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NOVEL TWO PARAMETER DISTRIBUTION: PROPERTIES, SIMULATION AND APPLICATION IN INTERACTIVE DATA ANALYSIS AND FINANCE

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Abstract. This paper introduces a new two parameter generalized New XLindley distribution called Lazri Zeghdoudi(LZ) distribution. The proposed model provides more flexibility in modeling data with increasing hazard rate functions. Several statistical properties of the model were derived, such as shape, moments, order statistics, stochastic ordering, Lorenz Curve, stress-strength reliability, and actuarial measures. The unknown parameters of the new distribution were explored using several frequentest estimation approaches. The performance of the proposed distribution is illustrated using two real datasets from the fields of interactive data analysis and finance.

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1. Introduction

Numerous applied domains, including engineering, finance, and medicine, depend heavily on modeling and longevity data analysis. Many lifespan distributions have been used to model these kinds of data. The assumed probability model or distributions have a significant impact on the effectiveness of the procedures used in a statistical investigation. As a result, a great deal of effort has gone into developing large classes of traditional probability distributions and useful statistical methods. However, the data contradicts every one of the accepted probability models, leaving a number of important questions unanswered. Statistical models can be used to describe and predict real-world events. In recent years, a variety of distributions has been employed for data modeling in a variety of domains. Recent advances have centered on establishing new families that extend well known distributions while still allowing for a great deal of flexibility in data modeling in practice. Several distributions have

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been proposed in the statistical literature to modify lifetime data, including the Lindley, exponential, gamma, Weibull, Zeghdoudi, Xgamma, XLindley, new XLindley, *gamma Lindley, quasi*, *Lindley, new quasi Lindley, two parameter Lindley distribution I, two parameter Lindley distribution II, Pseudo Lindley, and Power XLindley* distributions. In this paper, we investigate a new polynomial exponential family that includes the distributions of XLindley and Xgamma, XLindley as well as Zeghdoudi as special instances, to introduce a new family of single-parameter continuous distributions. The existing literature on modeling survival data, biological sciences, and actuarial sciences will benefit from this new family of distributions.

In this study, a flexible extension of the XLindley distribution is introduced. The new distribution was derived using two parameters polynomial exponential family(see Belili et al.(2023)), with the probability density function is expressed as

$$f(t;\theta,\gamma) = b(\theta,\gamma)\left(a_0\left(\theta,\gamma\right) + a_1\left(\theta,\gamma\right)t\right)exp(-c\left(\theta,\gamma\right)t)$$

where $a_0\left(\theta,\gamma\right) = \gamma$, $a_1\left(\theta,\gamma\right) = \theta$, $c\left(\theta,\gamma\right) = \frac{\theta}{\gamma}$ and $b(\theta,\gamma) = \frac{\theta}{2\gamma^2}$, and the resultant model is named the "Lazri Zeghoudi(LZ)" distribution. Some important statistical properties are derived, including mode, quantile function, moments and their associated measures, actuarial (risk) measures, and reliability features such as survival, hazard (failure) rate, and mean residual life function. The parameters of the proposed distribution are estimated using the maximum likelihood approach. A comprehensive simulation study is also carried out to access the behavior-derived estimators. Two datasets are utilized to demonstration the applicability and usefulness of the new model. It is concluded that the Lazri Zeghoudi distribution is more flexible and efficiently analyzes both datasets as compared to competitive continus distributions.

The paper is organized as follows. Section 2 presents a new two parameter distribution with the study of its main properties such as: shape, survival function and failure rate, quantile function, moments, moment generating function, order statistics, stochastic ordering, Lorenz Curve, stress-strength reliability and actuarial measures. Then, estimation of the parameter is discussed in Section 3. Simulation study and applications are developed in Sections 4 and 5. Some concluding remarks are given in Section 6.

2. Main Results

2.1. **The shape of the LZ distribution.** Assume X is a random variable with values in the range $]0,\infty[$, and the distribution of X depends on indeterminate parameters θ and γ with values in the range $]0,\infty[$. The density function of the LZ distribution, given by

$$f(x;\theta,\gamma) = \begin{cases} \frac{\theta}{2\gamma^2}\left(\gamma + \theta x\right)\exp\left(-\frac{\theta}{\gamma}x\right) & x,\theta,\gamma > 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The behavior of $f_{LZ}\left(x\right)$ at $x=0$ and $x=\infty$, respectively, is given by

$$\lim_{x\to 0} f(x) = \frac{\theta}{2\gamma},$$

$$\lim_{x\to\infty} f(x) = 0.$$

The derivative with respect to x of Eq. (1) is given by

$$\frac{d}{dx}f(x) = -\frac{\theta^3}{2\gamma^3}x\exp\left(-\frac{\theta}{\gamma}x\right) < 0, \tag{2}$$

then $f(x)$ is decreasing

and the second derivative of LZ distribution is

$$\frac{d^2}{dx^2}f(x) = \frac{\theta^3}{2\gamma^4}(\theta x-\gamma)\exp\left(-\frac{\theta}{\gamma}x\right) \tag{3}$$

with $\frac{d^2}{dx^2}f(x) < 0, x < \frac{\gamma}{\theta}$ and $\frac{d^2}{dx^2}f(x) > 0, x > \frac{\gamma}{\theta}$.

