Asia Pac. J. Math. 2025 **12**:45

QUADRATIC APPROXIMATIONS OF AN EXTENDED SEIR MODEL IN COVID-19: A CASE STUDY

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Received Mar. 3, 2025

Abstract. Nonlinear ordinary differential equations have been increasingly gaining interest in modeling complex dynamical systems. While existing models focus on the linear approach in approximating the exact solution, a more complex model would require deeper insights into system behavior through approximating using a non-linear approach. This paper explores the recently proposed quadratic approximations using the second-order terms from the Taylor series expansion to the extended SEIR COVID-19 model. The results revealed the existence of an approximate solution to the model and we have demonstrated that the system solution converges to the disease-free equilibrium. This study suggests that of the nonlinear models, quadratic approximations offer an interesting result representing the dynamics of a model and encourage further studies into its applications to approximate the solutions of other models represented by a system of nonlinear ordinary differential equations.

2020 Mathematics Subject Classification. 34A34; 92D30.

Key words and phrases. nonlinear ordinary differential equations; quadratic approximations; COVID-19 model.

1. Introduction

Nonlinear ordinary differential equations are important in modeling complex systems where the interactions and dependencies exhibit nonlinear behavior. There are several studies in various disciplines introducing models represented by systems of nonlinear ordinary differential equations.

Researchers have explored different social issues using mathematical models providing insights into their underlying dynamics. Some studies investigated the spread of moral corruption among adolescents [1], modeling of divorce epidemic [2] and teenage pregnancy [3]. Mathematical modeling has also been applied in tourism sustainability [4]. These studies demonstrate how mathematical modeling helps in understanding address social issues. Numerous researchers have also incorporated

DOI: 10.28924/APJM/12-45

nonlinear ordinary differential equations to model the spread of infectious diseases providing insights into the transmission patterns. Some studies examined the dynamics of Ebola Zaire Virus [5] while others investigated the spread of HIV/AIDS [6], transmission of tuberculosis [7] and the most recent is the transmission of the COVID-19 pandemic.

Several mathematical models of COVID-19 have been created, of which a susceptible-infected-removed (SIR) model attempted to evaluate how well a particular approach a modeling technique would work for the analysis of the COVID-19 pandemic, its disease spread, population changes over time, and important parameters that govern the spread of the infection in different societies [8]. A model of COVID-19 virus propagation in island countries was proposed with special emphasis on the Philippines using movement in the region and local transmission [9]. Another mathematical model aimed at understanding the dynamics of COVID-19 in the Philippines considering the impact of contact tracing and vaccination during the era of severe pandemic effects such as quarantines, testing, and tracing [10].

These studies involve systems of nonlinear ordinary differential equations. The nonlinear terms make it hard to find the exact solution of the system. Linear approximations are a common approach for solving nonlinear models such as Jacobian matrix [11] and perturbation techniques [12] but they often fail to capture the full complexity of these systems. On the other hand, nonlinear approximations provide a more precise approximation compared to other estimating methods, which is why it is an appealing research area. Some studies have looked into alternate techniques such as the variational iteration method [13] and the Adomian decomposition method [14]. Another study explored quadratic approximations using the second-order terms from the Taylor series expansion which demonstrate the existence of nonzero real solutions and offer interesting results in approximating solutions of a system of nonlinear ordinary differential equations [15].

This paper focuses on applying quadratic approximations using the second-order terms from the Taylor series expansion to solve an extended SIR model represented by a system of nonlinear ordinary differential equations. We aim to investigate the consistency of this method to find the approximate solution of a real-world epidemiological model.

The structure of this paper is organized as follows: Section 2 presents the extended SIR COVID-19 model and approximation of this system using quadratic approximations. Section 3 concludes the paper with a summary of our findings and a discussion of their wider implications.

2. The Extended $SEIR$ model and the Quadratic Approximations

We use the COVID-19 model, in which the impact of contact tracing, testing, and vaccination is considered [10]. This model is an extension of the SEIR model with nine compartments: Susceptible individuals S, Exposed individuals E, Exposed and tested individuals E_d, Exposed and contact traced individuals E_c, Critical and severe infected individuals I_c, Moderate and mild infected individuals I_m, Asymptomatic infected individuals I_a, Recovered individuals R, and Vaccinated individuals V. The $SE(E_dE_c)I(I_cI_mI_a)RV$ model is given by

$$\begin{cases} \dfrac{dS}{dt} &= \theta N - (\alpha_s E + \nu + \mu) S \\ \dfrac{dE}{dt} &= (\alpha_s S + \alpha_v V) E - (\beta_d + \beta_c + \mu) E \\ \dfrac{dE_d}{dt} &= \beta_d E + \gamma E_c - (\sigma_c + \sigma_m + \sigma_a + \mu) E_d \\ \dfrac{dE_c}{dt} &= \beta_c E - (\gamma + \mu) E_c \\ \dfrac{dI_c}{dt} &= \sigma_c E_d + \phi I_m - (\rho_c + \delta + \mu) I_c \\ \dfrac{dI_m}{dt} &= \sigma_m E_d - (\rho_m + \phi + \mu) I_m \\ \dfrac{dI_a}{dt} &= \sigma_a E_d - (\rho_a + \mu) I_a \\ \dfrac{dR}{dt} &= \rho_c I_c + \rho_m I_m + \rho_a I_a - \mu R \\ \dfrac{dV}{dt} &= \nu S - (\alpha_v E + \mu) V \end{cases} \tag{1}$$

with the positive initial conditions:

$$S(0) = S_0 \geq 0, E(0) = E_0 \geq 0, E_d(0) = E_{d0} \geq 0, E_c(0) = E_{c0} \geq 0, I_c(0) = I_{c0} \geq 0,$$
$$I_m(0) = I_{m0} \geq 0, I_a(0) = I_{a0} \geq 0, R(0) = R_0 \geq 0, V(0) = V_0 \geq 0.$$

The parameters of the model are the following: the birth rate θ, vaccination rate ν of S, transmission rate α_s from S to E from contact with E, transmission rate α_v from V to E from contact with E, testing rate β_d of E, contact tracing rate β_c of E, testing rate γ of E_c, incubation rate σ_c of E_d to I_c, incubation rate σ_m of E_d to I_m, incubation rate σ_a of E_d to I_a, transfer rate ϕ from I_m to I_c, recovery rate ρ_c of E_c, recovery rate ρ_m of E_m, recovery rate ρ_a of E_a, induced death rate δ by COVID-19 and natural death rate μ.

Nonlinear approximations of nonlinear ordinary differential equations became an interesting study. A study explored quadratic approximations, focusing on the inclusion of the second-order terms from the Taylor series expansion [15]. This approximation demonstrated the existence of nonzero real solutions for systems of ordinary differential equations. It was applied to some systems of nonlinear ordinary differential equations involving two or three variables which demonstrated a better approximation of solutions of the systems. Hence, it becomes more interesting to approximate the solution of a system of nonlinear differential equations with more than three variables.

We consider the following theorem which can be used in the following proofs of the succeeding section.

Theorem 2.1. *The system of differential equations involving quadratic terms*

$$\begin{cases} \dfrac{dx_1}{dt} &= \sum_{j=1}^n a_{1j} x_j^2 \\ \dfrac{dx_2}{dt} &= \sum_{j=1}^n a_{2j} x_j^2 \\ &\vdots \\ \dfrac{dx_n}{dt} &= \sum_{j=1}^n a_{nj} x_j^2 \end{cases}$$

has a nonzero real solution whenever the system of equations

$$\begin{cases} \sum_{j=1}^n a_{1j} y_j^2 + y_1 &= 0 \\ \sum_{j=1}^n a_{2j} y_j^2 + y_2 &= 0 \\ &\vdots \\ \sum_{j=1}^n a_{nj} y_j^2 + y_n &= 0 \end{cases}$$

has a nonzero real solution.

Here, $(x_1, x_2, \ldots, x_n) = \left(\dfrac{\lambda_1}{t}, \dfrac{\lambda_2}{t}, \ldots, \dfrac{\lambda_n}{t}\right)$ is a nonzero real solution to the system of differential equations involving quadratic terms since $(y_1, y_2, \ldots, y_n) = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is nonzero.

3. Approximation of $SE(E_dE_c)I(I_cI_mI_a)RV$ model

We focus on approximating the solution of the formulated extended SIR model which is the $SE(E_dE_c)I(I_cI_mI_a)RV$ model using the quadratic approximations. Then we will show that there exists a solution of the approximate system converges to the actual disease-free equilibrium point.

