174253	0.00970797	0.0316703
Δ_9	0.0234025	0.0171738	0.00852936	0.0200536	0.00916356	0.015	0.0210528	0.0151575	0.00915517	0.0215617	0.00943451	0.0159542

Table 8. The analysis results and ranking based on entropy.

	Non-probabilistic (MOORA)				Probabilistic (MOORA)				The TOPSIS method			
	B_i	NB_i	P_i	Ranking	B_i	NB_i	P_i	Ranking	B_i^+	NB_i^-	R_i	Ranking
Δ_1	0.0797866	0.0307819	1	1	0.0851419	0.000709724	1	1	0.0617	0.1745	0.7390	1
Δ_2	0.070537	0.0899314	-0.395766	8	0.0806338	0.00207351	0.930455	2	0.1714	0.0328	0.1605	8
Δ_3	0.0634334	0.040439	0.469228	3	0.0759456	0.000932382	0.888443	8	0.1668	0.0299	0.1522	9
Δ_4	0.0490043	0.0307819	0.37185	5	0.0770694	0.000709724	0.90439	5	0.1656	0.0382	0.1875	5
Δ_5	0.0624337	0.040439	0.448829	4	0.0760072	0.000932382	0.889172	7	0.1697	0.0357	0.1740	6
Δ_6	0.0637595	0.040439	0.475883	2	0.0743997	0.000932382	0.870134	9	0.1651	0.0324	0.1639	7
Δ_7	0.0521596	0.040439	0.239174	7	0.0777777	0.000932382	0.910143	4	0.1752	0.0600	0.2552	3
Δ_8	0.0586066	0.0802744	-0.442157	9	0.077491	0.00185085	0.895869	6	0.1747	0.0437	0.2000	4
Δ_9	0.0557324	0.040439	0.312082	6	0.0792781	0.000932382	0.927913	3	0.1749	0.0618	0.2612	2

	Non-probabilistic (COPRAS)				Probabilistic (COPRAS)				The TOPSIS method			
	B_i	NB_i	P_i	Ranking	B_i	NB_i	P_i	Ranking	B_i^+	NB_i^-	R_i	Ranking
Δ_1	0.158263	0.011126	1	1	0.147963	0.000256526	1	1	0.0617	0.1745	0.7390	1
Δ_2	0.101888	0.0325053	0.604065	4	0.111096	0.000749458	0.749309	2	0.1714	0.0328	0.1605	8
Δ_3	0.0921624	0.0146165	0.605899	3	0.104479	0.000337005	0.706319	7	0.1656	0.0382	0.1875	5
Δ_4	0.0712947	0.011126	0.522878	7	0.105584	0.000256526	0.71465	6	0.1656	0.0382	0.1875	5
Δ_5	0.0899462	0.0146165	0.593741	5	0.104341	0.000337005	0.705391	8	0.1697	0.0357	0.1740	6
Δ_6	0.0926426	0.0146165	0.608534	2	0.10229	0.000337005	0.69158	9	0.1651	0.0324	0.1639	7
Δ_7	0.0739573	0.0146165	0.506024	9	0.106137	0.000337005	0.717484	4	0.1752	0.0600	0.2552	3
Δ_8	0.0839044	0.0290148	0.51083	8	0.106212	0.000668979	0.716582	5	0.1747	0.0437	0.2000	4
Δ_9	0.079087	0.0146165	0.534166	6	0.10828	0.000337005	0.731914	3	0.1749	0.0618	0.2612	2

TABLE 9. The analysis results and ranking based on fractional entropy, $\theta_1 = 0.1$.