2.2. **Survival function and failure rate.** The cumulative distribution function of the L-Z distribution is

$$F(x;\theta,\gamma) = 1-\left[1+\frac{\theta}{2\gamma}x\right]\exp\left(-\frac{\theta}{\gamma}x\right) \tag{4}$$

The survival function of the new distribution is given by

$$S\left(x\right) = 1-F(x) = \left[1+\frac{\theta}{2\gamma}x\right]\exp\left(-\frac{\theta}{\gamma}x\right) \tag{5}$$

The hazard function associated with LZ distribution is

$$h\left(x\right) = \frac{f(x)}{1-F(x)} = \frac{\theta(\gamma+\theta x)}{\gamma(2\gamma+\theta x)}, x>0, \theta, \gamma > 0. \tag{6}$$

The behavior of $h_{LZ}\left(x\right)$ at $x=0$ and $x=\infty$, respectively, is given by

$$\lim_{x\to 0} h(x) = \frac{\theta}{2\gamma},$$

$$\lim_{x\to\infty} h(x) = \frac{\theta}{\gamma}.$$

Proposition 1. *The HRT $h_{LZ}(x)$ of the LZ distribution is: Increasing*

Proof.

$\Psi(x) = -\frac{\acute{f}(x)}{f(x)} = \frac{\theta^2}{\gamma}\frac{x}{(\gamma+\theta x)}$

The first derivative of $\Psi(x)$ is

$\acute{\Psi}(t) = \frac{\theta^2}{\gamma}\frac{\gamma}{(\gamma+\theta x)^2} > 0$

It follows from Theorem (b) of Glaser (1980) that the failure rate is increasing.

2.3. **The quantile function of LZ distribution.** F_X is continuous and strictly increasing, so the quantile function of X is $Q_X(u) = F_X^{-1}(u)$; $0 < u < 1$: In the following theorem, we give an explicit expression for Q_X in terms of the Lambert W function. For more details on Lambert W function, we refer the reader to Jodra(2010).

Theorem 2. *The quantile function of the LZ distribution is defined as follows*

$$x_u = F_X^{-1}(u) = -\frac{\gamma}{\theta}\left[2 + W_{-1}\left[2\left(u-1\right)e^{-2}\right]\right]. \tag{7}$$

where $W_{-1}(\cdot)$ *denotes the negative branch of the Lambert* W *function.* ($W(z)e^{(W(z))} = z$*, where* z *is a complex number*)

Proof.

For any fixed $\theta, \gamma > 0$, let $u \in (0, 1)$. We have to solve the equation $F_X(x) = u$ with respect to x, for $x > 0$.We have to solve the following equation

$$-(2 + \frac{\theta}{\gamma}x)e^{-\left(\frac{\theta}{\gamma}x\right)} = 2\left(u-1\right)$$

Multiplying by e^{-2} both sides, we obtain:

$$-(2 + \frac{\theta}{\gamma}x)e^{-\left(2+\frac{\theta}{\gamma}x\right)} = 2\left(u-1\right)e^{-2}$$

we see that $-(2 + \frac{\theta}{\gamma}x)$ is the Lambert W function of the real argument $2\left(u-1\right)e^{-2}$. Thus, we have

$$LambertW\left(2\left(u-1\right)e^{-2}\right) = -(2 + \frac{\theta}{\gamma}x) \tag{8}$$

Moreover , for any, $\theta, \gamma > 0$ and $x > 0$ it is immediate that $(2 + \frac{\theta}{\gamma}x) > 1$ and it can also be checked that$-(2 + \frac{\theta}{\gamma}x)e^{-\left(2+\frac{\theta}{\gamma}x\right)} = 2\left(u-1\right)e^{-2} \in (-\frac{1}{e}, 0)$ since $u \in (0, 1)$ .There for,by taking into account the properties of the negative branch of the Lambert W function, equation (2.8) becomes

$$LambertW\left(-1, 2\left(u-1\right)e^{-2}\right) = -(2 + \frac{\theta}{\gamma}x)$$

Again, solving for x, we get

$$Q_X\left(u\right) = x_u = -\frac{\gamma}{\theta}\left[2 + W_{-1}\left[2\left(u-1\right)e^{-2}\right]\right]$$

. Setting $u = 0.25, 0.50, and\ 0.75$ in (2.4), the three quartiles of the LZD can be obtained.

$F_X^{-1}(0.25) = -\frac{\gamma}{\theta}\left[2 + W_{-1}\left[2\left(\frac{1}{4}-1\right)e^{-2}\right]\right] = -\frac{\gamma}{\theta}\left(-0.518\right)$

$F_X^{-1}(0.5) = -\frac{\gamma}{\theta}\left[2 + W_{-1}\left[2\left(\frac{1}{2}-1\right)e^{-2}\right]\right] = -\frac{\gamma}{\theta}\left(-1.1461\right)$

$F_X^{-1}(0.75) = -\frac{\gamma}{\theta}\left[2 + W_{-1}\left[2\left(\frac{3}{4}-1\right)e^{-2}\right]\right] = -\frac{\gamma}{\theta}\left(-2.1055\right)$

Table 1 shows some quantiles of the LZ distribution, which have been calculated from the closed form expression for Q_X given in **Theorem 2.2**.

u	$\theta=0.1,\gamma=0.5$	$\theta=0.5,\gamma=0.2$	$\theta=1,\gamma=2$	$\theta=2,\gamma=1.5$	$\theta=5,\gamma=5$
0.01	0.1	0.008	0.04	0.0150	0.0200
0.05	0.5007	0.0401	0.2003	0.0751	0.1001
0.1	1.0061	0.0805	0.4025	0.1509	0.2012
0.2	2.0468	0.1637	0.8187	0.3070	0.4094
0.25	2.59	0.2072	1.036	0.3885	0.518
0.3	3.1541	0.2523	1.2617	0.4731	0.6308
0.4	4.3653	0.3492	1.7461	0.6548	0.8731
0.5	5.731	0.4585	2.2924	0.8596	1.1462
0.6	7.3310	0.5865	2.9324	1.0997	1.4662
0.75	10.5273	0.8422	4.2109	1.5791	2.1055
0.8	11.9864	0.9589	4.7946	1.798	2.3973
0.99	29.9512	2.3961	11.9805	4.4927	5.9902