Theorem 3.1. *The system (1) has an approximate solution*

$$\begin{cases} S^+ &= \left(\dfrac{\sqrt{e_1 + \alpha_s \theta N \lambda_2^2} - \sqrt{e_1}}{(\nu+\mu)t} + \sqrt{e_1}\right)^2 - e_1 + \dfrac{\theta N}{\nu+\mu} \\ E^+ &= \left(\dfrac{\mu(\nu+\mu)\sqrt{e_2}}{(\mu\alpha_s\theta N + \alpha_v\theta\nu N - \mu(\nu+\mu)(\beta_d+\beta_c+\mu))t} + \sqrt{e_2}\right)^2 - e_2 \\ E_d^+ &= \left(\dfrac{\sqrt{e_3 - (\sigma_c+\sigma_m+\sigma_a+\mu)(\beta_d\lambda_2^2 + \gamma\lambda_4^2)} - \sqrt{e_3}}{(\sigma_c+\sigma_m+\sigma_a+\mu)t} + \sqrt{e_3}\right)^2 - e_3 \\ E_c^+ &= \left(\dfrac{\sqrt{e_4 - \beta_c(\gamma+\mu)\lambda_2^2} - \sqrt{e_4}}{(\gamma+\mu)t} + \sqrt{e_4}\right)^2 - e_4 \\ I_c^+ &= \left(\dfrac{\sqrt{e_5 - (\rho_c+\delta+\mu)(\sigma_c\lambda_2^2 + \phi\lambda_6^2)} - \sqrt{e_5}}{(\rho_c+\delta+\mu)t} + \sqrt{e_5}\right)^2 - e_5 \\ I_m^+ &= \left(\dfrac{\sqrt{e_6 - \sigma_m(\rho_m+\phi+\mu)\lambda_3^2} - \sqrt{e_6}}{(\rho_m+\phi+\mu)t} + \sqrt{e_6}\right)^2 - e_6 \\ I_a^+ &= \left(\dfrac{\sqrt{e_7 - \sigma_a(\rho_a+\mu)\lambda_3^2} - \sqrt{e_7}}{(\rho_a+\mu)t} + \sqrt{e_7}\right)^2 - e_7 \\ R^+ &= \left(\dfrac{\sqrt{e_8 - \mu(\rho_c\lambda_5^2 + \rho_m\lambda_6^2 + \rho_a\lambda_7^2)} - \sqrt{e_8}}{\mu t} + \sqrt{e_8}\right)^2 - e_8 \\ V^+ &= \left(\dfrac{\sqrt{e_9 - (\nu\mu(\nu+\mu)\lambda_1^2 - \alpha_v\theta\nu N\lambda_2^2)} - \sqrt{e_9}}{\mu t} + \sqrt{e_9}\right)^2 - e_9 + \dfrac{\theta\nu N}{\mu(\nu+\mu)} \end{cases}$$

where

$$\begin{aligned}
e_1 &> \frac{\theta N}{\nu+\mu}, \\
e_2 &> 0, \\
e_3 &> (\sigma_c+\sigma_m+\sigma_a+\mu)(\beta_d\lambda_2^2+\gamma\lambda_4^2), \\
e_4 &> \beta_c(\gamma+\mu)\lambda_2^2, \\
e_5 &> (\rho_c+\delta+\mu)(\sigma_c\lambda_2^2+\phi\lambda_6^2), \\
e_6 &> \sigma_m(\rho_m+\phi+\mu)\lambda_3^2, \\
e_7 &> \sigma_a(\rho_a+\mu)\lambda_3^2, \\
e_8 &> \mu(\rho_c\lambda_5^2+\rho_m\lambda_6^2+\rho_a\lambda_7^2), \\
e_9 &> \frac{\theta\nu N}{\mu(\nu+\mu)},
\end{aligned}$$

and

$$\begin{aligned}
\lambda_1 &= \frac{\sqrt{e_1+\alpha_s\theta N\lambda_2^2}-\sqrt{e_1}}{\nu+\mu} \\
\lambda_2 &= \frac{\mu(\nu+\mu)\sqrt{e_2}}{\mu\alpha_s\theta N+\alpha_v\theta\nu N-\mu(\nu+\mu)(\beta_d+\beta_c+\mu)} \\
\lambda_3 &= \frac{\sqrt{e_3-(\sigma_c+\sigma_m+\sigma_a+\mu)(\beta_d\lambda_2^2+\gamma\lambda_4^2)}-\sqrt{e_3}}{\sigma_c+\sigma_m+\sigma_a+\mu} \\
\lambda_4 &= \frac{\sqrt{e_4-\beta_c(\gamma+\mu)\lambda_2^2}-\sqrt{e_4}}{\gamma+\mu} \\
\lambda_5 &= \frac{\sqrt{e_5-(\rho_c+\delta+\mu)(\sigma_c\lambda_2^2+\phi\lambda_6^2)}-\sqrt{e_6}}{\rho_c+\delta+\mu} \\
\lambda_6 &= \frac{\sqrt{e_6-\sigma_m(\rho_m+\phi+\mu)\lambda_3^2}-\sqrt{e_6}}{\rho_m+\phi+\mu} \\
\lambda_7 &= \frac{\sqrt{e_7-\sigma_a(\rho_a+\mu)\lambda_3^2}-\sqrt{e_7}}{\rho_a+\mu} \\
\lambda_8 &= \frac{\sqrt{e_8-\mu(\rho_c\lambda_5^2+\rho_m\lambda_6^2+\rho_a\lambda_7^2)}-\sqrt{e_8}}{\mu} \\
\lambda_9 &= \frac{\sqrt{e_9-(\nu\mu(\nu+\mu)\lambda_1^2-\alpha_v\theta\nu N\lambda_2^2)}-\sqrt{e_9}}{\mu}
\end{aligned}$$

Proof. We approximate the solution of the model using the Taylor series expansion and focusing on its quadratic terms.

Notice that every second partial derivative with respect to each variable is zero. So we proceed to the changing of variables involving arbitrary constants.

We let the following:

$$\begin{aligned}
z_1 &= \sqrt{S-\frac{\theta N}{\nu+\mu}+e_1}, \quad z_2=\sqrt{E+e_2}, \quad z_3=\sqrt{E_d+e_3} \\
z_4 &= \sqrt{E_c+e_4}, \qquad z_5=\sqrt{I_c+e_5}, \quad z_6=\sqrt{I_m+e_6} \\
z_7 &= \sqrt{I_a+e_7}, \qquad z_8=\sqrt{R+e_8}, \quad z_9=\sqrt{V-\frac{\theta\nu N}{\mu(\nu+\mu)}+e_9}
\end{aligned}$$

We convert the system in terms of z_i for $i = 1, 2, \ldots, 9$. Then we get

$$
\left\{
\begin{array}{rcl}
\dfrac{dz_1}{dt} & = & \dfrac{1}{2z_1}\left[\theta N - (\alpha_s(z_2^2 - e_2) + \nu + \mu)\left(z_1^2 + \dfrac{\theta N}{\nu + \mu} - e_1\right)\right] \\
\dfrac{dz_2}{dt} & = & \dfrac{1}{2z_2}\left[\left(\alpha_s\left(z_1^2 + \dfrac{\theta N}{\nu + \mu} - e_1\right) + \alpha_v\left(z_9^2 + \dfrac{\theta \nu N}{\mu(\nu + \mu)} - e_9\right) - (\beta_d + \beta_c + \mu)\right)(z_2^2 - e_2)\right] \\
\dfrac{dz_3}{dt} & = & \dfrac{1}{2z_3}\left[\beta_d(z_2^2 - e_2) + \gamma(z_4^2 - e_4) - (\sigma_c + \sigma_m + \sigma_a + \mu)(z_3^2 - e_3)\right] \\
\dfrac{dz_4}{dt} & = & \dfrac{1}{2z_4}\left[\beta_c(z_2^2 - e_2) - (\gamma + \mu)(z_4^2 - e_4)\right] \\
\dfrac{dz_5}{dt} & = & \dfrac{1}{2z_5}\left[\sigma_c(z_3^2 - e_3) + \phi(z_6^2 - e_6) - (\rho_c + \delta + \mu)(z_5^2 - e_5)\right] \\
\dfrac{dz_6}{dt} & = & \dfrac{1}{2z_6}\left[\sigma_m(z_3^2 - e_3) - (\rho_m + \phi + \mu)(z_6^2 - e_6)\right] \\
\dfrac{dz_7}{dt} & = & \dfrac{1}{2z_7}\left[\sigma_a(z_3^2 - e_3) - (\rho_a + \mu)(z_7^2 - e_7)\right] \\
\dfrac{dz_8}{dt} & = & \dfrac{1}{2z_8}\left[\rho_c(z_5^2 - e_5) + \rho_m(z_6^2 - e_6) + \rho_a(z_7^2 - e_7) - \mu(z_8^2 - e_8)\right] \\
\dfrac{dz_9}{dt} & = & \dfrac{1}{2z_9}\left[\nu\left(z_1^2 + \dfrac{\theta N}{\nu + \mu} - e_1\right) - (\alpha_v(z_2^2 - e_2) + \mu)\left(z_9^2 + \dfrac{\theta \nu N}{\mu(\nu + \mu)} - e_9\right)\right]
\end{array}
\right. \tag{2}
$$

Set $\dfrac{dz_i}{dt} = 0$ for $i = 1, 2, \ldots, 9$. Then we solve the equilibrium point.