	Non-probabilistic (MOORA)				Probabilistic (MOORA)				The TOPSIS method			
	B_i	NB_i	P_i	Ranking	B_i	NB_i	P_i	Ranking	B_i^+	NB_i^-	R_i	Ranking
Δ_1	0.0827296	0.0237338	1	1	0.101458	0.000167403	1	1	0.0617	0.1745	0.7390	1
Δ_2	0.0716584	0.0693398	0.0393018	8	0.0806263	0.000489079	0.79116	2	0.1714	0.0328	0.1605	8
Δ_3	0.0656513	0.0311796	0.584307	2	0.0751031	0.000219921	0.73929	4	0.1668	0.0299	0.1522	9
Δ_4	0.0558855	0.0237338	0.544984	5	0.0773391	0.000167403	0.761883	3	0.1656	0.0382	0.1875	5
Δ_5	0.0649341	0.0311796	0.57215	4	0.0730622	0.000219921	0.719141	7	0.1697	0.0357	0.1740	6
Δ_6	0.0655725	0.0311796	0.582972	3	0.0720915	0.000219921	0.709557	9	0.1651	0.0324	0.1639	7
Δ_7	0.0583324	0.0311796	0.46025	7	0.0736645	0.000219921	0.725086	6	0.1752	0.0600	0.2552	3
Δ_8	0.0624101	0.0618939	0.00874949	9	0.0730931	0.000436561	0.717307	8	0.1747	0.0437	0.2000	4
Δ_9	0.0611539	0.0311796	0.508075	6	0.0742816	0.000219921	0.73118	5	0.1749	0.0618	0.2612	2
	Non-probabilistic (COPRAS)				Probabilistic (COPRAS)				The TOPSIS method			
	B_i	NB_i	P_i	Ranking	B_i	NB_i	P_i	Ranking	B_i^+	NB_i^-	R_i	Ranking
Δ_1	0.159882	0.00857844	1	1	0.175156	0.0000605069	1	1	0.0617	0.1745	0.7390	1
Δ_2	0.102368	0.0250625	0.609345	2	0.111039	0.000176775	0.633729	2	0.1714	0.0328	0.1605	8
Δ_3	0.0941788	0.0112697	0.606917	3	0.103377	0.0000794895	0.590326	4	0.1668	0.0299	0.1522	9
Δ_4	0.0799221	0.00857844	0.551787	7	0.106325	0.0000605069	0.607326	3	0.1656	0.0382	0.1875	5
Δ_5	0.0924822	0.0112697	0.597406	5	0.100486	0.0000794895	0.573838	7	0.1697	0.0357	0.1740	6
Δ_6	0.0941022	0.0112697	0.606487	4	0.0992013	0.0000794895	0.566506	9	0.1651	0.0324	0.1639	7
Δ_7	0.081874	0.0112697	0.537943	8	0.101114	0.0000794895	0.577418	6	0.1752	0.0600	0.2552	3
Δ_8	0.0883371	0.0223712	0.534967	9	0.100467	0.000157793	0.573444	8	0.1747	0.0437	0.2000	4
Δ_9	0.0859135	0.0112697	0.560586	6	0.101982	0.0000794895	0.582367	5	0.1749	0.0618	0.2612	2

TABLE 10. The analysis results and ranking based on Tsallis entropy, $\theta_2 = 1.5$.

	Non-probabilistic (MOORA)				Probabilistic (MOORA)				The TOPSIS method			
	B_i	NB_i	P_i	Ranking	B_i	NB_i	P_i	Ranking	B_i^+	NB_i^-	R_i	Ranking
Δ_1	0.0793235	0.0315897	1	1	0.0783605	0.0335992	1	1	0.0617	0.1745	0.7390	1
Δ_2	0.0705509	0.0922915	-0.455456	8	0.0704341	0.0981624	-0.619471	8	0.1714	0.0328	0.1605	8
Δ_3	0.0632621	0.0415002	0.455902	3	0.0627424	0.0441402	0.415587	3	0.1668	0.0299	0.1522	9
Δ_4	0.0482158	0.0315897	0.348308	5	0.0462332	0.0335992	0.282252	5	0.1656	0.0382	0.1875	5
Δ_5	0.0622239	0.0415002	0.434151	4	0.0616059	0.0441402	0.390196	4	0.1697	0.0357	0.1740	6
Δ_6	0.0636227	0.0415002	0.463455	2	0.0631986	0.0441402	0.425779	2	0.1651	0.0324	0.1639	7
Δ_7	0.0514646	0.0415002	0.208748	7	0.0496861	0.0441402	0.123901	7	0.1752	0.0600	0.2552	3
Δ_8	0.058236	0.082381	-0.505827	9	0.0572216	0.0876215	-0.679157	9	0.1747	0.0437	0.2000	4
Δ_9	0.0551406	0.0415002	0.285759	6	0.0536021	0.0441402	0.211387	6	0.1749	0.0618	0.2612	2
	Non-probabilistic (COPRAS)				Probabilistic (COPRAS)				The TOPSIS method			
	B_i	NB_i	P_i	Ranking	B_i	NB_i	P_i	Ranking	B_i^+	NB_i^-	R_i	Ranking
Δ_1	0.157641	0.0114179	1	1	0.156687	0.0121443	1	1	0.0617	0.1745	0.7390	1
Δ_2	0.102004	0.0333583	0.605860	4	0.10212	0.0354803	0.607392	4	0.1714	0.0328	0.1605	8
Δ_3	0.092012	0.0150000	0.607680	3	0.0915535	0.0159542	0.609658	3	0.1668	0.0299	0.1522	9
Δ_4	0.0702711	0.0114179	0.520698	7	0.0677419	0.0121443	0.513692	7	0.1656	0.0382	0.1875	5
Δ_5	0.089733	0.0150000	0.595178	5	0.0891088	0.0159542	0.596291	5	0.1697	0.0357	0.1740	6
Δ_6	0.0925367	0.0150000	0.610559	2	0.0922062	0.0159542	0.613226	2	0.1651	0.0324	0.1639	7
Δ_7	0.0730429	0.0150000	0.503618	9	0.0707329	0.0159542	0.495821	9	0.1752	0.0600	0.2552	3
Δ_8	0.0834662	0.0297762	0.509730	8	0.0822776	0.0316703	0.504806	8	0.1747	0.0437	0.2000	4
Δ_9	0.0783228	0.0150000	0.532583	6	0.0763617	0.0159542	0.526596	6	0.1749	0.0618	0.2612	2