Table 1. Some quantiles values of the LZ distribution

2.4. **Moment generating function.** The moment generating function of a $X \sim LZ\,(\theta)$ random variable

is given as

$$\begin{aligned} M_X\,(t) &= E\left(e^{tX}\right) = \int_0^\infty e^{tx} f\,(x)\,dx \\ &= \frac{\theta}{2\gamma^2}\int_0^\infty e^{tx}(\gamma+\theta x)e^{-\frac{\theta}{\gamma}x}dx \\ &= \frac{\theta}{2\gamma^2}\int_0^\infty (\gamma+\theta x)e^{(t-\frac{\theta}{\gamma})x}dx \\ &= \frac{\theta}{2\gamma^2}\left[\int_0^\infty \gamma e^{(t-\frac{\theta}{\gamma})x}dx + \int_0^\infty \theta x e^{(t-\frac{\theta}{\gamma})x}dx\right] \\ &= \frac{\theta}{2}\left(\frac{2\theta-\gamma t}{(\gamma t-\theta)^2}\right) \end{aligned} \tag{9}$$

2.5. **Characteristic function.** The moment characteristic function of a $X \sim LZ\,(\theta)$ random variable is given by

$$\Phi_X\,(it) = \frac{\theta}{2}\left(\frac{2\theta-\gamma it}{(\gamma it-\theta)^2}\right)$$

2.6. **Moments.** The rth moment about the origin of the LZ distribution can be obtained as:

$$\mu_r' = E(X^r) = \int_0^\infty x^{(r)} f(x)dx = \frac{(r+1)!+r!}{2}\left(\frac{\gamma}{\theta}\right)^r$$

In particular, we have

$$E\,(X\ ) = \frac{3}{2}\,\frac{\gamma}{\theta} \tag{10}$$

$$E\left(X^2\right)=\frac{8}{2}\left(\frac{\gamma}{\theta}\right)^2 \tag{11}$$

The variance the LZ distribution is

$$Var(X)=\frac{7}{4}\left(\frac{\gamma}{\theta}\right)^2$$

The coefficient of variation β is :

$$\beta=\frac{\sqrt{Var(X)}}{E\left(X\ \right)}=\frac{\sqrt{7}}{3}$$

The skewness and the kurtosis for the LZ distribution are respectively expressed as

$$\gamma_1=\frac{E\left(X^{\ 3}\right)}{(Var(X))^{\frac{3}{2}}}=\frac{120}{\sqrt[3]{7}}$$

$$\gamma_2=\frac{E\left(X^{\ 4}\right)}{(Var(X))^{2}}=\frac{1152}{98}$$

Theorem 3. *Let* $X\sim LZ\left(\theta,\gamma\right)$ *. Then* $Median(X)<E(X)$

Proof.

Let $x_{0.5}=Median(X)$ and $\mu=E(X)$, $\mu=\frac{3}{2}\frac{\gamma}{\theta}$, $F\left(x_{0.5}\right)=0.5$.

It is easy to see that the theorem holds by the following substitution in the cdf in equation (2.2)

$$F\left(\mu\right)=1-\frac{7}{4}e^{-\frac{3}{2}}$$

Note that $0.5<1-\frac{7}{4}e^{-\frac{3}{2}}$.Finally, since $F\left(x\right)$ is an increasing function in $x>0$ for all $\theta>0$, we have $x_{0.5}<\mu$.

2.7. **Information measure and asymptotic behaviour of LZ distribution.** Entropy is the quantity of uncertainty or randomness in a system. It is an information measure for non-negative $s\neq 1$. The Rény Entropy for LZ distributed random variable X is

$$R_s(x)=\frac{1}{1-s}\log\left\{\int_0^{\infty} f^s(x)dx\right\}$$

where $s>0$ and $s\neq 1$.

$$\begin{aligned} R_s(x) &= \frac{1}{1-s}\log\int_0^{\infty}\left(\frac{\theta}{2\gamma}\left(1+\frac{\theta}{\gamma}x\right)e^{\left(-\frac{\theta}{\gamma}x\right)}\right)^s dx \\ &= \frac{1}{1-s}\log\int_0^{\infty}\frac{\theta^s}{2^s\gamma^s}\left(1+\frac{\theta}{\gamma}x\right)^s e^{\left(-\frac{\theta}{\gamma}sx\right)}dx \end{aligned}$$

using binomial expansion, we find

$$\frac{\theta^s}{2^s\gamma^s}\int_0^\infty\left(1+\frac{\theta}{\gamma}x\right)^s e^{\left(-\frac{\theta s}{\gamma}x\right)}dx = \frac{\theta^s}{2^s\gamma^s}\sum_{j=0}^\infty \frac{s!}{j!(s-j)!}\int_0^\infty\left(\frac{\theta}{\gamma}x\right)^j e^{\left(-\frac{\theta s}{\gamma}x\right)}dx$$
$$= \frac{\theta^s}{2^s\gamma^s}\sum_{j=0}^\infty \frac{s!}{j!(s-j)!}\left(\frac{\theta}{\gamma}\right)^j\Gamma\left(j+1\right)\left(\frac{\theta s}{\gamma}\right)^{-j-1}$$