$$
\left\{
\begin{array}{rcl}
\dfrac{1}{2z_1}\left[\theta N - (\alpha_s(z_2^2 - e_2) + \nu + \mu)\left(z_1^2 + \dfrac{\theta N}{\nu + \mu} - e_1\right)\right] & = & 0 \\
\dfrac{1}{2z_2}\left[\left(\alpha_s\left(z_1^2 + \dfrac{\theta N}{\nu + \mu} - e_1\right) + \alpha_v\left(z_9^2 + \dfrac{\theta \nu N}{\mu(\nu + \mu)} - e_9\right) - (\beta_d + \beta_c + \mu)\right)(z_2^2 - e_2)\right] & = & 0 \\
\dfrac{1}{2z_3}\left[\beta_d(z_2^2 - e_2) + \gamma(z_4^2 - e_4) - (\sigma_c + \sigma_m + \sigma_a + \mu)(z_3^2 - e_3)\right] & = & 0 \\
\dfrac{1}{2z_4}\left[\beta_c(z_2^2 - e_2) - (\gamma + \mu)(z_4^2 - e_4)\right] & = & 0 \\
\dfrac{1}{2z_5}\left[\sigma_c(z_3^2 - e_3) + \phi(z_6^2 - e_6) - (\rho_c + \delta + \mu)(z_5^2 - e_5)\right] & = & 0 \\
\dfrac{1}{2z_6}\left[\sigma_m(z_3^2 - e_3) - (\rho_m + \phi + \mu)(z_6^2 - e_6)\right] & = & 0 \\
\dfrac{1}{2z_7}\left[\sigma_a(z_3^2 - e_3) - (\rho_a + \mu)(z_7^2 - e_7)\right] & = & 0 \\
\dfrac{1}{2z_8}\left[\rho_c(z_5^2 - e_5) + \rho_m(z_6^2 - e_6) + \rho_a(z_7^2 - e_7) - \mu(z_8^2 - e_8)\right] & = & 0 \\
\dfrac{1}{2z_9}\left[\nu\left(z_1^2 + \dfrac{\theta N}{\nu + \mu} - e_1\right) - (\alpha_v(z_2^2 - e_2) + \mu)\left(z_9^2 + \dfrac{\theta \nu N}{\mu(\nu + \mu)} - e_9\right)\right] & = & 0
\end{array}
\right.
$$

From the second equation, we get

$$
\begin{aligned}
&\frac{1}{2z_2}\left[\left(\alpha_s\left(z_1^2 + \frac{\theta N}{\nu + \mu} - e_1\right) + \alpha_v\left(z_9^2 + \frac{\theta \nu N}{\mu(\nu + \mu)} - e_9\right) - (\beta_d + \beta_c + \mu)\right)(z_2^2 - e_2)\right] = 0 \\
&\implies z_2 = \sqrt{e_2}
\end{aligned}
$$

From the first equation,

$$
\begin{aligned}
&\frac{1}{2z_1}\left[\theta N - (\alpha_s(z_2^2 - e_2) + \nu + \mu)\left(z_1^2 + \frac{\theta N}{\nu + \mu} - e_1\right)\right] = 0 \\
&\implies z_1 = \sqrt{e_1}
\end{aligned}
$$

From the fourth equation,

$$
\begin{aligned}
&\frac{1}{2z_4}\left[\beta_c(z_2^2 - e_2) - (\gamma + \mu)(z_4^2 - e_4)\right] = 0 \\
&\implies z_4 = \sqrt{e_4}
\end{aligned}
$$

From the third equation,

$$
\begin{aligned}
&\frac{1}{2z_3}\left[\beta_d(z_2^2 - e_2) + \gamma(z_4^2 - e_4) - (\sigma_c + \sigma_m + \sigma_a + \mu)(z_3^2 - e_3)\right] = 0 \\
&\implies z_3 = \sqrt{e_3}
\end{aligned}
$$

From the sixth equation,

$$\frac{1}{2z_6}\left[\sigma_m(z_3^2-e_3)-(\rho_m+\phi+\mu)(z_6^2-e_6)\right]=0$$
$$\implies z_6=\sqrt{e_6}$$

From the fifth equation,

$$\frac{1}{2z_5}\left[\sigma_c(z_3^2-e_3)+\phi(z_6^2-e_6)-(\rho_c+\delta+\mu)(z_5^2-e_5)\right]=0$$
$$\implies z_7=\sqrt{e_7}$$

From the seventh equation,

$$\frac{1}{2z_7}\left[\sigma_a(z_3^2-e_3)-(\rho_a+\mu)(z_7^2-e_7)\right]=0$$
$$\implies z_7=\sqrt{e_7}$$

From the eighth equation,

$$\frac{1}{2z_8}\left[\rho_c(z_5^2-e_5)+\rho_m(z_6^2-e_6)+\rho_a(z_7^2-e_7)-\mu(z_8^2-e_8)\right]=0$$
$$\implies z_8=\sqrt{e_8}$$

From the ninth equation,

$$\frac{1}{2z_9}\left[\nu\left(z_1^2+\frac{\theta N}{\nu+\mu}-e_1\right)-\left(\alpha_v(z_2^2-e_2)+\mu\right)\left(z_9^2+\frac{\theta\nu N}{\mu(\nu+\mu)}-e_9\right)\right]=0$$
$$\implies z_9=\sqrt{e_9}$$

The equilibrium point of the new system is $(\sqrt{e_1},\sqrt{e_2},\ldots,\sqrt{e_9})$.

Let $z=(z_1,z_2,\ldots,z_9)$ and $e=(\sqrt{e_1},\sqrt{e_2},\ldots,\sqrt{e_9})$.

We form an equation by collecting the quadratic terms of the Taylor series expansion for each equation in the system.

Let $F_1(z)=\dfrac{1}{2z_1}\left[\theta N-\left(\alpha_s(z_2^2-e_2)+\nu+\mu\right)\left(z_1^2+\dfrac{\theta N}{\nu+\mu}-e_1\right)\right]$.

Now,

$$\frac{\partial^2F_1}{\partial z_1^2}(z)=-\alpha_s(z_2^2-e_2)\left(\frac{\theta N}{\nu+\mu}-e_1\right)z_1^{-3}+(\nu+\mu)e_1z_1^{-3}$$
$$\implies \frac{\partial^2F_1}{\partial z_1^2}(e)=\frac{\nu+\mu}{\sqrt{e_1}}$$

and

$$\frac{\partial^2F_1}{\partial z_2^2}(z)=-\alpha_s\left(z_1+\left(\frac{\theta N}{\nu+\mu}-e_1\right)z_1^{-1}\right)$$
$$\implies \frac{\partial^2F_1}{\partial z_2^2}(e)=-\frac{\alpha_s\theta N}{(\nu+\mu)\sqrt{e_1}}$$

Then the estimate quadratic equation of $F_1(z)$ is given by

$$G_1(z)=\frac{1}{2}\left[\frac{\partial^2F_1}{\partial z_1^2}(e)(z_1-\sqrt{e_1})^2+\frac{\partial^2F_1}{\partial z_2^2}(e)(z_2-\sqrt{e_2})^2\right]$$
$$\implies G_1(z)=\frac{\nu+\mu}{2\sqrt{e_1}}(z_1-\sqrt{e_1})^2-\frac{\alpha_s\theta N}{2(\nu+\mu)\sqrt{e_1}}(z_2-\sqrt{e_2})^2$$

Let $F_2(z) = \frac{1}{2z_2}\left[\left(\alpha_s\left(z_1^2 + \frac{\theta N}{\nu+\mu} - e_1\right) + \alpha_v\left(z_9^2 + \frac{\theta\nu N}{\mu(\nu+\mu)} - e_9\right) - (\beta_d + \beta_c + \mu)\right)(z_2^2 - e_2)\right]$.

Now,

$$\begin{aligned} &\frac{\partial^2 F_2}{\partial z_1^2}(z) = \alpha_s(z_2 - e_2 z_2^{-1}) \\ \implies \quad &\frac{\partial^2 F_2}{\partial z_1^2}(e) = 0, \end{aligned}$$

$$\begin{aligned} &\frac{\partial^2 F_2}{\partial z_2^2}(z) = -e_2 z_2^{-3}\left[\alpha_s\left(z_1^2 + \frac{\theta N}{\nu+\mu} - e_1\right) + \alpha_v\left(z_9^2 + \frac{\theta\nu N}{\mu(\nu+\mu)} - e_9\right) - (\beta_d + \beta_c + \mu)\right] \\ \implies \quad &\frac{\partial^2 F_2}{\partial z_2^2}(e) = -\frac{1}{\sqrt{e_2}}\left[\frac{\alpha_s\theta N}{\nu+\mu} + \frac{\alpha_v\theta\nu N}{\mu(\nu+\mu)} - (\beta_d + \beta_c + \mu)\right] \end{aligned}$$

and

$$\begin{aligned} &\frac{\partial^2 F_2}{\partial z_9^2}(z) = \alpha_v(z_2 - e_2 z_2^{-1}) \\ \implies \quad &\frac{\partial^2 F_2}{\partial z_9^2}(e) = 0 \end{aligned}$$

Then the estimate quadratic equation of $F_2(z)$ is given by

$$\begin{aligned} &G_2(z) = \frac{1}{2}\left[\frac{\partial^2 F_2}{\partial z_1^2}(e)(z_1 - \sqrt{e_1})^2 + \frac{\partial^2 F_2}{\partial z_2^2}(e)(z_2 - \sqrt{e_2})^2 + \frac{\partial^2 F_2}{\partial z_9^2}(e)(z_9 - \sqrt{e_9})^2\right] \\ \implies \quad &G_2(z) = -\frac{1}{\sqrt{e_2}}\left[\frac{\alpha_s\theta N}{\nu+\mu} + \frac{\alpha_v\theta\nu N}{\mu(\nu+\mu)} - (\beta_d + \beta_c + \mu)\right](z_2 - \sqrt{e_2})^2 \end{aligned}$$

Let $F_3(z) = \dfrac{1}{2z_3}\left[\beta_d(z_2^2 - e_2) + \gamma(z_4^2 - e_4) - (\sigma_c + \sigma_m + \sigma_a + \mu)(z_3^2 - e_3)\right]$.