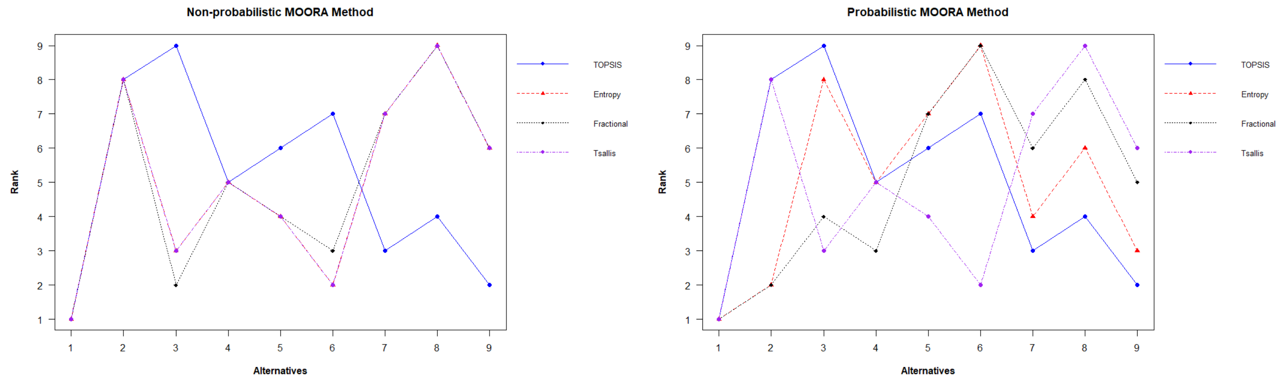

Figure 4. Comparative evaluation of ranks of MOORA and TOPSIS methods.

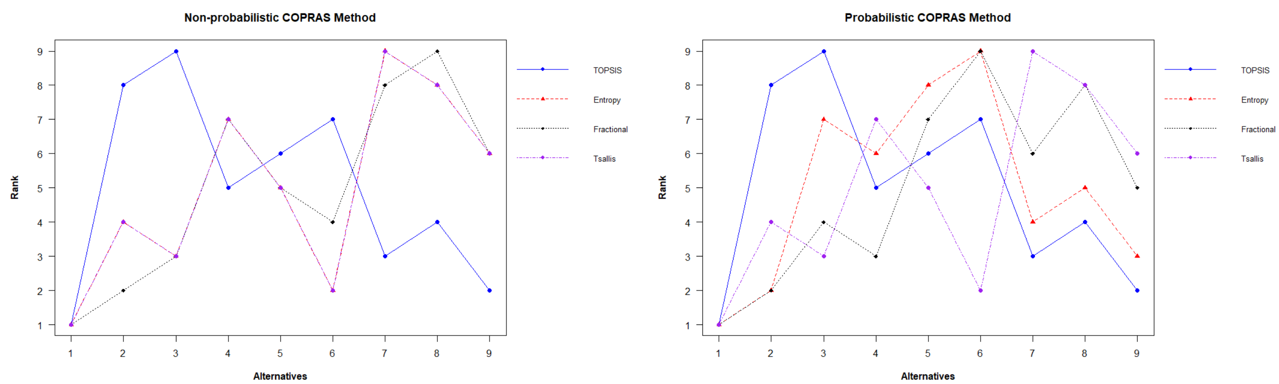

Figure 5. Comparative evaluation of ranks of COPRAS and TOPSIS methods.