Now, the Rényi entropy for the LZ model is determined as follows

$$R_s(x)=\frac{1}{1-s}\log\left(\frac{\theta^{s+1}}{2^s\gamma^{s+1}}\sum_{j=0}^\infty\frac{s!}{(s-j)!s^{j+1}}\right).$$

2.8. **Distribution of the order statistics.** Suppose $X_1, X_2, ..., X_n$ is a random sample of $X_{(r)}$; $(r = 1, 2, ..., n)$ are the r^{th} order statistics obtained by arranging X_r in ascending order of magnitude $X_1 \le X_2 \le ... \le X_r$ and $X_1 = \min(X_1, X_2, ..., X_r)$, $X_r = \max(X_1, X_2, .., X_r)$ then the probability density function of the r^{th} order statistics is given by

$$f_{r:n}(x;\theta)=\frac{n!}{(r-1)!\,(n-r)!}f_{LZ}(x;\theta,\gamma)\left[F_{LZ}(x;\theta,\gamma)\right]^{r-1}\left[1-F_{LZ}(x;\theta,\gamma)\right]^{n-r}$$

where $f(.)$ and $F(.)$ are the pdf and cdf of Lazri-Zegh distribution respectively. Hence, we have

$$f_{r:n}(x;\theta) = \frac{n!}{(r-1)!\,(n-r)!}\frac{\theta}{2\gamma^2}\left(\theta x+\gamma\right)e^{-\frac{\theta}{\gamma}x}\left[1-\left(1+\frac{\theta}{2\gamma}x\right)e^{-\frac{\theta}{\gamma}x}\right]^{r-1}$$
$$\left[\left(1+\frac{\theta}{2\gamma}x\right)e^{-\frac{\theta}{\gamma}x}\right]^{n-r}$$

The pdf of the largest order statistics is obtained by setting $r = n$

$$f_{n;n}(x;\theta)=\frac{n\theta}{2\gamma^2}\left(\theta x+\gamma\right)e^{-\frac{\theta}{\gamma}x}\left[1-\left(1+\frac{\theta}{2\gamma}x\right)e^{-\frac{\theta}{\gamma}x}\right]^{n-1}$$

The pdf of the smallest order statistics is obtained by setting $r = 1$

$$f_{1:n}(x;\theta)=\frac{n\theta}{2\gamma^2}\left(\theta x+\gamma\right)\left(1+\frac{\theta}{2\gamma}x\right)^{n-1}e^{-\frac{\theta}{\gamma}nx}$$

2.9. **Stochastic ordering of LZ distribution.** The stochastic ordering of a non-negative continuous random variable is a vital tool for comparing the behavior of system components. A random variable X is said to be smaller than another random variable Y in the Stochastic order $(X \le_{st} Y)$ if $F_x(x) \ge F_Y(x)$ $\forall x$ Hazard rate order $(X \le_{hr} Y)$ if $h_x(x) \ge h_y(x)$ $\forall x$ Mean residual life order $(X \le_{mrl} Y)$ if $m_X(x) \ge m_y(y)$ $\forall x$ Likelihood ratio order $X \le_{lr} Y$ if $\frac{f_X(x)}{f_Y(x)}$ decreases in x. This implies that

$$X \le_{lr} Y \Rightarrow X \le_{hr} Y \Rightarrow X \le_{st} Y \Rightarrow X \le_{mrl} Y$$

Theorem 4. *Let $X \sim LZ(\theta_1)$ and $Y \sim LZ(\theta_2)$. If $\theta_1 \succ \theta_2$ and $\gamma_1 = \gamma_2$ then $X \le_{lr} Y$ hence $X \le_{hr} Y, X \le_{mrl} Y$ and $X \le_{st} Y$*

Proof.
$\frac{f_X(x)}{f_Y(x)} = \frac{\frac{\theta_1}{2\gamma_1^2}(\gamma_1+\theta_1 x)\exp\left(-\frac{\theta_1}{\gamma_1}x\right)}{\frac{\theta_2}{2\gamma_2^2}(\gamma_2+\theta_2 x)\exp\left(-\frac{\theta_2}{\gamma_2}x\right)} = \frac{\theta_1\gamma_2^2}{\theta_2\gamma_1^2}\frac{(\gamma_1+\theta_1 x)}{(\gamma_2+\theta_2 x)}e^{\left(\frac{\theta_2}{\gamma_2}-\frac{\theta_1}{\gamma_1}\right)x}$

Taking natural log of the ratio will yield

$\ln\frac{f_X(x)}{f_Y(x)} = \ln\left(\frac{\theta_1\gamma_2^2}{\theta_2\gamma_1^2}\right) + \ln\left(\frac{\gamma_1+\theta_1 x}{\gamma_2+\theta_2 x}\right) + \left(\frac{\theta_2}{\gamma_2}-\frac{\theta_1}{\gamma_1}\right)x$

Differentiating the natural logarithm of the ratio with respect to x yields

$\frac{d}{dx}\ln\frac{f_X(x)}{f_Y(x)} = \frac{\theta_1\gamma_2-\theta_1\gamma_1}{(\gamma_1+\theta_1 x)(\gamma_2+\theta_2 x)} + \left(\frac{\theta_2\gamma_1-\theta_1\gamma_2}{\gamma_1\gamma_2}\right)$

If $\theta_1 \geq \theta_2$ and $\gamma_1 = \gamma_2$, we have $\frac{d}{dx}\ln\left(\frac{f_X(x)}{f_Y(x)}\right) \leq 0$. This means that $X \leq_{lr} Y$.