Now,

$$\begin{aligned} &\frac{\partial^2 F_3}{\partial z_2^2}(z) = \beta_d z_3^{-1} \\ \implies \quad &\frac{\partial^2 F_3}{\partial z_2^2}(e) = \frac{\beta_d}{\sqrt{e_3}}, \end{aligned}$$

$$\begin{aligned} &\frac{\partial^2 F_3}{\partial z_3^2}(z) = \beta_d(z_2^2 - e_2)z_3^{-3} + \gamma(z_4^2 - e_4)z_3^{-3} + (\sigma_c + \sigma_m + \sigma_a + \mu)e_3 z_3^{-3} \\ \implies \quad &\frac{\partial^2 F_3}{\partial z_3^2}(e) = \frac{\sigma_c + \sigma_m + \sigma_a + \mu}{\sqrt{e_3}} \end{aligned}$$

and

$$\begin{aligned} &\frac{\partial^2 F_3}{\partial z_4^2}(z) = \gamma z_3^{-1} \\ \implies \quad &\frac{\partial^2 F_3}{\partial z_4^2}(e) = \frac{\gamma}{\sqrt{e_3}} \end{aligned}$$

Then the estimate quadratic equation of $F_3(z)$ is given by

$$\begin{aligned} &G_3(z) = \frac{1}{2}\left[\frac{\partial^2 F_3}{\partial z_2^2}(e)(z_2 - \sqrt{e_2})^2 + \frac{\partial^2 F_3}{\partial z_3^2}(e)(z_3 - \sqrt{e_3})^2 + \frac{\partial^2 F_3}{\partial z_4^2}(e)(z_4 - \sqrt{e_4})^2\right] \\ \implies \quad &G_3(z) = \frac{\beta_d}{2\sqrt{e_3}}(z_2 - \sqrt{e_2})^2 + \frac{\sigma_c + \sigma_m + \sigma_a + \mu}{2\sqrt{e_3}}(z_3 - \sqrt{e_3})^2 + \frac{\gamma}{2\sqrt{e_3}}(z_4 - \sqrt{e_4})^2 \end{aligned}$$

Let $F_4(z) = \dfrac{1}{2z_4}\left[\beta_c(z_2^2 - e_2) - (\gamma + \mu)(z_4^2 - e_4)\right]$.

Now,

$$\begin{aligned} &\frac{\partial^2 F_4}{\partial z_2^2}(z) = \beta_c z_4^{-1} \\ \implies \quad &\frac{\partial^2 F_4}{\partial z_2^2}(e) = \frac{\beta_c}{\sqrt{e_4}} \end{aligned}$$

and

$$\begin{aligned} &\frac{\partial^2 F_4}{\partial z_4^2}(z) = \beta_c z_2^2 z_4^{-3} - \beta_c e_2 z_4^{-3} + (\gamma+\mu)e_4 z_4^{-3} \\ \Longrightarrow \quad &\frac{\partial^2 F_4}{\partial z_4^2}(e) = \frac{\gamma+\mu}{\sqrt{e_4}} \end{aligned}$$

Then the estimate quadratic equation of $F_4(z)$ is given by

$$\begin{aligned} &G_4(z) = \frac{1}{2}\left[\frac{\partial^2 F_4}{\partial z_2^2}(e)(z_2-\sqrt{e_2})^2 + \frac{\partial^2 F_4}{\partial z_4^2}(e)(z_4-\sqrt{e_4})^2\right] \\ \Longrightarrow \quad &G_4(z) = \frac{\beta_c}{2\sqrt{e_4}}(z_2-\sqrt{e_2})^2 + \frac{\gamma+\mu}{2\sqrt{e_4}}(z_4-\sqrt{e_4})^2 \end{aligned}$$

Let $F_5(z) = \dfrac{1}{2z_5}\left[\sigma_c(z_3^2-e_3) + \phi(z_6^2-e_6) - (\rho_c+\delta+\mu)(z_5^2-e_5)\right]$.

Now,

$$\begin{aligned} &\frac{\partial^2 F_5}{\partial z_3^2}(z) = \sigma_c z_5^{-1} \\ \Longrightarrow \quad &\frac{\partial^2 F_5}{\partial z_3^2}(e) = \frac{\sigma_c}{\sqrt{e_5}}, \end{aligned}$$

$$\begin{aligned} &\frac{\partial^2 F_5}{\partial z_5^2}(z) = \sigma_c(z_3^2-e_3)z_5^{-3} + \phi(z_6^2-e_6)z_5^{-3} + (\rho_c+\delta+\mu)e_5 z_5^{-3} \\ \Longrightarrow \quad &\frac{\partial^2 F_5}{\partial z_5^2}(e) = \frac{\rho_c+\delta+\mu}{\sqrt{e_5}} \end{aligned}$$

and

$$\begin{aligned} &\frac{\partial^2 F_5}{\partial z_6^2}(z) = \phi z_5^{-1} \\ \Longrightarrow \quad &\frac{\partial^2 F_5}{\partial z_6^2}(e) = \frac{\phi}{\sqrt{e_5}} \end{aligned}$$

Then the estimate quadratic equation of $F_5(z)$ is given by

$$\begin{aligned} &G_5(z) = \frac{1}{2}\left[\frac{\partial^2 F_5}{\partial z_3^2}(e)(z_3-\sqrt{e_3})^2 + \frac{\partial^2 F_5}{\partial z_5^2}(e)(z_5-\sqrt{e_5})^2 + \frac{\partial^2 F_5}{\partial z_6^2}(e)(z_6-\sqrt{e_6})^2\right] \\ \Longrightarrow \quad &G_5(z) = \frac{\sigma_c}{2\sqrt{e_5}}(z_3-\sqrt{e_3})^2 + \frac{\rho_c+\delta+\mu}{2\sqrt{e_5}}(z_5-\sqrt{e_5})^2 + \frac{\phi}{2\sqrt{e_5}}(z_6-\sqrt{e_6})^2 \end{aligned}$$

Let $F_6(z) = \dfrac{1}{2z_6}\left[\sigma_m(z_3^2-e_3) - (\rho_m+\phi+\mu)(z_6^2-e_6)\right]$.

Now,

$$\begin{aligned} &\frac{\partial^2 F_6}{\partial z_3^2}(z) = \sigma_m z_6^{-1} \\ \Longrightarrow \quad &\frac{\partial^2 F_6}{\partial z_3^2}(e) = \frac{\sigma_m}{\sqrt{e_6}} \end{aligned}$$

and

$$\begin{aligned} &\frac{\partial^2 F_6}{\partial z_6^2}(z) = \sigma_m(z_3^2-e_3)z_6^{-3} + (\rho_m+\phi+\mu)e_6 z_6^{-3} \\ \Longrightarrow \quad &\frac{\partial^2 F_6}{\partial z_6^2}(e) = \frac{\rho_m+\phi+\mu}{\sqrt{e_6}} \end{aligned}$$

Then the estimate quadratic equation of $F_6(z)$ is given by

$$\begin{aligned} &G_6(z) = \frac{1}{2}\left[\frac{\partial^2 F_6}{\partial z_3^2}(e)(z_3-\sqrt{e_3})^2 + \frac{\partial^2 F_6}{\partial z_6^2}(e)(z_6-\sqrt{e_6})^2\right] \\ \Longrightarrow \quad &G_6(z) = \frac{\sigma_m}{2\sqrt{e_6}}(z_3-\sqrt{e_3})^2 + \frac{\rho_m+\phi+\mu}{2\sqrt{e_6}}(z_6-\sqrt{e_6})^2 \end{aligned}$$

Let $F_7(z) = \dfrac{1}{2z_7}\left[\sigma_a(z_3^2 - e_3) - (\rho_a + \mu)(z_7^2 - e_7)\right]$.

Now,

$$\begin{aligned}
&\frac{\partial^2 F_7}{\partial z_3^2}(z) = \sigma_a z_7^{-1}\\
\implies \quad &\frac{\partial^2 F_7}{\partial z_3^2}(e) = \frac{\sigma_a}{\sqrt{e_7}}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial^2 F_7}{\partial z_7^2}(z) = \sigma_a(z_3^2 - e_3)z_7^{-3} + (\rho_a + \mu)e_7 z_7^{-3}\\
\implies \quad &\frac{\partial^2 F_7}{\partial z_7^2}(e) = \frac{\rho_a + \mu}{\sqrt{e_7}}
\end{aligned}$$

Then the estimate quadratic equation of $F_7(z)$ is given by

$$\begin{aligned}
&G_7(z) = \frac{1}{2}\left[\frac{\partial^2 F_7}{\partial z_3^2}(e)(z_3 - \sqrt{e_3})^2 + \frac{\partial^2 F_7}{\partial z_7^2}(e)(z_7 - \sqrt{e_7})^2\right]\\
\implies \quad &G_7(z) = \frac{\sigma_a}{2\sqrt{e_7}}(z_3 - \sqrt{e_3})^2 + \frac{\rho_a + \mu}{2\sqrt{e_7}}(z_7 - \sqrt{e_7})^2
\end{aligned}$$

Let $F_8(z) = \dfrac{1}{2z_8}\left[\rho_c(z_5^2 - e_5) + \rho_m(z_6^2 - e_6) + \rho_a(z_7^2 - e_7) - \mu(z_8^2 - e_8)\right]$.