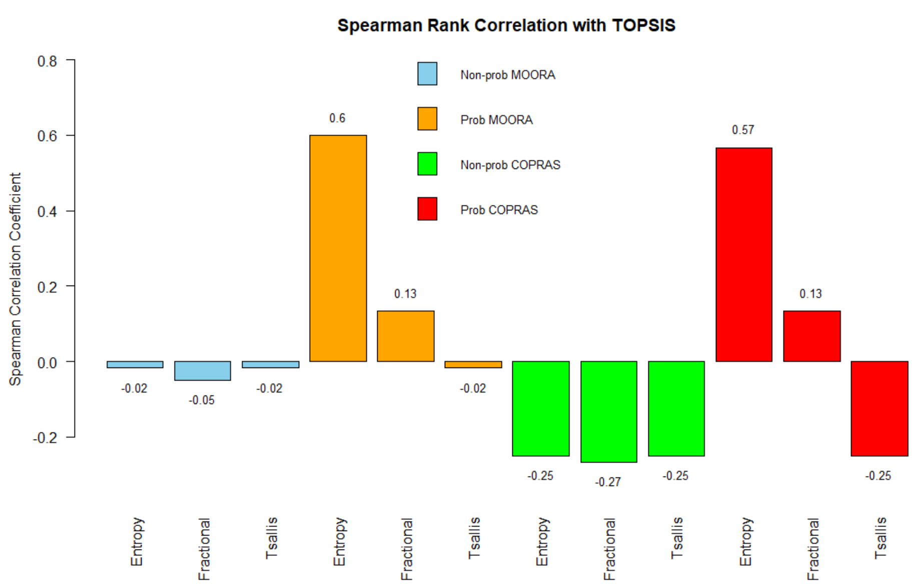

Figure 6. Correlation between TOPSIS and other methods.

4.1. **Principal component analysis technique.** In this subsection, we will discuss the Principal Component Analysis (PCA) technique under the case study that utilized by Rathod and Kanzaria [17]. By splitting a dataset into a collection of uncorrelated components while keeping the majority of the variance, PCA is a statistical method for reducing a dataset's dimensionality. It does this by converting the original variables into principle components (PCs), that serve as linear combinations of the basic variables and a fresh set of uncorrelated variables. The flow of this technique is depicted in Figure 7. Numerous scholarly works have addressed this process; for instance, [30] and [4].

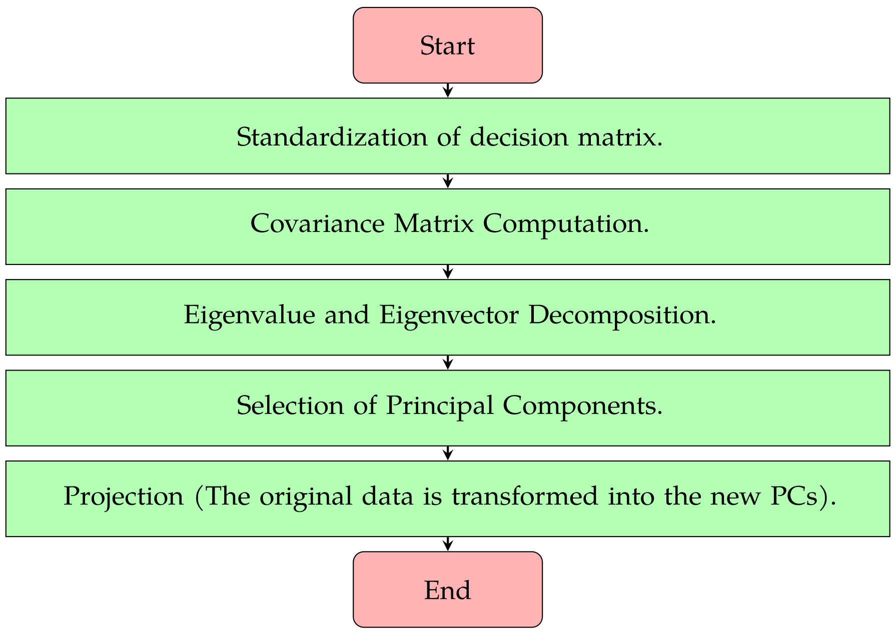

FIGURE 7. Key Steps in PCA.

The amount of variation that each PC captures is displayed in the PCA output summary. The proportion of variance indicates the percentage of the data's overall volatility that each PC can account for. Since the significant positive or negative values show criteria closely related with the PC, loadings indicate the influence of each criterion T_j, $j = 1, 2, ..., t$, to every PC. Each option Δ_i, $i = 1, 2, ..., r$, is represented by its PCA score in the new PC space. The coordinates of each option along the PCs (e.g., PC1, PC2) serve as its representation. Alternatives that are near one another share comparable traits.

