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FRACTIONAL VERSION OF STOKES' THEOREM

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Abstract. In this paper, we utilize the conformable derivative to introduce the fractional versions of some fundamental concepts related to vector analysis. The fractional normal vector to a given surface as well as the fractional curl operator in order to define the fractional surface integrals. Moreover, we utilize the fractional version of Green's theorem [1] to discuss and prove the fractional form of the Stokes' theorem.

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1. Introduction

In vector field theory, Stokes' theorem is considered as one of the fundamental theorems that relates the anticlockwise line integral of a vector field around a closed curve (boundary of a surface) with the surface integral of the curl of that vector field over the open surface. This theorem together with other tools in vector calculus topics are assumed to be the language of partial differential equations. They can be used to construct and manipulate some physical quantities such as the well known conservation laws of energy, mass, and momentum in order to model some dynamical systems using partial differential equations.

Definition 1.1. (*Surface Integral*) *Let* $\overrightarrow{F}(x,y,z) = P(x,y,z)\widehat{i} + Q(x,y,z)\widehat{j} + R(x,y,z)\widehat{k}$ *be a continuous vector field defined on an oriented surface* $S = \{(x,y,z) |\, z = g(x,y), (x,y) \in D = \underset{(x,y)}{\Pr}\, S\}$ *with a unit outward normal vector* $\overrightarrow{n}(x,y,z)$. *Then the surface integral of* F *over* S *is given by:*

$$\iint\limits_{S} \overrightarrow{F}\cdot ds = \iint\limits_{S} \left(\overrightarrow{F}\cdot\overrightarrow{n}\right) ds = \iint\limits_{D} \left(-P\frac{\partial z}{\partial x} - Q\frac{\partial z}{\partial y} + R\right) dA, \tag{1}$$

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where $\overrightarrow{n}(x,y,z) = \frac{-\frac{\partial z}{\partial x}\widehat{i} - \frac{\partial z}{\partial y}\widehat{j} + \widehat{k}}{\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}}$ *and* $ds = \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dA.$

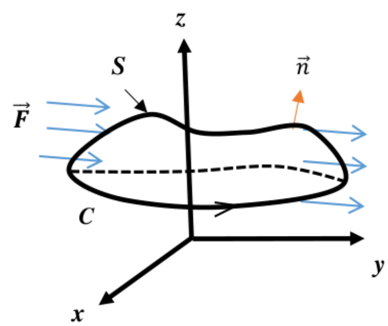

FIGURE 1. Stoke's Theorem relates the surface integral of curl $\overrightarrow{F}$ over an open surface S to the line integral of the $\overrightarrow{F}$ around the boundary of S.

Theorem 1.1. (*Stokes' Theorem*) *The line integral of a vector field* $\overrightarrow{F}(x,y,z) = P(x,y,z)\widehat{i} + Q(x,y,z)\widehat{j} + R(x,y,z)\widehat{k}$ *(with components function* P, Q, *and* R, *have continuous partial derivatives on an open region that contains surface* S*) around the boundary* C *of the positively oriented piece-wise smooth surface* S *is equal to the integral of the vector field* curl $\overrightarrow{F}$ *over the surface* S *(Figure 1). That is*

$$\int_C \overrightarrow{F} \cdot d\overrightarrow{r} = \iint_S \operatorname{curl} \overrightarrow{F} \cdot d\overrightarrow{s}, \tag{2}$$

where C *is given by its vector equation* $\overrightarrow{r}(t) = x(t)\widehat{i} + y(t)\widehat{j} + z(t)\widehat{k}$, $a \le t \le b$ *and the* curl $\overrightarrow{F}$ *is given by*

$$\operatorname{curl} \overrightarrow{F} = \nabla \times \overrightarrow{F} = \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}. \tag{3}$$

2. Fractional Vector Analysis

In [2] the definition of α-conformable fractional derivative was introduced as follows.