Now,

$$\begin{aligned}
&\frac{\partial^2 F_8}{\partial z_5^2}(z) = \rho_c z_8^{-1}\\
\implies \quad &\frac{\partial^2 F_8}{\partial z_5^2}(e) = \frac{\rho_c}{\sqrt{e_8}},
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial^2 F_8}{\partial z_6^2}(z) = \rho_m z_8^{-1}\\
\implies \quad &\frac{\partial^2 F_8}{\partial z_6^2}(e) = \frac{\rho_m}{\sqrt{e_8}},
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial^2 F_8}{\partial z_7^2}(z) = \rho_a z_8^{-1}\\
\implies \quad &\frac{\partial^2 F_8}{\partial z_7^2}(e) = \frac{\rho_a}{\sqrt{e_8}}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial^2 F_8}{\partial z_8^2}(z) = \rho_c(z_5^2 - e_5)z_8^{-3} + \rho_m(z_6^2 - e_6)z_8^{-3}\\
&\qquad + \rho_a(z_7^2 - e_7)z_8^{-3} + \mu e_8 z_8^{-3}\\
\implies \quad &\frac{\partial^2 F_8}{\partial z_8^2}(e) = \frac{\mu}{\sqrt{e_8}}
\end{aligned}$$

Then the estimate quadratic equation of $F_7(z)$ is given by

$$\begin{aligned}
&G_8(z) = \frac{1}{2}\left[\frac{\partial^2 F_8}{\partial z_5^2}(e)(z_5 - \sqrt{e_5})^2 + \frac{\partial^2 F_8}{\partial z_6^2}(e)(z_6 - \sqrt{e_6})^2 + \frac{\partial^2 F_8}{\partial z_7^2}(e)(z_7 - \sqrt{e_7})^2 + \frac{\partial^2 F_8}{\partial z_8^2}(e)(z_8 - \sqrt{e_8})^2\right]\\
\implies \quad &G_8(z) = \frac{\rho_c}{2\sqrt{e_8}}(z_5 - \sqrt{e_5})^2 + \frac{\rho_m}{2\sqrt{e_8}}(z_6 - \sqrt{e_6})^2 + \frac{\rho_a}{2\sqrt{e_8}}(z_7 - \sqrt{e_7})^2 + \frac{\mu}{2\sqrt{e_8}}(z_8 - \sqrt{e_8})^2
\end{aligned}$$

Let

$$F_9(z) = \frac{1}{2z_9}\left[\nu\left(z_1^2 + \frac{\theta N}{\nu+\mu} - e_1\right)\right.$$
$$\left. - \left(\alpha_v(z_2^2 - e_2) + \mu\right)\left(z_9^2 + \frac{\theta\nu N}{\mu(\nu+\mu)} - e_9\right)\right].$$

Now,

$$\frac{\partial^2 F_9}{\partial z_1^2}(z) = \nu z_9^{-1}$$
$$\implies \quad \frac{\partial^2 F_9}{\partial z_1^2}(e) = \frac{\nu}{\sqrt{e_9}},$$
$$\frac{\partial^2 F_9}{\partial z_2^2}(z) = -\alpha_v\left(z_9 + \left(\frac{\theta\nu N}{\mu(\nu+\mu)} - e_9\right)z_9^{-1}\right)$$
$$\implies \quad \frac{\partial^2 F_9}{\partial z_2^2}(e) = -\frac{\alpha_v\theta\nu N}{\mu(\nu+\mu)\sqrt{e_9}}$$

and

$$\frac{\partial^2 F_9}{\partial z_9^2}(z) = \nu\left(z_1^2 + \frac{\theta N}{\nu+\mu} - e_1\right)z_9^{-3} - \left(\alpha_v(z_2^2 - e_2) + \mu\right)\left(\frac{\theta\nu N}{\mu(\nu+\mu)} - e_9\right)z_9^{-3}$$
$$\implies \quad \frac{\partial^2 F_9}{\partial z_9^2}(e) = \frac{\mu}{\sqrt{e_9}}$$

Then the estimate quadratic equation of $F_9(z)$ is given by

$$G_9(z) = \frac{1}{2}\left[\frac{\partial^2 F_9}{\partial z_1^2}(e)(z_1 - \sqrt{e_1})^2 + \frac{\partial^2 F_9}{\partial z_2^2}(e)(z_2 - \sqrt{e_2})^2 + \frac{\partial^2 F_9}{\partial z_9^2}(e)(z_9 - \sqrt{e_9})^2\right]$$
$$\implies \quad G_9(z) = \frac{\nu}{2\sqrt{e_9}}(z_1 - \sqrt{e_1})^2 - \frac{\alpha_v\theta\nu N}{2\mu(\nu+\mu)\sqrt{e_9}}(z_2 - \sqrt{e_2})^2 + \frac{\mu}{2\sqrt{e_9}}(z_9 - \sqrt{e_9})^2$$

Let $x_i = z_i - \sqrt{e_i}$ for each $i = 1, 2, \ldots, 9$. Observe that $\frac{dz_i}{dt} = \frac{dx_i}{dt}$. Thus, the system of differential equations of quadratic terms becomes

$$\begin{cases} \frac{dx_1}{dt} &= \left(\frac{\nu+\mu}{2\sqrt{e_1}}\right)x_1^2 - \left(\frac{\alpha_s\theta N}{2(\nu+\mu)\sqrt{e_1}}\right)x_2^2 \\ \frac{dx_2}{dt} &= -\left(\frac{1}{\sqrt{e_2}}\left[\frac{\alpha_s\theta N}{\nu+\mu} + \frac{\alpha_v\theta\nu N}{\mu(\nu+\mu)} - (\beta_d+\beta_c+\mu)\right]\right)x_2^2 \\ \frac{dx_3}{dt} &= \left(\frac{\beta_d}{2\sqrt{e_3}}\right)x_2^2 + \left(\frac{\sigma_c+\sigma_m+\sigma_a+\mu}{2\sqrt{e_3}}\right)x_3^2 + \left(\frac{\gamma}{2\sqrt{e_3}}\right)x_4^2 \\ \frac{dx_4}{dt} &= \left(\frac{\beta_c}{2\sqrt{e_4}}\right)x_2^2 + \left(\frac{\gamma+\mu}{2\sqrt{e_4}}\right)x_4^2 \\ \frac{dx_5}{dt} &= \left(\frac{\sigma_c}{2\sqrt{e_5}}\right)x_3^2 + \left(\frac{\rho_c+\delta+\mu}{2\sqrt{e_5}}\right)x_5^2 + \left(\frac{\phi}{2\sqrt{e_5}}\right)x_6^2 \\ \frac{dx_6}{dt} &= \left(\frac{\sigma_m}{2\sqrt{e_6}}\right)x_3^2 + \left(\frac{\rho_m+\phi+\mu}{2\sqrt{e_6}}\right)x_6^2 \\ \frac{dx_7}{dt} &= \left(\frac{\sigma_a}{2\sqrt{e_7}}\right)x_3^2 + \left(\frac{\rho_a+\mu}{2\sqrt{e_7}}\right)x_7^2 \\ \frac{dx_8}{dt} &= \left(\frac{\rho_c}{2\sqrt{e_8}}\right)x_5^2 + \left(\frac{\rho_m}{2\sqrt{e_8}}\right)x_6^2 + \left(\frac{\rho_a}{2\sqrt{e_8}}\right)x_7^2 + \left(\frac{\mu}{2\sqrt{e_8}}\right)x_8^2 \\ \frac{dx_9}{dt} &= \left(\frac{\nu}{2\sqrt{e_9}}\right)x_1^2 - \left(\frac{\alpha_v\theta\nu N}{2\mu(\nu+\mu)\sqrt{e_9}}\right)x_2^2 + \left(\frac{\mu}{2\sqrt{e_9}}\right)x_9^2 \end{cases} \tag{3}$$

Using Theorem 2.1, the system (3) has the solution

$$(x_1, x_2, \ldots, x_9) = \left(\frac{\lambda_1}{t}, \frac{\lambda_2}{t}, \ldots, \frac{\lambda_9}{t}\right)$$