Both loadings (criteria) and scores (alternatives) are displayed in the biplot. The following points provide a summary of the biplot's interpretation:

(1) The two most significant principal components are shown by the axes (PC1 on the x-axis and PC2 on the y-axis, for example).
(2) A point's value in the PC space becomes more extreme the further it is from the origin.

(3) Every red arrow denotes the criteria, and every blue point indicates an alternative.

(4) The criterion values of alternatives that are near to one another are comparable.

(5) The arrow's direction indicates how much the criteria affects the PCs; that is, if a criterion points in the same direction as a PC axis, it makes a significant contribution to that component. A criteria affects both components if it distinguishes between two PCs.

(6) Criteria that significantly impact the distribution of alternatives are indicated by longer arrows.

(7) The degree to which an alternative aligns with the criterion is shown by the projection of that alternative's point onto the criterion arrow.

Tables 11, 12 and 13 show the PCA results. In Table 11, the percentage of variance that each PC accounts for is displayed by the proportion of variance. For example, PC1 explains 43.89% of the variance, and PC2 explains 33.06%. Therefore, the first two PCs account for the majority (76.95%) of the variance. Thus, a two-dimensional representation captures most of the data's structure. Figure 8 shows the PCA bipolt which visualize the results in Tables 12 and 13. The best alternative for a maximization problem is Δ_7, which has the highest score in PC1 $(PC1 = 1.592)$, making it the best choice. In a minimization problem, the best alternative is Δ_1 with a score of $PC1 = -3.83$.

Table 11. Importance of Principal Components.

Component	PC1	PC2	PC3	PC4	PC5	PC6
Standard Deviation	1.6228	1.4083	0.9424	0.6923	0.0971	0.0791
Proportion of Variance	0.4389	0.3306	0.1480	0.0799	0.0016	0.0010
Cumulative Proportion	0.4389	0.7695	0.9175	0.9974	0.9990	1.0000

Table 12. PCA Loadings (Contribution of Variables to PCs).

Variable	PC1	PC2	PC3	PC4	PC5	PC6
T_1	0.5373	0.2381	0.2360	-0.3924	-0.6472	-0.1615
T_2	-0.5700	0.1909	0.0769	-0.3634	-0.3129	0.6346
T_3	-0.2395	-0.6232	-0.1946	0.2925	-0.6386	-0.1515
T_4	-0.0429	-0.3378	0.9270	0.1073	0.0980	0.0605
T_5	-0.5655	0.2330	0.1358	-0.2522	0.0028	-0.7375
T_6	0.0864	-0.5917	-0.1510	-0.7439	0.2564	-0.0256

TABLE 13. PCA Scores for Alternatives.

Alternative	PC1	PC2	PC3	PC4	PC5	PC6
Δ_1	-3.8296	1.2943	0.4290	-0.4689	-0.0040	-0.0129
Δ_2	-0.5511	-2.6748	-0.3072	-0.4936	0.0000	0.1228
Δ_3	-0.3153	-0.5133	-0.2930	0.9073	-0.0320	0.0155
Δ_4	0.6031	1.8135	-2.0270	0.1270	-0.0109	0.0384
Δ_5	0.2264	-0.3887	0.4839	0.7042	-0.0617	-0.0152
Δ_6	-0.1104	-0.5567	0.3261	0.8925	0.1107	-0.0800
Δ_7	1.5920	1.1822	0.9172	-0.3990	0.1661	0.0690
Δ_8	0.8818	-0.9766	-0.5830	-1.0049	0.0068	-0.1470
Δ_9	1.5031	0.8201	1.0540	-0.2646	-0.1750	0.0095