Definition 2.1. [2] *Let* $\alpha \in (0,1)$, *and* $u : I \subseteq (0,\infty) \to \mathbb{R}$. *For* $x \in I$, *let*

$$D^\alpha u(x) = \lim_{\epsilon \to 0} \frac{u(x + \epsilon x^{1-\alpha}) - u(x)}{\epsilon}. \tag{4}$$

If the limit exists, then it is called the α-conformable fractional derivative of u at x. Moreover, if u is α-differentiable on $(0,r)$ for some $r > 0$, and $\lim_{\epsilon \to 0^+} D^\alpha u(x)$ exists then we write

$$D^\alpha u(0) = \lim_{x \to 0} D^\alpha u(x). \tag{5}$$

For $\alpha \in (0,1]$ and u, υ *are* α-differentiable at a point x, one can easily see that the conformable derivative satisfies

$$\begin{aligned} &(i)\ D^{\alpha}(c_1u + c_2\upsilon) = c_1D^{\alpha}(u) + c_2D^{\alpha}(\upsilon), \text{ for all } c_1, c_2 \in \mathbb{R}, \\ &(ii)\ D^{\alpha}(k) = 0, \text{ for all constant functions } f(x) = k, \\ &(iii)\ D^{\alpha}(u\upsilon) = uD^{\alpha}(\upsilon) + \upsilon D^{\alpha}(u), \\ &(iv)\ D^{\alpha}(\tfrac{u}{\upsilon}) = \tfrac{\upsilon D^{\alpha}(u) - uD^{\alpha}(\upsilon)}{\upsilon^2}, \upsilon(x) \neq 0. \end{aligned} \tag{6}$$

In the following, we provide the α-conformable fractional derivatives of some basic functions,

$$\begin{aligned} &(i)\ D^{\alpha}(x^p) = px^{p-\alpha}, \\ &(ii)\ D^{\alpha}(\sin(\tfrac{1}{\alpha}x^{\alpha})) = \cos(\tfrac{1}{\alpha}x^{\alpha}), \\ &(iii)\ D^{\alpha}(\cos(\tfrac{1}{\alpha}x^{\alpha})) = -\sin(\tfrac{1}{\alpha}x^{\alpha}), \\ &(iv)\ D^{\alpha}(e^{\frac{1}{\alpha}x^{\alpha}}) = e^{\frac{1}{\alpha}x^{\alpha}}. \end{aligned} \tag{7}$$

On letting $\alpha = 1$ in these derivatives, we get the corresponding classical rules for ordinary derivatives. For more on fractional calculus and its applications we refer to [3], [4], [5], [6], [7], [8], [9] and [10]. Many differential equations can be transformed to fractional form and can have many applications in many branches of science.

If u is a function of two variables s and t, then we write $D_s^{\alpha}u$ and $D_t^{\alpha}u$ to denote the partial α-conformable fractional derivative with respect to s and t respectively. Moreove, we write $D_s^{2\alpha}u$ to denote $D_s^{\alpha}D_s^{\alpha}u$ and similarly, for $D_t^{2\alpha}u$.

Definition 2.2. [2] *The α-fractional integral of a function f starting from $\mathbf{a} \geq 0$ is denoted by $I_{\alpha}^{\mathbf{a}}(u)(x)$ such that*

$$I_{\alpha}^{\mathbf{a}}(u)(x) = I_1^{\mathbf{a}}\left(x^{\alpha-1}u\right) = \int_{\mathbf{a}}^{x} \frac{u(t)}{t^{1-\alpha}}\,dt,, \tag{8}$$

where the integral is the usual Riemann improper integral, and $\alpha \in (0,1)$.

Recently, in 2020, utilizing the conformable derivative, the fractional form of some concepts related to vector analysis theory were proposed [1]. The fractional gradient and fractional curl were introduced as follows.

Definition 2.3. [1] *The fractional gradient vector of a scalar field $f(x_1, x_2, x_3)$ is given by*

$$\overrightarrow{\nabla}^{\alpha} f = D_x^{\alpha} f\ \ \widehat{i} + D_y^{\alpha} f\ \ \widehat{j} + D_z^{\alpha} f\ \ \widehat{k}. \tag{9}$$

Definition 2.4. [1] *The fractional* curl *of a vector field $\overrightarrow{F}(x,y,z) = P(x,y,z)\widehat{i} + Q(x,y,z)\widehat{j} + R(x,y,z)\widehat{k}$ is given by*

$$\overrightarrow{\nabla}^{\alpha} \times \overrightarrow{F} = \left(D_y^{\alpha}R - D_z^{\alpha}Q\right)\widehat{i} - \left(D_x^{\alpha}R - D_z^{\alpha}P\right)\widehat{j} + \left(D_x^{\alpha}Q - D_y^{\alpha}P\right)\widehat{k}. \tag{10}$$

Moreovre, the fractional line integral and the fractional version of Green's Theorem were also discussed in [1].

3. Main Results

In this section, the goal is to provide the concept of both fractional surface integral as well as the fractional version of the Stoke's Theorem.