We solve for the point $(\lambda_1, \lambda_2, \ldots, \lambda_9)$ such that

$$\begin{cases} \left(\frac{\nu+\mu}{2\sqrt{e_1}}\right)\lambda_1^2 - \left(\frac{\alpha_s\theta N}{2(\nu+\mu)\sqrt{e_1}}\right)\lambda_2^2 + \lambda_1 & = 0 \\ -\left(\frac{1}{\sqrt{e_2}}\left[\frac{\alpha_s\theta N}{\nu+\mu} + \frac{\alpha_v\theta\nu N}{\mu(\nu+\mu)} - (\beta_d+\beta_c+\mu)\right]\right)\lambda_2^2 + \lambda_2 & = 0 \\ \left(\frac{\beta_d}{2\sqrt{e_3}}\right)\lambda_2^2 + \left(\frac{\sigma_c+\sigma_m+\sigma_a+\mu}{2\sqrt{e_3}}\right)\lambda_3^2 + \left(\frac{\gamma}{2\sqrt{e_3}}\right)\lambda_4^2 + \lambda_3 & = 0 \\ \left(\frac{\beta_c}{2\sqrt{e_4}}\right)\lambda_2^2 + \left(\frac{\gamma+\mu}{2\sqrt{e_4}}\right)\lambda_4^2 + \lambda_4 & = 0 \\ \left(\frac{\sigma_c}{2\sqrt{e_5}}\right)\lambda_3^2 + \left(\frac{\rho_c+\delta+\mu}{2\sqrt{e_5}}\right)\lambda_5^2 + \left(\frac{\phi}{2\sqrt{e_5}}\right)\lambda_6^2 + \lambda_5 & = 0 \\ \left(\frac{\sigma_m}{2\sqrt{e_6}}\right)\lambda_3^2 + \left(\frac{\rho_m+\phi+\mu}{2\sqrt{e_6}}\right)\lambda_6^2 + \lambda_6 & = 0 \\ \left(\frac{\sigma_a}{2\sqrt{e_7}}\right)\lambda_3^2 + \left(\frac{\rho_a+\mu}{2\sqrt{e_7}}\right)\lambda_7^2 + \lambda_7 & = 0 \\ \left(\frac{\rho_c}{2\sqrt{e_8}}\right)\lambda_5^2 + \left(\frac{\rho_m}{2\sqrt{e_8}}\right)\lambda_6^2 + \left(\frac{\rho_a}{2\sqrt{e_8}}\right)\lambda_7^2 + \left(\frac{\mu}{2\sqrt{e_8}}\right)\lambda_8^2 + \lambda_8 & = 0 \\ \left(\frac{\nu}{2\sqrt{e_9}}\right)\lambda_1^2 - \left(\frac{\alpha_v\theta\nu N}{2\mu(\nu+\mu)\sqrt{e_9}}\right)\lambda_2^2 + \left(\frac{\mu}{2\sqrt{e_9}}\right)\lambda_9^2 + \lambda_9 & = 0 \end{cases}$$

Solve for λ_2 from the second equation.

$$\lambda_2 = \frac{\mu(\nu+\mu)\sqrt{e_2}}{\mu\alpha_s\theta N + \alpha_v\theta\nu N - \mu(\nu+\mu)(\beta_d+\beta_c+\mu)}$$

Substitute λ_2 and solve for λ_1 from the first equation.

$$\lambda_1 = \frac{\sqrt{e_1 + \alpha_s\theta N\lambda_2^2} - \sqrt{e_1}}{\nu+\mu}$$

Substitute λ_2 and solve for λ_4 from the fourth equation.

$$\lambda_4 = \frac{\sqrt{e_4 - \beta_c(\gamma+\mu)\lambda_2^2} - \sqrt{e_4}}{\gamma+\mu}$$

Substitute λ_2 and λ_4 to the third equation. Then solve for λ_3.

$$\lambda_3 = \frac{\sqrt{e_3 - (\sigma_c+\sigma_m+\sigma_a+\mu)(\beta_d\lambda_2^2 + \gamma\lambda_4^2)} - \sqrt{e_3}}{\sigma_c+\sigma_m+\sigma_a+\mu}$$

Substitute λ_3 and solve for λ_6 from the sixth equation.

$$\lambda_6 = \frac{\sqrt{e_6 - \sigma_m(\rho_m+\phi+\mu)\lambda_3^2} - \sqrt{e_6}}{\rho_m+\phi+\mu}$$

Substitute λ_3 and λ_6 to the fifth equation. Then solve for λ_5.

$$\lambda_5 = \frac{\sqrt{e_5 - (\rho_c+\delta+\mu)(\sigma_c\lambda_2^2 + \phi\lambda_6^2)} - \sqrt{e_5}}{\rho_c+\delta+\mu}$$

Substitute λ_3 and solve for λ_7 from the seventh equation.

$$\lambda_7 = \frac{\sqrt{e_7 - \sigma_a(\rho_a+\mu)\lambda_3^2} - \sqrt{e_7}}{\rho_a+\mu}$$

Substitute λ_5, λ_6 and λ_7 to the eighth equation. Then solve for λ_8.

$$\lambda_8 = \frac{\sqrt{e_8 - \mu(\rho_c\lambda_5^2 + \rho_m\lambda_6^2 + \rho_a\lambda_7^2)} - \sqrt{e_8}}{\mu}$$

Substitute λ_1 and λ_2 to the ninth equation. Then solve for λ_9.

$$\lambda_9 = \frac{\sqrt{e_9 - (\nu\mu(\nu+\mu)\lambda_1^2 - \alpha_v\theta\nu N\lambda_2^2)} - \sqrt{e_9}}{\mu}$$

So we have

$$\begin{cases} \lambda_1 &= \dfrac{\sqrt{e_1 + \alpha_s\theta N\lambda_2^2} - \sqrt{e_1}}{\nu+\mu} \\ \lambda_2 &= \dfrac{\mu(\nu+\mu)\sqrt{e_2}}{\mu\alpha_s\theta N + \alpha_v\theta\nu N - \mu(\nu+\mu)(\beta_d+\beta_c+\mu)} \\ \lambda_3 &= \dfrac{\sqrt{e_3 - (\sigma_c+\sigma_m+\sigma_a+\mu)(\beta_d\lambda_2^2 + \gamma\lambda_4^2)} - \sqrt{e_3}}{\sigma_c+\sigma_m+\sigma_a+\mu} \\ \lambda_4 &= \dfrac{\sqrt{e_4 - \beta_c(\gamma+\mu)\lambda_2^2} - \sqrt{e_4}}{\gamma+\mu} \\ \lambda_5 &= \dfrac{\sqrt{e_5 - (\rho_c+\delta+\mu)(\sigma_c\lambda_2^2 + \phi\lambda_6^2)} - \sqrt{e_6}}{\rho_c+\delta+\mu} \\ \lambda_6 &= \dfrac{\sqrt{e_6 - \sigma_m(\rho_m+\phi+\mu)\lambda_3^2} - \sqrt{e_6}}{\rho_m+\phi+\mu} \\ \lambda_7 &= \dfrac{\sqrt{e_7 - \sigma_a(\rho_a+\mu)\lambda_3^2} - \sqrt{e_7}}{\rho_a+\mu} \\ \lambda_8 &= \dfrac{\sqrt{e_8 - \mu(\rho_c\lambda_5^2 + \rho_m\lambda_6^2 + \rho_a\lambda_7^2)} - \sqrt{e_8}}{\mu} \\ \lambda_9 &= \dfrac{\sqrt{e_9 - (\nu\mu(\nu+\mu)\lambda_1^2 - \alpha_v\theta\nu N\lambda_2^2)} - \sqrt{e_9}}{\mu} \end{cases}$$

We change the variable back to the variables used in the original system.
Since $x_i = \dfrac{\lambda_i}{t}$ for $i = 1, 2, \ldots, 9$, the solution of the system (3) is given by

$$\begin{cases} x_1 &= \dfrac{\sqrt{e_1 + \alpha_s\theta N\lambda_2^2} - \sqrt{e_1}}{(\nu+\mu)t} \\ x_2 &= \dfrac{\mu(\nu+\mu)\sqrt{e_2}}{(\mu\alpha_s\theta N + \alpha_v\theta\nu N - \mu(\nu+\mu)(\beta_d+\beta_c+\mu))t} \\ x_3 &= \dfrac{\sqrt{e_3 - (\sigma_c+\sigma_m+\sigma_a+\mu)(\beta_d\lambda_2^2 + \gamma\lambda_4^2)} - \sqrt{e_3}}{(\sigma_c+\sigma_m+\sigma_a+\mu)t} \\ x_4 &= \dfrac{\sqrt{e_4 - \beta_c(\gamma+\mu)\lambda_2^2} - \sqrt{e_4}}{(\gamma+\mu)t} \\ x_5 &= \dfrac{\sqrt{e_5 - (\rho_c+\delta+\mu)(\sigma_c\lambda_2^2 + \phi\lambda_6^2)} - \sqrt{e_6}}{(\rho_c+\delta+\mu)t} \\ x_6 &= \dfrac{\sqrt{e_6 - \sigma_m(\rho_m+\phi+\mu)\lambda_3^2} - \sqrt{e_6}}{(\rho_m+\phi+\mu)t} \\ x_7 &= \dfrac{\sqrt{e_7 - \sigma_a(\rho_a+\mu)\lambda_3^2} - \sqrt{e_7}}{(\rho_a+\mu)t} \\ x_8 &= \dfrac{\sqrt{e_8 - \mu(\rho_c\lambda_5^2 + \rho_m\lambda_6^2 + \rho_a\lambda_7^2)} - \sqrt{e_8}}{\mu t} \\ x_9 &= \dfrac{\sqrt{e_9 - (\nu\mu(\nu+\mu)\lambda_1^2 - \alpha_v\theta\nu N\lambda_2^2)} - \sqrt{e_9}}{\mu t} \end{cases}$$