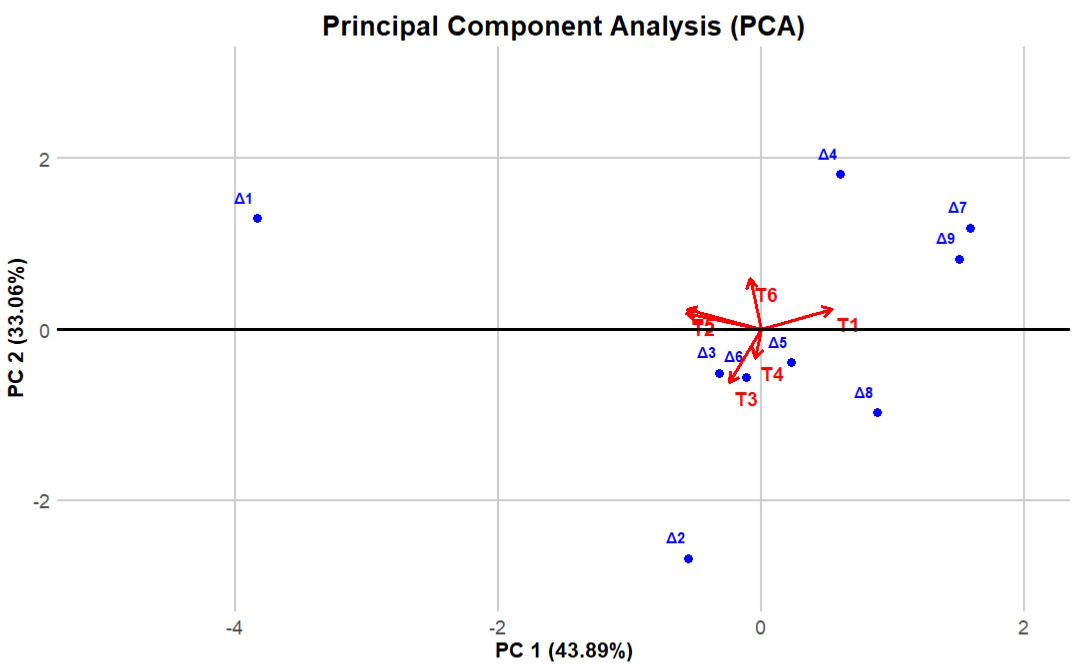

FIGURE 8. The PCA biplot.

4.2. **Classification with pattern recognition.** In this subsection, the probabilistic informational measurements (entropy, fractional entropy, and Tsallis entropy) in the issues of classification involving recognition of the pattern will be covered. Uncertainty measurements are a useful tool for classification challenges; for example, you may observe [2] and [20]. In 145 non-fat people, Reaven and Miller [19] examined the relationship involving insulin and blood chemistry markers of glucose tolerances. Moreover, the 145 observations are calculate into 76 in Normal group, 36 in Chemical diabetic group, and 33 in Overt diabetic group. Figure 9 shows the visualize of the correlation between the five variables.

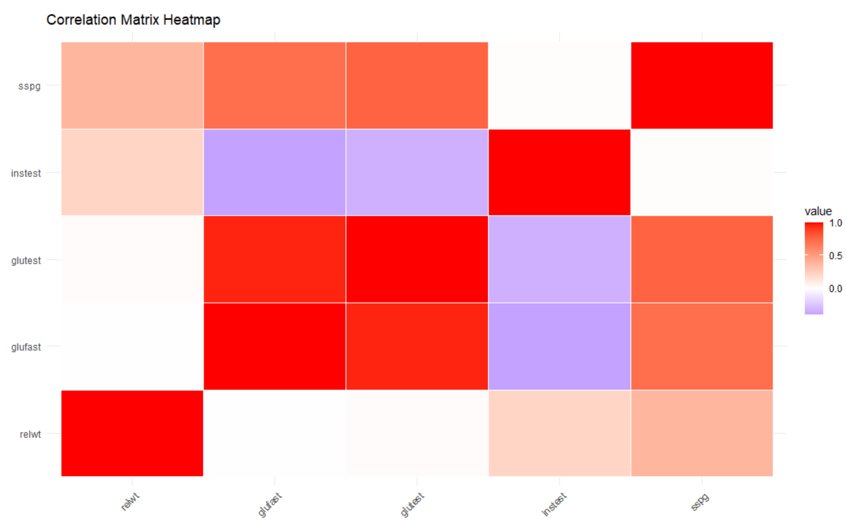

Figure 9. Correlation matrix heatmap between the variables.

To create an interval number model, we choose 30 samples for each classification of the data and then find a sample that has both the highest and lowest values, as shown in Table 14(a). Each potential entry in the dataset is shown as an anonymous test sample. Assuming the Normal group is the source of the chosen singleton data sample (1.04, 90, 356, 199, 108) (A singleton is a value that appears just once in a dataset).

Table 14. (a) The interval numbers of the given groups. (b) Probability distributions based on the interval numbers.