Definition 3.1. (*Fractional Surface Integral of a Vector Field*) *Let* $\overrightarrow{F}(x,y,z) = P(x,y,z)\widehat{i} + Q(x,y,z)\widehat{j} + R(x,y,z)\widehat{k}$ *be a continuous vector field defined on an oriented open surface* $S = \{(x,y,z)|\ z = g(x,y),\ (x,y) \in D = \underset{(x,y)}{\Pr}\ S\}$ *with a unit fractional outward normal vector* $\overrightarrow{n}_\alpha(x,y,z)$. *Then the fractional surface integral of* F *over* S *is*

$$\iint_S \overrightarrow{F} \cdot d\overrightarrow{s^\alpha} = \iint_S \left(\overrightarrow{F} \cdot \overrightarrow{n}_\alpha\right) ds^\alpha, \tag{11}$$

where

$(i)\ \overrightarrow{n}_\alpha(x,y,z) = \frac{\nabla^\alpha h(x,y,z)}{|\nabla^\alpha h(x,y,z)|} = \frac{-x^{1-\alpha}g_x\,\widehat{i} - y^{1-\alpha}g_y\,\widehat{j} + z^{1-\alpha}\ \widehat{k}}{\sqrt{(x^{1-\alpha}g_x)^2 + (y^{1-\alpha}g_y)^2 + (z^{1-\alpha})^2}}$,

such that the surface S *is the level surface* $h(x,y,z) = 0$ *of the function*

$h(x,y,z) = z - g(x,y)$,

$$(ii)\ ds^\alpha = \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2} dx^\alpha dy^\alpha = \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2} x^{1-\alpha} y^{1-\alpha} dxdy, \tag{12}$$

is the fractional surface area element.

Now, we are interested in finding the fractional form of the Stokes' Theorem. To do so, let us consider the following theorem.

Theorem 3.1. (*Fractional Stoke's Theorem*) *Let* S *be an open orientable surface bounded by a closed curve* C. *If* $\overrightarrow{F}(x,y,z) = P(x,y,z)\widehat{i} + Q(x,y,z)\widehat{j} + R(x,y,z)\widehat{k}$ *is a differentiable vector field, then*

$$\int_C \overrightarrow{F} \cdot d\overrightarrow{r^\alpha} = \iint_S \operatorname{curl}^\alpha \overrightarrow{F} \cdot d\overrightarrow{s^\alpha}, \tag{13}$$

where C *is given by the parameterized position vector* $\overrightarrow{r^\alpha} = x(t)\widehat{i} + y(t)\widehat{j} + z(t)\widehat{k}$, $t \in [a,\ b]$ *and the fractional* $\operatorname{curl}^\alpha \overrightarrow{F}$ *is given by*

$$\operatorname{curl}^\alpha \overrightarrow{F} = \overrightarrow{\nabla}^\alpha \times \overrightarrow{F} = \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ \frac{\partial^\alpha}{\partial x^\alpha} & \frac{\partial^\alpha}{\partial y^\alpha} & \frac{\partial^\alpha}{\partial z^\alpha} \\ P & Q & R \end{vmatrix}. \tag{14}$$

Proof.: Let $S = \{(x,y,z)|\ z = g(x,y),\ (x,y) \in D = \underset{(x,y)}{\Pr}\ S\}$ be an open orientable surface above the $xy-$plane bounded by a closed curve C in $\mathbb{R}^3$ with a unit fractional outward normal vector $\overrightarrow{n}_\alpha(x,y,z)$. Further, we will assume that $\underset{(x,y)}{\Pr}\ S = D$, where Pr is the projection on the $xy-$plane (Figure 2).

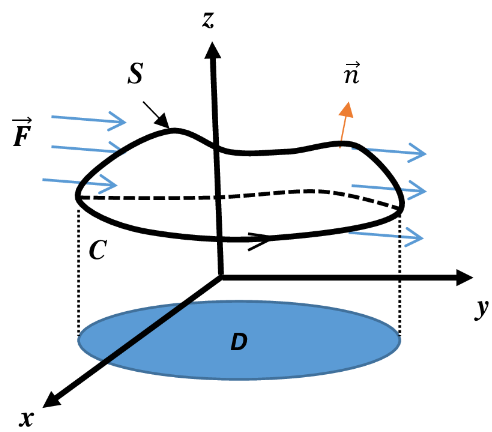

FIGURE 2. The projection D of the surface S on the $xy-$plane

Also, since $\overrightarrow{F}(x,y,z) = P(x,y,z)\widehat{i} + Q(x,y,z)\widehat{j} + R(x,y,z)\widehat{k}$, we have