2.10. **Lorenz Curve.** Let X be a random variable pdf $f(x)$ and the cdf $F(x)$,the Lorenz curve L is given by

$$L(F(x)) = \frac{\int\limits_{-\infty}^{x} tf(t)dt}{E(X)}$$

where $E(X)$ denotes the average.The Lorenz curve $L(F)$ may then be plotted as a function parametric in x: $L(x)$ vs$F(x)$.In other contexts, the quantity computed here is known as size-biased distribution; it also has an important role in renewal theory

We have it for LZD

$$\int\limits_{0}^{x} tf(t)dt = \frac{3\gamma}{2\theta} - \frac{1}{2\theta\gamma}e^{-\frac{\theta}{\gamma}x}\left(3\gamma^2+3\theta\gamma x+\theta^2x^2\right)$$

We obtain the Lorenz curve for the LZD as follows

$$L(p) = 1-(1-p)\frac{\left(\frac{\theta^2}{3\gamma}x^2+\theta x+\gamma\right)}{\left(1+\frac{\theta}{2\gamma}x\right)}$$

where $x = F^{-1}(p)$ with $F(.)$ given by (2.4).

2.11. **Stress-Strength reliability.** We examine the Stress-Strength Reliability of LZ distribution. The stress-strength reliability measures the life of a component that possesses random strength X and subjected to random stress Y.

Theorem 5. *Suppose X and Y are independent random variables denoting strength and stress of a component. We assume further that X and Y follow LZ distribution with pdf given in equation* (2.1), *with parameter θ_1 and θ_2 respectively. Then, the stress-strength reliability is obtained as follows*

$$R = P(Y < X) = 1 - \frac{\theta_1}{4\gamma}\left[\frac{(2\gamma+\theta_2)(2\theta_1+\theta_2)}{(\theta_1+\theta_2)^2}\right].$$

Proof.

We have

$$R \;=\; P(Y<X) = \int_0^\infty P(Y<X/X=x)f_X(x)dx$$

$$= \int_0^\infty f(x,\theta_1,\gamma)F(x,\theta_2,\gamma)dx$$
$$= \frac{\theta_1}{2\gamma^2}\int_0^\infty (\gamma+\theta_1 x)\, e^{\left(-\frac{\theta_1}{\gamma}x\right)} \left(1-\left(1+\frac{\theta_2}{2\gamma}x\right)e^{\left(-\frac{\theta_2}{\gamma}x\right)}\right)dx$$
$$= 1-\frac{\theta_1}{4\gamma}\left[\frac{(2\gamma+\theta_2)(2\theta_1+\theta_2)}{(\theta_1+\theta_2)^2}\right].$$

2.12. **The quantile function or value at risk of the LZ distribution.** The quantile function of the LZ distribution is defined as follows

$$VaR = x_u = F_X^{-1}(u) = -\frac{\gamma}{\theta}\left[2+W_{-1}\left[2(u-1)e^{-2}\right]\right]. \tag{12}$$

where $W_{-1}(\cdot)$ denotes the negative branch of the Lambert W function.

Definition 1. *Risk managers use value at risk (VaR) to measure and control the level of risk exposure. The mathematical definition is*

$$VaR = \inf\left\{x\in\mathbb{R}, P(X>x\leq 1-p)\right\},$$

where $p\in(0,1)$ is the level. The formula tells us what the maximum loss we can expect tomorrow, with normal market conditions, or what amount of loss we should not exceed with a given level of probability, thus VaR is also known as a quantile risk measure and is defined as $VaR = F^{-1}(p)$ for a continuous distribution is.

2.13. **Mean excess function.** For a claim amount random variable X, the mean excess or residual life function is the expected payment per claim on a policy with a fixed amount deductible of x, where claims with amounts less than or equal to x is thoroughly ignored. It is defined for the LZ distribution as follows

$$e(x) = E(X-x/X>x) = \frac{1}{1-F(x)}\int_x^\infty (1-F(u))du,$$

where

$$\int_x^\infty (1-F(u))du = \int_x^\infty \left(1+\frac{\theta}{2\gamma}u\right)e^{-\frac{\theta}{\gamma}u}du = \frac{(3\gamma+\theta x)}{2\theta}e^{-\frac{\theta}{\gamma}x}.$$

Then, we have

$$e(x) = \frac{\gamma(3\gamma+\theta x)}{\theta(2\gamma+\theta x)}.$$

2.14. **Limited expected value function.** The limited expected value function L of a claim size variable X, or of the corresponding c.d.f $F(x)$, is defined as follows

$$L(u) = E\{\min(X,u)\} = \int_0^u xf(x)dx + u(1-F(u)), u>0.$$

The value of the function L at point x is equal to the expectation of the c.d.f $F(x)$ truncated at this point. Given a policy limit or deductible from a reinsurance perspective, say u, a limited loss random variable is defined as follows

$$X\Lambda u=\min(X,u)=\begin{cases}X, & X\leq u\\ u, & X>u\end{cases}$$

The limited expected value function is defined as the expectation of the limited which is calculated as follows

$$\begin{aligned}E\left(X\Lambda u\right) &= \int_0^u xf(x)dx+u(1-F(u))\\ &= m_1(u)+u\left(1-F(u)\right)\end{aligned}$$

where

$$m_1(u)=\int_0^u xf(x)dx=\frac{3\gamma}{2\theta}-e^{-\frac{\theta}{\gamma}u}\left(\frac{\theta}{2\gamma}u^2+\frac{3}{2}u+\frac{3\gamma}{2\theta}\right)$$