(i) Item	relwt	glufast	glutest	instest	sspg
Normal	[0.74, 1.2]	[74, 112]	[269, 418]	[81, 267]	[29, 273]
Chemical Diabetic	[0.83, 1.2]	[75, 114]	[413, 643]	[109, 748]	[60, 300]
Overt Diabetic	[0.74, 1.2]	[120, 353]	[538, 1520]	[10, 460]	[150, 458]
(ii) Item	relwt	glufast	glutest	instest	sspg
$\mathbb{P}$(Normal)	0.319438	0.497341	0.766072	0.609048	0.452434
$\mathbb{P}$(Chemical Diabetic)	0.361124	0.467	0.186743	0.122842	0.373827
$\mathbb{P}$(Overt Diabetic)	0.319438	0.0356591	0.0471852	0.268109	0.173739

Next, we use the method of Kang et al. [12], which is based on the similarity between interval numbers, to generate five different probability distributions. The two ranges $I_1 = [\alpha_1, \alpha_2]$ and $I_2 = [\beta_1, \beta_2]$ are taken into consideration. Next, we determine the separation across the ranges I_1 and I_2 by

$$\mho(I_1, I_2) = \left[\left(\frac{\alpha_1+\alpha_2}{2}\right) - \left(\frac{\beta_1+\beta_2}{2}\right)\right]^2 + \frac{1}{3}\left[\left(\frac{\alpha_1-\alpha_2}{2}\right)^2 + \left(\frac{\beta_1-\beta_2}{2}\right)^2\right]. \tag{28}$$

Additionally, their resemblance $Simy(X, Y)$ is described as

$$Simy(I_1, I_2) = \frac{1}{1+\psi\, \mho(I_1, I_2)}, \tag{29}$$

where ψ is the coefficient of supporting; setting ψ to 5 is an example of how to use it. We use the intervals specified in Table 14(a) for interval I_1; for interval I_2, we generate the provided probability distributions using individual values from the chosen sample. Each of the five assessed characteristics yields three similarity values, as indicated in Table 14(b). Our entropy measure, fractional entropy measure (with $\theta_1 = 0.5$), and Tsallis entropy measure (with $\theta_2 = 2$) are then assessed for these probability distributions, which are listed in Table 15(a). Next, in the case of glufast for the entropy (E), fractional entropy FE_{θ_1}, and Tsallis entropy measures TE_{θ_2}, the process yields, respectively,

$$\Omega_1(glufast) = \frac{e^{-E(glufast)}}{e^{-E(relwt)} + e^{-E(glufast)} + e^{-E(glutest)} + e^{-E(instest)} + e^{-E(sspg)}},$$

$$\Omega_2(glufast) = \frac{e^{-FE_{\theta_1}(glufast)}}{e^{-FE_{\theta_1}(relwt)} + e^{-FE_{\theta_1}(glufast)} + e^{-FE_{\theta_1}(glutest)} + e^{-FE_{\theta_1}(instest)} + e^{-FE_{\theta_1}(sspg)}},$$

$$\Omega_3(glufast) = \frac{e^{-TE_{\theta_2}(glufast)}}{e^{-TE_{\theta_2}(relwt)} + e^{-TE_{\theta_2}(glufast)} + e^{-TE_{\theta_2}(glutest)} + e^{-TE_{\theta_2}(instest)} + e^{-TE_{\theta_2}(sspg)}}.$$

TABLE 15. (a) Measures of entropy, fractional entropy, and Tsallis entropy (b) the weights Ω_i, $i = 1, 2, 3$, corresponding to the five attributes.

(a) Item	relwt	glufast	glutest	instest	sspg
Entropy	1.0969	0.821847	0.661589	0.912512	1.03074
Fractional: $\theta_1 = 0.1$	1.00916	0.974476	0.920597	0.975654	0.999032
$\theta_1 = 0.5$	1.04694	0.888264	0.719818	0.914366	1.00359
$\theta_1 = 0.9$	1.08666	0.830878	0.659426	0.906505	1.0232
Tsallis: $\theta_2 = 0.5$	1.46263	1.15487	1.04923	1.29739	1.40173
$\theta_2 = 1.5$	0.843805	0.646788	0.477087	0.685619	0.789395
$\theta_2 = 2$	0.665508	0.533291	0.376035	0.542087	0.625372
(b) Item	Ω_i(relwt)	Ω_i(glufast)	Ω_i(glutest)	Ω_i(instest)	Ω_i(sspg)
Entropy	0.163055	0.214677	0.251991	0.19607	0.174207
Fractional: $\theta_1 = 0.1$	0.193344	0.200167	0.211247	0.199931	0.195311
$\theta_1 = 0.5$	0.174064	0.203996	0.241423	0.198741	0.181776
$\theta_1 = 0.9$	0.164274	0.212154	0.251833	0.196702	0.175037
Tsallis: $\theta_2 = 0.5$	0.163539	0.222473	0.247261	0.192921	0.173806
$\theta_2 = 1.5$	0.169835	0.206819	0.245071	0.198942	0.179332
$\theta_2 = 2$	0.17701	0.202032	0.236437	0.200262	0.184259