$$\begin{aligned}
\operatorname{curl}^\alpha \overrightarrow{F} &= \operatorname{curl}^\alpha \left(P\widehat{i}\right) + \operatorname{curl}^\alpha \left(Q\widehat{j}\right) + \operatorname{curl}^\alpha \left(R\widehat{k}\right) \\
&= \left\langle 0, z^{1-\alpha}\frac{\partial P}{\partial z}, -y^{1-\alpha}\frac{\partial P}{\partial y}\right\rangle \\
&+ \left\langle -z^{1-\alpha}\frac{\partial Q}{\partial z}, 0, -x^{1-\alpha}\frac{\partial Q}{\partial x}\right\rangle \\
&+ \left\langle y^{1-\alpha}\frac{\partial R}{\partial y}, -x^{1-\alpha}\frac{\partial R}{\partial x}, 0\right\rangle .
\end{aligned} \tag{15}$$

Hence, the right hand side of (13) can be expanded as

$$\begin{aligned}
&\iint\limits_S \operatorname{curl}^\alpha \overrightarrow{F} \cdot \overrightarrow{ds^\alpha} \\
&= \iint\limits_S \operatorname{curl}^\alpha \left(P\widehat{i}\right) \cdot \overrightarrow{ds^\alpha} + \iint\limits_S \operatorname{curl}^\alpha \left(Q\widehat{j}\right) \cdot \overrightarrow{ds^\alpha} \\
&+ \iint\limits_S \operatorname{curl}^\alpha \left(R\widehat{k}\right) \cdot \overrightarrow{ds^\alpha}.
\end{aligned} \tag{16}$$

Therefore, if we can prove that

$$\begin{aligned}
&\text{(i)} \iint\limits_S \operatorname{curl}^\alpha \left(P\widehat{i}\right) \cdot \overrightarrow{ds^\alpha} = \int\limits_C P\widehat{i} \cdot \overrightarrow{dr^\alpha}, \\
&\text{(ii)} \iint\limits_S \operatorname{curl}^\alpha \left(Q\widehat{j}\right) \cdot \overrightarrow{ds^\alpha} = \int\limits_C Q\widehat{j} \cdot \overrightarrow{dr^\alpha}, \\
&\text{(iii)} \iint\limits_S \operatorname{curl}^\alpha \left(R\widehat{k}\right) \cdot \overrightarrow{ds^\alpha} = \int\limits_C R\widehat{k} \cdot \overrightarrow{dr^\alpha},
\end{aligned} \tag{17}$$

then the proof is complete.

In the following, we will prove (i) and a similar argument can be established for both (i) and (ii).

Now, by (11) and (12), the left hand side of (i) can be reduced as follows

$$\iint_S \operatorname{curl}^\alpha\left(P\widehat{i}\right)\cdot d\overrightarrow{s^\alpha} = \iint_S \operatorname{curl}^\alpha\left(P\widehat{i}\right)\cdot \overrightarrow{n}_\alpha ds^\alpha, \tag{18}$$

where $\overrightarrow{n}_\alpha$ is unit fractional outward normal vector of S (Figure 2). For simplicity, we assume S to be parameterized with respect to the u, v coordinate system as follows:

$$\left.\begin{array}{c} S=\{(x,y,z)|\ x=x(u,v),\ y=y(u,v),\ z=z(u,v) \\ \text{where } (u,v)\in\ D=\Pr\limits_{(u,v)} S \end{array}\right\}.$$

Thus, the position vector $\overrightarrow{r^\alpha}$ of the curve C can be represented by the following prameterized equation $\overrightarrow{r^\alpha}=x(u,v)\widehat{i}+y(u,v)\widehat{j}+z(u,v)\widehat{k}$. Hence, by noting that