Then, we have

$$E\left(X\Lambda u\right)=\frac{3\gamma}{2\theta}-\left(\frac{1}{2}u+\frac{3\gamma}{2\theta}\right)e^{-\frac{\theta}{\gamma}u}$$

2.15. **Tail value at risk.** The tail value at risk $(TVaR)$ also known as the tail conditional expectation is a risk measure associated with the general value at risk. $TVaR$ measures the expectation of the losses beyond VaR. The $TVaR$ is defined for the LZ distribution as follows

$$TVaR=E\left(X/X>VaR\right)=\frac{1}{1-p}\int_{VaR}^{\infty}xf(x)dx$$

Where

$$\begin{aligned}\int_{VaR}^{\infty}xf(x)dx &= \frac{\theta}{2\gamma^2}\int_{VaR}^{\infty}x\left(\gamma+\theta x\right)\exp\left(-\frac{\theta}{\gamma}x\right)dx\\ &= \frac{\theta}{2\gamma^2}\left(VaR^2+\frac{3\gamma}{\theta}VaR+\frac{3\gamma^2}{\theta^2}\right)e^{-\frac{\theta}{\gamma}VaR}\end{aligned}$$

Then, we have

$$TVaR=\frac{\theta}{2\gamma^2\left(1-p\right)}\left(VaR^2+\frac{3\gamma}{\theta}VaR+\frac{3\gamma^2}{\theta^2}\right)e^{-\frac{\theta}{\gamma}VaR}$$

Although it virtually always represents a loss, VaR is conventionally reported as a positive number.

2.16. **Tail variance.** Tail variance (TV) measures losses' conditional variance, given that they exceed VaR at a given probability P. TV is defined for the LZ distribution as follows

$$TV=E\left(X^2/X>VaR\right)-(TVaR)^2=\frac{1}{1-p}\int_{VaR}^{\infty}x^2f(x)dx-(TVaR)^2$$

Where

$$\frac{1}{1-p}\int_{VaR}^{\infty}x^2f(x)dx=\frac{\theta}{2\gamma\left(1-p\right)}\left(VaR^3+\frac{4\gamma}{\theta}VaR^2+\frac{8\gamma^2}{\theta^2}VaR+\frac{8\gamma^3}{\theta^3}\right)e^{-\frac{\theta}{\gamma}VaR}$$

Then, we have

$$\begin{aligned} TV &= \frac{\theta}{2\gamma(1-p)}\left(VaR^3+\frac{4\gamma}{\theta}VaR^2+\frac{8\gamma^2}{\theta^2}VaR+\frac{8\gamma^3}{\theta^3}\right)e^{-\frac{\theta}{\gamma}VaR} \\ &\quad -\frac{\theta^2}{4\gamma^4(1-p)^2}\left(VaR^2+\frac{3\gamma}{\theta}VaR+\frac{3\gamma^2}{\theta^2}\right)^2 e^{-\frac{2\theta}{\gamma}VaR} \end{aligned}$$

3. Estimation Methods

3.1. **Maximum Likelihood Estimates (MLE).** Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from the PDF of LZ model. A sorted random sample from the suggested distribution is $x_{1:n}, x_{2:n}, \ldots, x_{n:n}$. The ln-likelihood function, $\ln l(x_i;\theta,\gamma)$ is:

$$\ln l(x;\theta,\gamma) = n\ln\theta - n\ln 2 - 2n\ln\gamma + \sum \ln(\theta x_i+\gamma) - \frac{\theta}{\gamma}\sum x_i$$

The derivatives of $\ln l(x;\theta,\gamma)$ with respect to θ and γ are:

$$\begin{aligned} \frac{d\ln l(x;\theta,\gamma)}{d\theta} &= \frac{n}{\theta} + \sum \frac{x_i}{(\theta x_i+\gamma)} - \frac{1}{\gamma}\sum x_i \\ \frac{d\ln l(x;\theta,\gamma)}{d\gamma} &= -\frac{2n}{\gamma} + \sum \frac{1}{(\theta x_i+\gamma)} + \frac{\theta}{\gamma^2}\sum x_i \end{aligned}$$

The MLE is implemented using Newton-Raphson's numerical iterative method since it has no closed-form solution.

3.2. **Least-Squares Estimate (OLSE).** Another major technique that is employed in lieu of MLE is the conventional least-squares estimate (OLSE) by minimizing the following formula

$$O = \sum_{i=1}^{n}\left[F(x_{i:n}) - \frac{i}{n+1}\right]^2.$$

3.3. **Anderson Darling Estimate (ADE).** Anderson Darling estimate (ADE) is another key technique that is employed in place of MLE by minimizing the following formula

$$A = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)[\log F(x_{i:n}) + \log S(x_{i:n})].$$