The weighted values Ω_i, $i = 1, 2, 3$, corresponding to the five attributes, are outlined in Table 15(b). Therefore, the final probability distribution of the entropy measure is given by:

$$\mathbb{P}(\text{Normal}) = 0.55013,\ \mathbb{P}(\text{Chemical Diabetic}) = 0.295404,\ \mathbb{P}(\text{Overt Diabetic}) = 0.154466.$$

For fractional entropy measure with $\theta_1 = 0.1$,

$$\mathbb{P}(\text{Normal}) = 0.533276,\ \mathbb{P}(\text{Chemical Diabetic}) = 0.30032,\ \mathbb{P}(\text{Overt Diabetic}) = 0.30032,$$

and $\theta_1 = 0.5$,

$$\mathbb{P}(\text{Normal}) = 0.54529,\ \mathbb{P}(\text{Chemical Diabetic}) = 0.295576,\ \mathbb{P}(\text{Overt Diabetic}) = 0.159134,$$

and $\theta_1 = 0.9$,

$$\mathbb{P}(\text{Normal}) = 0.549904,\ \mathbb{P}(\text{Chemical Diabetic}) = 0.295024,\ \mathbb{P}(\text{Overt Diabetic}) = 0.155072.$$

For Tsallis entropy measure with $\theta_2 = 0.5$,

$$\mathbb{P}(\text{Normal}) = 0.548439,\ \mathbb{P}(\text{Chemical Diabetic}) = 0.297799,\ \mathbb{P}(\text{Overt Diabetic}) = 0.153762,$$

and $\theta_2 = 1.5$,

$$\mathbb{P}(\text{Normal}) = 0.547155,\ \mathbb{P}(\text{Chemical Diabetic}) = 0.295159,\ \mathbb{P}(\text{Overt Diabetic}) = 0.157686,$$

and $\theta_2 = 2$,

$$\mathbb{P}(\text{Normal}) = 0.543484,\ \mathbb{P}(\text{Chemical Diabetic}) = 0.295906,\ \mathbb{P}(\text{Overt Diabetic}) = 0.16061.$$

Subsequently, it was determined that the selected sample belongs to the Normal group, which has the highest probability. Consequently, in this instance, an accurate conclusion was reached. Using this approach, we looked at all 145 samples, including 33 in the overt diabetic group, 36 in the chemical diabetic group, and 76 in the normal group, for the entropy method, fractional entropy method across various values of θ_1, and Tsallis entropy method across various values of θ_2. Table 16 shows the recognition rates as the entropy approach gives 72.03%. Meanwhile, the fractional entropy and Tsallis entropy approaches give 59.23. In comparison to the other two approaches, the entropy strategy clearly performs somewhat better.

TABLE 16. The recognition rates of different approaches.

Approach	Normal	Chemical Diabetic	Overt Diabetic	Overall
Entropy approach	100%	55.5%	60.6%	72.03%
Fractional entropy approach	100%	44.4%	33.3%	59.23%
Tsallis entropy approach	100%	44.4%	33.3%	59.23%

Conclusion

Effective use of the latent heat thermal energy storage system results from the proper selection of the phase change material. Depending on their background or the material's accessibility, the majority of researchers employ PTM in the specific application. However, in a PTM selection challenge, a number of options need to be taken into account and assessed using a wide range of competing criteria. Therefore, to increase the quality of decisions, an efficient assessment method is necessary. In order to solve the PTM selection problem, the current study suggests two MCDM techniques: MOORA and COPRAS. Both approaches provide weights to the criteria employed in PTM selection using non-probabilistic and probabilistic entropy, fractional entropy, and Tsallis entropy. As can be seen, all of the methods demonstrate that, after considering six criteria, calcium hexahydrate chloride, also known as alternative Δ_1, is the first best choice for the given design application under the given conditions. This is also the outcome of the TOPSIS technique, which was employed by Rathod and Kanzaria [17] for the same case study. Furthermore, the Tsallis entropy-based approach yields the same rankings for probabilistic and non-probabilistic MOORA or COPRAS processes. Additionally, the rankings of the COPRAS and MOORA techniques based on Tsallis entropy and non-probabilistic entropy are identical. Finally, we used the PCA technique to analyze the data in the case study, and it indicates that the alternatives with a high $PC1$ score likely perform well on the maximization criteria, while alternatives with a low $PC1$ score perform better on the minimization criterion. Additionally, a classification challenge involving pattern recognition for diabetes data using the entropy method, fractional entropy method across various values of θ_1, and Tsallis entropy method across various values of θ_2 is presented. We can see that the entropy method performs better than the other two methods.

Authors' Contributions. All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgment. This study is supported via funding from Prince sattam bin Abdulaziz University project number (PSAU/2025/R/1446).

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