Since $z_i = x_i + \sqrt{e_i}$ for $i = 1, 2, \ldots, 9$, the solution of the system (2) is given by

$$\begin{cases}
z_1 &= \dfrac{\sqrt{e_1 + \alpha_s \theta N \lambda_2^2} - \sqrt{e_1}}{(\nu+\mu)t} + \sqrt{e_1} \\
z_2 &= \dfrac{\mu(\nu+\mu)\sqrt{e_2}}{(\mu\alpha_s\theta N + \alpha_v \theta \nu N - \mu(\nu+\mu)(\beta_d + \beta_c + \mu))t} + \sqrt{e_2} \\
z_3 &= \dfrac{\sqrt{e_3 - (\sigma_c + \sigma_m + \sigma_a + \mu)(\beta_d \lambda_2^2 + \gamma\lambda_4^2)} - \sqrt{e_3}}{(\sigma_c + \sigma_m + \sigma_a + \mu)t} + \sqrt{e_3} \\
z_4 &= \dfrac{\sqrt{e_4 - \beta_c(\gamma+\mu)\lambda_2^2} - \sqrt{e_4}}{(\gamma+\mu)t} + \sqrt{e_4} \\
z_5 &= \dfrac{\sqrt{e_5 - (\rho_c + \delta + \mu)(\sigma_c \lambda_2^2 + \phi\lambda_6^2)} - \sqrt{e_6}}{(\rho_c + \delta + \mu)t} + \sqrt{e_5} \\
z_6 &= \dfrac{\sqrt{e_6 - \sigma_m(\rho_m + \phi + \mu)\lambda_3^2} - \sqrt{e_6}}{(\rho_m + \phi + \mu)t} + \sqrt{e_6} \\
z_7 &= \dfrac{\sqrt{e_7 - \sigma_a(\rho_a + \mu)\lambda_3^2} - \sqrt{e_7}}{(\rho_a + \mu)t} + \sqrt{e_7} \\
z_8 &= \dfrac{\sqrt{e_8 - \mu(\rho_c\lambda_5^2 + \rho_m\lambda_6^2 + \rho_a\lambda_7^2)} - \sqrt{e_8}}{\mu t} + \sqrt{e_8} \\
z_9 &= \dfrac{\sqrt{e_9 - (\nu\mu(\nu+\mu)\lambda_1^2 - \alpha_v\theta\nu N\lambda_2^2)} - \sqrt{e_9}}{\mu t} + \sqrt{e_9}
\end{cases}$$

We solve for $S, E, E_d, E_c, I_c, I_m, I_a, R$ and V from the following equations:

$$\begin{aligned}
z_1 &= \sqrt{S - \frac{\theta N}{\nu+\mu} + e_1}, \quad z_2 = \sqrt{E + e_2}, \quad z_3 = \sqrt{E_d + e_3} \\
z_4 &= \sqrt{E_c + e_4}, \qquad z_5 = \sqrt{I_c + e_5}, \quad z_6 = \sqrt{I_m + e_6} \\
z_7 &= \sqrt{I_a + e_7}, \qquad z_8 = \sqrt{R + e_8}, \quad z_9 = \sqrt{V - \frac{\theta\nu N}{\mu(\nu+\mu)} + e_9}
\end{aligned}$$

So we obtain an approximate solution for the system (1).

$$\begin{cases}
S^+ &= \left(\dfrac{\sqrt{e_1 + \alpha_s\theta N\lambda_2^2} - \sqrt{e_1}}{(\nu+\mu)t} + \sqrt{e_1}\right)^2 - e_1 + \dfrac{\theta N}{\nu+\mu} \\
E^+ &= \left(\dfrac{\mu(\nu+\mu)\sqrt{e_2}}{(\mu\alpha_s\theta N + \alpha_v\theta\nu N - \mu(\nu+\mu)(\beta_d + \beta_c + \mu))t} + \sqrt{e_2}\right)^2 - e_2 \\
E_d^+ &= \left(\dfrac{\sqrt{e_3 - (\sigma_c + \sigma_m + \sigma_a + \mu)(\beta_d\lambda_2^2 + \gamma\lambda_4^2)} - \sqrt{e_3}}{(\sigma_c + \sigma_m + \sigma_a + \mu)t} + \sqrt{e_3}\right)^2 - e_3 \\
E_c^+ &= \left(\dfrac{\sqrt{e_4 - \beta_c(\gamma+\mu)\lambda_2^2} - \sqrt{e_4}}{(\gamma+\mu)t} + \sqrt{e_4}\right)^2 - e_4 \\
I_c^+ &= \left(\dfrac{\sqrt{e_5 - (\rho_c + \delta + \mu)(\sigma_c\lambda_2^2 + \phi\lambda_6^2)} - \sqrt{e_5}}{(\rho_c + \delta + \mu)t} + \sqrt{e_5}\right)^2 - e_5 \\
I_m^+ &= \left(\dfrac{\sqrt{e_6 - \sigma_m(\rho_m + \phi + \mu)\lambda_3^2} - \sqrt{e_6}}{(\rho_m + \phi + \mu)t} + \sqrt{e_6}\right)^2 - e_6 \\
I_a^+ &= \left(\dfrac{\sqrt{e_7 - \sigma_a(\rho_a + \mu)\lambda_3^2} - \sqrt{e_7}}{(\rho_a + \mu)t} + \sqrt{e_7}\right)^2 - e_7 \\
R^+ &= \left(\dfrac{\sqrt{e_8 - \mu(\rho_c\lambda_5^2 + \rho_m\lambda_6^2 + \rho_a\lambda_7^2)} - \sqrt{e_8}}{\mu t} + \sqrt{e_8}\right)^2 - e_8 \\
V^+ &= \left(\dfrac{\sqrt{e_9 - (\nu\mu(\nu+\mu)\lambda_1^2 - \alpha_v\theta\nu N\lambda_2^2)} - \sqrt{e_9}}{\mu t} + \sqrt{e_9}\right)^2 - e_9 + \dfrac{\theta\nu N}{\mu(\nu+\mu)}
\end{cases}$$

$\square$

Consider the set of parameter values—adopted, estimated, and assumed—for the extended SEIR model as outlined in [10]. These values were used to simulate the model, and they are as follows: $\delta = 1.80\times10^{-3}$, $\alpha_s = 2.02\times10^{-9}$, $\alpha_v = 4.05\times10^{-10}$, $\rho_c = 1.23\times10^{-1}$, $\rho_m = 1.23\times10^{-1}$, $\rho_a = 1.23\times10^{-1}$, $\theta = 2.93\times10^{-5}$, $\mu = 1.56\times10^{-5}$, $\nu = 9.97\times10^{-4}$, $\phi = 5.00\times10^{-2}$, $\sigma_c = 1.60\times10^{-3}$, $\sigma_m = 1.29\times10^{-1}$, $\sigma_a = 4.33\times10^{-3}$, $\gamma = 2.00\times10^{-1}$, $\beta_d = 4.78\times10^{-4}$, and $\beta_c = 4.32\times10^{-2}$.

Table 1 shows the minimum required values and chosen constants e_i's and λ_i's. These were determined by plugging the parameter values into the system's equations and checking which values satisfy the conditions. Once the values were verified, we used them to form the approximate solutions of the model.

Table 1. Parameter values of the approximate solution

Parameters	Range of values	Values	Parameters	Values
e_1	$> 3.19\times10^{6}$	3.20×10^{6}	λ_1	7.22×10^{-6}
e_2	> 0	1.00×10^{-1}	λ_2	2.00×10^{1}
e_3	$> 2.72\times10^{-3}$	1.00×10^{-1}	λ_3	-3.20×10^{-2}
e_4	$> 3.46\times10^{-2}$	1.00×10^{-1}	λ_4	-3.02×10^{-2}
e_5	$> 7.95\times10^{-4}$	1.00×10^{-1}	λ_5	-1.01×10^{-2}
e_6	$> 2.30\times10^{-5}$	1.00×10^{-1}	λ_6	-2.11×10^{-4}
e_7	$> 5.50\times10^{-7}$	1.00×10^{-1}	λ_7	-7.09×10^{-6}
e_8	$> 1.96\times10^{-10}$	1.00×10^{-1}	λ_8	-1.99×10^{-5}
e_9	$> 2.04\times10^{8}$	2.05×10^{8}	λ_9	1.15×10^{-5}

With these, we can now express the approximate solution by substituting the parameter values and constants from the table. The system looks like this:

$$\begin{cases} S^+ &= \left(\dfrac{0.00000722}{t} + \sqrt{3200000}\right)^2 - 7220.854882 \\ E^+ &= \left(\dfrac{6.99}{t} + \sqrt{0.1}\right)^2 - 0.1 \\ E_d^+ &= \left(\dfrac{-0.032167519}{t} + \sqrt{0.1}\right)^2 - 0.1 \\ E_c^+ &= \left(\dfrac{-0.302417597}{t} + \sqrt{0.1}\right)^2 - 0.1 \\ I_c^+ &= \left(\dfrac{-0.010114134}{t} + \sqrt{0.1}\right)^2 - 0.1 \\ I_m^+ &= \left(\dfrac{-0.000210905}{t} + \sqrt{0.1}\right)^2 - 0.1 \\ I_a^+ &= \left(\dfrac{-0.0000070867}{t} + \sqrt{0.1}\right)^2 - 0.1 \\ R^+ &= \left(\dfrac{-0.0000198551}{t} + \sqrt{0.1}\right)^2 - 0.1 \\ V^+ &= \left(\dfrac{0.00000000001}{t} + \sqrt{204000000}\right)^2 - 149326 \end{cases}$$

We can obtain different systems of approximate solutions by specifying another value for e_i's. Here, we use $N = 110500000$. Consequently, the approximate solution converges to the disease-free equilibrium as $t \to \infty$. That is,

$$
\begin{aligned}
& \lim_{t\to\infty} \left(S^+(t), E^+(t), E_d^+(t), E_c^+(t), I_c^+(t), I_m^+(t), I_a^+(t), R^+(t), V^+(t)\right) \\
= \quad & \left(S^0, E^0, E_d^0, E_c^0, I_c^0, I_m^0, I_a^0, R^0, V^0\right) \\
= \quad & (3192779, 0, 0, 0, 0, 0, 0, 0, 203850674)
\end{aligned}
$$

This is supported with the graphical representation of the solution of the system below.