$$\begin{aligned}
\overrightarrow{n}_\alpha ds^\alpha &= \left(\overrightarrow{r_u^\alpha}\times\overrightarrow{r_v^\alpha}\right)du^\alpha dv^\alpha \\
&= \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ u^{1-\alpha}x_u & u^{1-\alpha}y_u & u^{1-\alpha}z_u \\ v^{1-\alpha}x_v & v^{1-\alpha}y_v & v^{1-\alpha}z_v \end{vmatrix} du^\alpha dv^\alpha \\
&= \Big[\left(u^{1-\alpha}v^{1-\alpha}y_u z_v - u^{1-\alpha}v^{1-\alpha}y_v z_u\right)\widehat{i} \\
&- \left(u^{1-\alpha}v^{1-\alpha}x_u z_v - u^{1-\alpha}v^{1-\alpha}x_v z_u\right)\widehat{j} \\
&+ \left(u^{1-\alpha}v^{1-\alpha}x_u y_v - u^{1-\alpha}v^{1-\alpha}x_v y_u\right)\widehat{k}\Big] du^\alpha dv^\alpha.
\end{aligned} \tag{19}$$

Therefore, using both (15) and (19) , the surface integral (18) becomes

$$\begin{aligned}
&\iint_S \operatorname{curl}^\alpha\left(P\widehat{i}\right)\cdot\overrightarrow{n}_\alpha ds^\alpha \\
&= \iint_D \left[\left(u^{1-\alpha}v^{1-\alpha}z^{1-\alpha}\frac{\partial P}{\partial z}\right)(x_v z_u - x_u z_v)\right. \\
&+ \left.\left(u^{1-\alpha}v^{1-\alpha}y^{1-\alpha}\frac{\partial P}{\partial y}\right)(x_v y_u - x_u y_v)\right] dA^\alpha
\end{aligned} \tag{20}$$

Finally, the right hand side of ((i), (17)) can be expanded as follows:

$$\begin{aligned}
\int_C P\widehat{i}\cdot d\overrightarrow{r^\alpha} &= \int_\Gamma P d\overrightarrow{x^\alpha} \\
&= \int_\Gamma P\left(u^{1-\alpha}x_u du^\alpha + v^{1-\alpha}x_v dv^\alpha\right) \\
&= \int_\Gamma \left\langle u^{1-\alpha}x_u P,\ v^{1-\alpha}x_v P\right\rangle \left\langle du^\alpha,\ dv^\alpha\right\rangle,
\end{aligned} \tag{21}$$

where Γ represents the boundary of the region D on the $u\upsilon-$plane that is obtained by the projection of the surface S.

Now, by applying the fractional Green's theorem [1] on the line integral (21) and utilizing the chain rule, we get

$$\begin{aligned}
&\int_{\Gamma} \langle u^{1-\alpha}x_u P,\ \upsilon^{1-\alpha}x_\upsilon P\rangle \langle du^{\alpha},\ d\upsilon^{\alpha}\rangle \\
&= \iint_{D} \left(\frac{\partial^{\alpha}\left(\upsilon^{1-\alpha}x_\upsilon P\right)}{\partial u^{\alpha}} - \frac{\partial^{\alpha}\left(u^{1-\alpha}x_u P\right)}{\partial \upsilon^{\alpha}} \right) du^{\alpha} d\upsilon^{\alpha} \\
&= \iint_{D} \left[u^{1-\alpha}\upsilon^{1-\alpha}x_\upsilon \left(x^{1-\alpha}x_u P_x + y^{1-\alpha}y_u P_y + z^{1-\alpha}z_u P_z \right) \right. \\
&- u^{1-\alpha}\upsilon^{1-\alpha}x_u \left(x^{1-\alpha}x_\upsilon P_x + y^{1-\alpha}y_\upsilon P_y + z^{1-\alpha}z_\upsilon P_z \right) \\
&\left. + P\upsilon^{1-\alpha}u^{1-\alpha}x_{\upsilon u} - Pu^{1-\alpha}\upsilon^{1-\alpha}x_{u\upsilon} \right] dA^{\alpha} \\
&= \iint_{D} \left[\left(u^{1-\alpha}\upsilon^{1-\alpha}z^{1-\alpha}P_z \right)\left(x_\upsilon z_u - x_u z_\upsilon \right) \right. \\
&\left. + \left(u^{1-\alpha}\upsilon^{1-\alpha}y^{1-\alpha}P_y \right)\left(x_\upsilon y_u - x_u y_\upsilon \right) \right] dA^{\alpha}. \qquad (22)
\end{aligned}$$

where $dA^{\alpha} = du^{\alpha}d\upsilon^{\alpha}$. Therefore, by (20) and (22), we get ((i), (17)). Hence, the proof is complete.

4. Conclusions

In this paper, utilizing the concept of conformable derivative, we establish fractional versions of the Stokes' theorem and some related vector calculus concepts.

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Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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