4. Simulation Study

For this purpose, 5000 trials were used to estimate MLE, OLSE and ADE, and estimate the mean square errors (MSEs), for the sample size $n = 20, 40, 60, 100$, and 200.The MSEs is derived using the following equalities

$$MSE\left(\hat{\theta},\hat{\gamma}\right) = \frac{1}{5000}\sum_{i=1}^{5000}\left(\left(\hat{\theta},\hat{\gamma}\right) - (\theta,\gamma)\right)^2.$$

n	MLE		MSE		$OLSE$		MSE		ADE		MSE	
	$\hat{\theta}$	$\hat{\gamma}$	$\hat{\theta}$	$\hat{\gamma}$	$\hat{\theta}$	$\hat{\gamma}$	$\hat{\theta}$	$\hat{\gamma}$	$\hat{\theta}$	$\hat{\gamma}$	$\hat{\theta}$	$\hat{\gamma}$
20	0.5108	0.1039	0.0004	0.0011	0.5212	0.1122	0.0007	0.0016	0.5109	0.1038	0.0004	0.0011
40	0.5095	0.1028	0.0003	0.0008	0.5132	0.1099	0.0005	0.0012	0.5096	0.1021	0.0003	0.0008
60	0.5077	0.1019	0.0002	0.0006	0.5105	0.1057	0.0003	0.0008	0.5077	0.1019	0.0002	0.0006
100	0.5052	0.1010	0.0001	0.0003	0.5098	0.1043	0.0001	0.0005	0.5051	0.1012	0.0001	0.0003
300	0.5036	0.1006	0.0001	0.0001	0.5045	0.104	0.0001	0.0001	0.5035	0.1005	0.0001	0.0001

Table 2: Simulation values using MLE ,OLSE and ADE methods and MSE for $(\theta, \gamma) = (0.5, 0.1)$

n	MLE		MSE		$OLSE$		MSE		ADE		MSE	
	$\hat{\theta}$	$\hat{\gamma}$	$\hat{\theta}$	$\hat{\gamma}$	$\hat{\theta}$	$\hat{\gamma}$	$\hat{\theta}$	$\hat{\gamma}$	$\hat{\theta}$	$\hat{\gamma}$	$\hat{\theta}$	$\hat{\gamma}$
20	0.5285	0.9279	0.0014	0.0012	0.5298	0.9388	0.0017	0.0015	0.5267	0.927	0.0013	0.0011
40	0.5177	0.9157	0.0012	0.0010	0.5205	0.9198	0.0014	0.0012	0.5155	0.9151	0.0012	0.0010
60	0.5154	0.9128	0.0009	0.0008	0.5178	0.9145	0.0010	0.0009	0.5142	0.9125	0.0008	0.0008
100	0.5076	0.9036	0.0005	0.0005	0.5096	0.9089	0.0007	0.0006	0.5031	0.9033	0.0005	0.0005
300	0.5026	0.9024	0.0003	0.0003	0.5042	0.9051	0.0004	0.0003	0.5024	0.9022	0.0003	0.0003

Table 3: Simulation values using MLE, OLSE and ADE methods and MSE for $(\theta, \gamma) = (0.5, 0.9)$

n	MLE		MSE		$OLSE$		MSE		ADE		MSE	
	$\hat{\theta}$	$\hat{\gamma}$	$\hat{\theta}$	$\hat{\gamma}$	$\hat{\theta}$	$\hat{\gamma}$	$\hat{\theta}$	$\hat{\gamma}$	$\hat{\theta}$	$\hat{\gamma}$	$\hat{\theta}$	$\hat{\gamma}$
20	1.0128	0.4946	0.0062	0.0051	1.0281	0.4935	0.0075	0.006	1.0127	0.4945	0.0061	0.0051
40	1.0086	0.4989	0.0051	0.0038	1.0192	0.4958	0.0062	0.0047	1.0085	0.4988	0.0050	0.0038
60	1.0078	0.5006	0.0035	0.0026	1.0098	0.5001	0.0046	0.0032	1.0076	0.5005	0.0035	0.0026
100	1.0037	0.5015	0.0018	0.0013	1.0055	0.5005	0.0026	0.0021	1.0035	0.5014	0.0018	0.0013
300	1.0008	0.5038	0.0012	0.0010	1.0018	0.5017	0.0015	0.0014	1.0006	0.5035	0.0012	0.0010

Table 4: Simulation values using MLE ,OLSE and ADE methods and MSE for $(\theta, \gamma) = (1, 0.5)$

n	MLE		MSE		$OLSE$		MSE		ADE		MSE	
	$\hat{\theta}$	$\hat{\gamma}$	$\hat{\theta}$	$\hat{\gamma}$	$\hat{\theta}$	$\hat{\gamma}$	$\hat{\theta}$	$\hat{\gamma}$	$\hat{\theta}$	$\hat{\gamma}$	$\hat{\theta}$	$\hat{\gamma}$
20	2.0132	3.0119	0.0142	0.0121	2.0145	3.0124	0.0146	0.0124	2.0130	3.0117	0.0141	0.0120
40	2.0121	3.0113	0.0121	0.0108	2.0133	3.0116	0.0124	0.0109	2.0120	3.0111	0.0120	0.0106
60	2.0110	3.0107	0.0105	0.0096	2.0118	3.0104	0.0107	0.0098	2.0110	3.0105	0.0102	0.0094
100	2.0102	3.0102	0.0098	0.0053	2.0107	3.0101	0.0099	0.0055	2.0101	3.0101	0.0095	0.0051
300	2.0065	3.0058	0.0062	0.0022	2.0071	3.006	0.0066	0.0024	2.0062	3.0055	0.0060	0.0020

Table 5: Simulation values using MLE ,OLSE and ADE methods and MSE for $(\theta, \gamma) = (2, 3)$

Remark 1. *The simulation findings are given in Tables 2-4. It is observed that the MLE and ADE methods are good then OLSE method. Also, all methods are asymptotically unbiased as the sample size increases. Its is also seen that the MSEs are approaching zero with an increase in sample size.*

5. Applications

In this section, two datasets from various disciplines are utilized for application purposes. These datasets are based on "Finance Data (Ratio)" and "*Interactive Data Analysis*". The proposed distribution is compared with some famous count models available in literature such *two-parameter L1, gamma Lindley, quasi Lindley,* new *quasi Lindley, two parameter L2, Pseudo Lindley, Xlindley and* Power XLindley distributions.