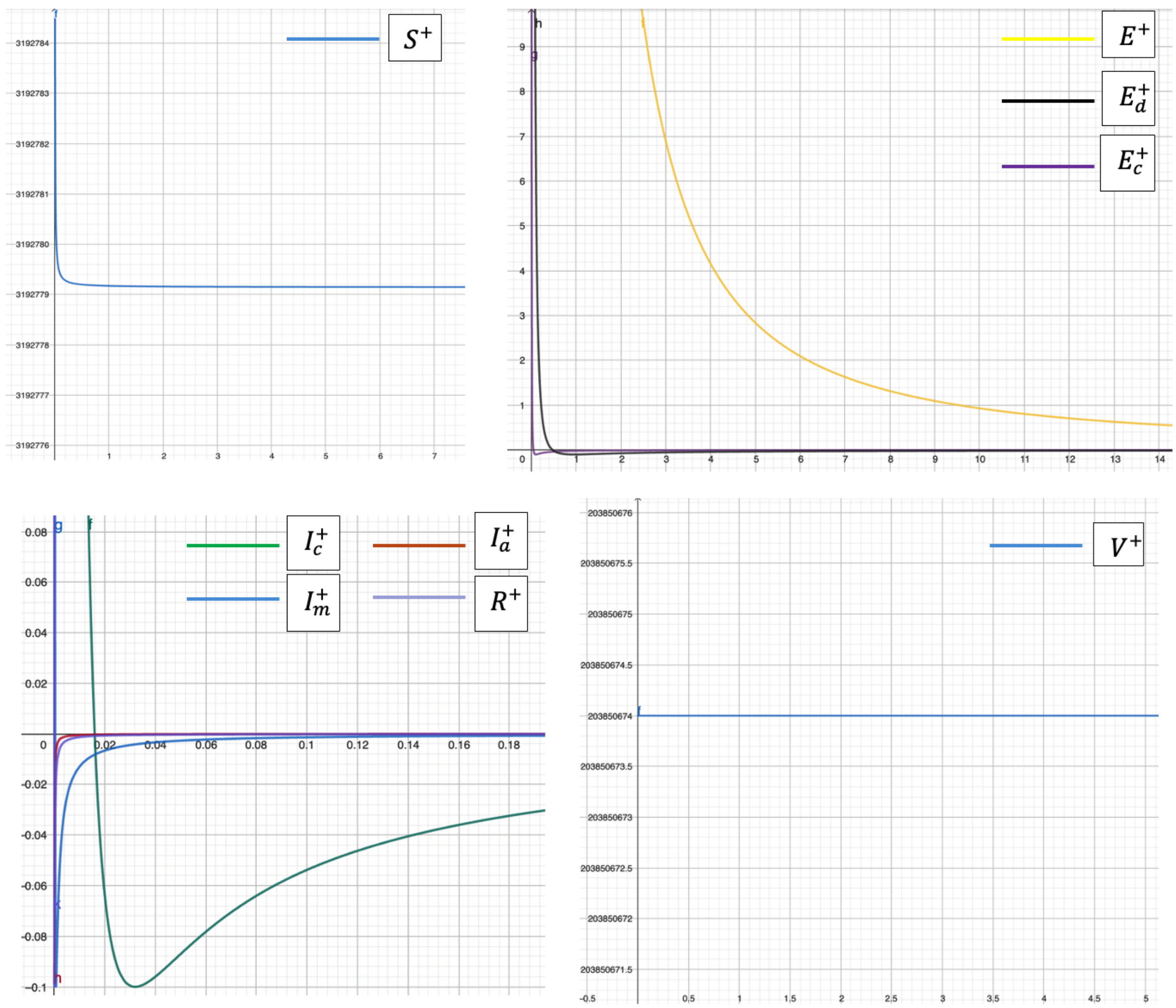

FIGURE 1. Graphical representation of the solutions of the system

By analyzing the structure of the approximate solutions derived from the parameter values and constants in Table 1, we observe a consistent trend toward stability over time. This behavior suggests that the system approaches a steady state in the long run. The following theorem demonstrates that the approximate solution of system (1) converges to the disease-free equilibrium as $t \to \infty$. This result confirms that the quadratic approximation offers a reliable estimate of the system's behavior near the point where the disease disappears. Notably, the approximate solution near the equilibrium closely resembles the actual disease-free state of the system.

Theorem 3.2. *There exists a solution* $\left(S^+, E^+, E_d^+, E_c^+, I_c^+, I_m^+, I_a^+, R^+, V^+\right)$ *of the approximate system that converges to* $\left(S^0, E^0, E_d^0, E_c^0, I_c^0, I_m^0, I_a^0, R^0, V^0\right)$ *of the system* (1). *That is,*

$$\begin{aligned} & \lim_{t\to\infty} \left(S^+(t), E^+(t), E_d^+(t), E_c^+(t), I_c^+(t), I_m^+(t), I_a^+(t), R^+(t), V^+(t)\right) \\ = \quad & \left(S^0, E^0, E_d^0, E_c^0, I_c^0, I_m^0, I_a^0, R^0, V^0\right) \end{aligned}$$

Proof. The following is a solution of the approximate system

$$\begin{cases} S^+ &= \left(\dfrac{\sqrt{e_1 + \alpha_s \theta N \lambda_2^2} - \sqrt{e_1}}{(\nu+\mu)t} + \sqrt{e_1}\right)^2 - e_1 + \dfrac{\theta N}{\nu + \mu} \\ E^+ &= \left(\dfrac{\mu(\nu+\mu)\sqrt{e_2}}{(\mu\alpha_s\theta N + \alpha_v \theta\nu N - \mu(\nu+\mu)(\beta_d+\beta_c+\mu))t} + \sqrt{e_2}\right)^2 - e_2 \\ E_d^+ &= \left(\dfrac{\sqrt{e_3 - (\sigma_c+\sigma_m+\sigma_a+\mu)(\beta_d\lambda_2^2 + \gamma\lambda_4^2)} - \sqrt{e_3}}{(\sigma_c+\sigma_m+\sigma_a+\mu)t} + \sqrt{e_3}\right)^2 - e_3 \\ E_c^+ &= \left(\dfrac{\sqrt{e_4 - \beta_c(\gamma+\mu)\lambda_2^2} - \sqrt{e_4}}{(\gamma+\mu)t} + \sqrt{e_4}\right)^2 - e_4 \\ I_c^+ &= \left(\dfrac{\sqrt{e_5 - (\rho_c+\delta+\mu)(\sigma_c\lambda_2^2 + \phi\lambda_6^2)} - \sqrt{e_5}}{(\rho_c+\delta+\mu)t} + \sqrt{e_5}\right)^2 - e_5 \\ I_m^+ &= \left(\dfrac{\sqrt{e_6 - \sigma_m(\rho_m+\phi+\mu)\lambda_3^2} - \sqrt{e_6}}{(\rho_m+\phi+\mu)t} + \sqrt{e_6}\right)^2 - e_6 \\ I_a^+ &= \left(\dfrac{\sqrt{e_7 - \sigma_a(\rho_a+\mu)\lambda_3^2} - \sqrt{e_7}}{(\rho_a+\mu)t} + \sqrt{e_7}\right)^2 - e_7 \\ R^+ &= \left(\dfrac{\sqrt{e_8 - \mu(\rho_c\lambda_5^2 + \rho_m\lambda_6^2 + \rho_a\lambda_7^2)} - \sqrt{e_8}}{\mu t} + \sqrt{e_8}\right)^2 - e_8 \\ V^+ &= \left(\dfrac{\sqrt{e_9 - (\nu\mu(\nu+\mu)\lambda_1^2 - \alpha_v\theta\nu N\lambda_2^2)} - \sqrt{e_9}}{\mu t} + \sqrt{e_9}\right)^2 - e_9 + \dfrac{\theta\nu N}{\mu(\nu+\mu)} \end{cases}$$

Observe that as $t \to \infty$, $S^+(t) \to \dfrac{\theta N}{\nu+\mu}$. We also have $E^+(t)$, $E_d^+(t)$, $E_c^+(t)$, $I_c^+(t)$, $I_m^+(t)$, $I_a^+(t)$, $R^+(t) \to 0$ as $t \to \infty$ while $V^+(t) \to \dfrac{\theta\nu N}{\mu(\nu+\mu)}$.
Therefore,

$$\begin{aligned} & \lim_{t\to\infty} \left(S^+(t), E^+(t), E_d^+(t), E_c^+(t), I_c^+(t), I_m^+(t), I_a^+(t), R^+(t), V^+(t)\right) \\ = \quad & \left(S^0, E^0, E_d^0, E_c^0, I_c^0, I_m^0, I_a^0, R^0, V^0\right) \end{aligned}$$

□

4. Conclusion

This study highlights the importance of quadratic approximations in the analysis of the system of nonlinear ordinary differential equations, particularly in the framework of an extended SIR COVID-19 model. We showed that there is an approximate solution and that the system solution converges to the disease-free equilibrium by including only the second-order terms from the Taylor series expansion. The findings suggest that quadratic approximations are a possible substitute for conventional linear approaches since quadratic approximations can represent the dynamics of nonlinear models. This study encourages further research into nonlinear approximation methods particularly the quadratic approximations, especially into their application to other complex dynamical systems in various scientific disciplines.

Authors' Contributions. All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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