The parameters of all competitive distributions are estimated using the ML estimation method. For comparison and identification of the best-fitted model, we used the following information and goodness-of-fit criteria, maximum log-likelihood, AIC, BIC, and AICC.

The first data set contains *Finance Data* (*Ratio*) counts from *https://data.world › datasets › finance*. Table 6 shows the parameter estimates and goodness-of-fit metrics for all fitted distributions. The suggested distribution matches this data set fairly well, as seen in Table 6.

Data set 1: Finance Data (*Ratio*) 0.09784662, 0.1002, 0.09420406, 0.09563, 0.11433458, 0.1164, 0.10697413, 0.1085, 0.08633513, 0.08743, 0.11151292, 0.1132, 0.09633629, 0.09831, 0.07243469, 0.07431, 0.08905936, 0.09026.

$Model$	θ	γ	AIC	BIC	$-2L$	$AICC$
two-parameter L1	12.02589	0.000562286	48.71469	46.93395	52.71469	47.91469
gamma Lindley	2.476847	0.0007199642	258.3068	256.5261	262.3068	257.5068
quasi Lindley	27.0735	0.0004977422	173.1456	171.3649	177.1456	172.3456
new quasi Lindley	12.07276	48.41638	44.86238	43.08163	48.86238	44.06238
two parameter L2	4.097152	0.0009746497	48.49152	46.71077	52.49152	47.69152
Pseudo Lindley	10.4013	77.09507	43.84285	42.06211	47.84285	43.04285
Xlindley	10.34574	/	45.84092	44.95055	47.84092	45.59092
Power XLindley	474.6577	2.646587	77.07403	75.29328	81.07403	76.27403
Lazri Zeghdoudi	1.625896	0.09805427	32.39848	30.61773	36.39848	31.59848

Table 6: Parameter estimates and model comparison characteristics for the first data set

The second data set is interactive data analysis given in Table 7 taken from *McNeil, D. R.* (*1977*). Table 7 displays the MLEs for all fitted models. The LZ distribution is proven to produce more efficient fits than competitor distributions.

Data set 2:

3.93, 5.31, 7.24, 9.64, 12.90, 17.10, 23.20, 31.40, 39.80, 50.20, 62.90, 76.00, 92.00, 105.70, 122.80, 131.70, 151.30, 179.30, 203.20.

$Model$	θ	γ	AIC	BIC	$-2L$	$AICC$
two-parameter L1	0.02166821	44.2597	203.7026	205.5914	199.7026	204.4526
gamma Lindley	0.01797983	0.02366775	203.6523	205.54126	199.6523	204.4023
quasi Lindley	0.04057428	0.2043486	336.5028	338.3917	332.5028	337.2528
new quasi Lindley	0.02478781	0.001029806	204.5499	206.4388	200.5499	205.2999
two parameter L2	0.01801368	0.006201614	203.2340	205.1228	199.234	203.9840
Pseudo Lindley	0.0194321	$0.031274 < 1$	-	-	-	-
Xlindley	0.02790613	-	206.9240	207.8684	204.9240	207.1593
Power XLindley	1.055696	0.1935496	251.3029	253.1917	247.3029	252.0529
Lazri Zeghdoudi	0.02778157	1.314093	203.2332	205.1221	199.2332	203.9831

Table 7: Parameter estimates and model comparison characteristics for the second data set

6. Conclusion

In this article, we introduced and studied Lazri Zeghdoudi (LZ) distribution. The LZ distribution is an important model that can be utilized to analyzed overdispersed datasets. Aside from distribution function and PMF other important mathematical properties were derived such as moments and related measurements, reliability characteristics, and two risk or actuarial measures. The LZ distribution was compared to the two-parameter L1, gamma Lindley, quasi Lindley, new quasi Lindley, two parameter L2, Pseudo Lindley, Xlindley and Power XLindley distributions on two datasets from distinct areas. The results demonstrate that the LZ distribution outperforms the competition.

Appendix

$Distribution$	$Density$
two-parameter L1	$\frac{\theta^2(\gamma+x)e^{-\theta x}}{\gamma\theta+1}$
gamma Lindley	$\frac{\theta^2((\gamma+\gamma\theta-\theta)x+1)e^{-\theta x}}{\gamma(1+\theta)}$
quasi Lindley	$\frac{\theta(\gamma+x\theta)e^{-\theta x}}{\gamma+1}$
new *quasi Lindley*	$\frac{\theta^2(\theta+\gamma x)e^{-\theta x}}{\gamma+\theta^2}$
two parameter L2	$\frac{\theta^2}{\theta+\gamma}(1+\gamma x)e^{-\theta x}$
Pseudo Lindley	$\frac{\theta(\gamma-1+\theta x)e^{-\theta x}}{\gamma}$
Xlindley	$\frac{\theta^2(2+\theta+x)}{(1+\theta)^2}e^{-\theta x}$
Power XLindley	$\frac{\alpha\theta^2(2+\theta+x^\alpha)x^{\alpha-1}}{(1+\theta)^2}e^{-\theta x^\alpha}$

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- Nouara Lazri: Investigation; formal analysis; methodology, writing—original draft; simulation; software; interpretation of results; writing—review and editing; supervision

- Amine Sakri: Writing—original draft ; formal analysis; comparison and interpretation of results; validation

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