Asia Pac. J. Math. 2025 **12**:48

SOME INTEGRAL FORMULAE FOR GRADIENT RICCI SOLITON

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Received Mar. 11, 2025

Abstract. In this paper, We consider Ricci soliton hypersurfaces embedded in weighted manifold. Using the weighted symmetric fuunctions σ_k^∞ and the weighted Newton transformations T_k^∞ [8], we give some new Minkowski type integral formulae for Ricci soliton and gradient Ricci soliton manifolds.

2020 Mathematics Subject Classification. 53C42; 53A10.

Key words and phrases. weighted manifolds; weighted Newton transformations; weighted mean curvature; Minkowski integral formulae.

1. Introduction

In differential geometry and geometric analysis, integral formulae has been a crucial and extensively-studied tool and has led to lots of interesting applications.

The first known integral formulae were published in 1903 by H. Minkowski [14] for compact surfaces in three-dimensional Euclidean space.

In [11] C.C.Hsiung derived a serie of higher order Minkwski formulae for hypersurfaces in Euclidean space. He obtained the following result.

Theorem 1. *Let $x : M^n \longrightarrow \mathbb{R}^{n+1}$ be an isometric immersion of a compact oriented Riemannian manifold M^n into the Euclidean space $\mathbb{R}^{n+1}$, then we have:*

$$\int_{M^n} (1 + H \langle x, N\rangle)\, dM = 0.$$

Where N is a unit normal vector field on M^n and H is the normalized mean curvature of the hypersurface M^n given by

$$H = \frac{1}{n} traceA = \frac{1}{n} S_1 = \frac{1}{n}\sum^{n} \kappa_i,$$

A is the shape operator corresponding to the second fundamental form with eigenvalues $\kappa_1, ..., \kappa_n$.

DOI: 10.28924/APJM/12-48

This result were generalized later in Riemannian manifolds. The idea is to consider a manifold addmiting a conformal (Killing or homothetic) vector filed, computing the divergence of certain quantities and then applying the divergence theorem.

In 70s, Reilly established an important Minkowski type formulae in Euclidean spaces, and later in Riemannian manifolds [17]. Surprisly, these formulae were used to obtain some rigidity results for hypersurfaces isometrically immersed in Riemannian space.

Over time, many generalization of Minkowski integral formulae appear in literature for Riemannian and pseudo-Riemannian cases (see for instance [4,5,7,12,13,17])

It is interesting to know if theorem 1, can be extended to other cases, and applied to generalize the aforementioned results.

In [1–3], the authors derived a serie of integral formulae on weighted manifolds using the weighted elementary symmetric functions σ_k^∞ and the weighted Newton transformations T_k^∞ [8]. These quantities are natural generalization of the well known $k-$mean curvature σ_k and Newtons transformations T_k, and they are defined as:

$$\begin{cases} \sigma_0^\infty(u,\mu) = 1, \\ k\sigma_k^\infty(u,\mu) = u\sigma_{k-1}^\infty(u,\mu) + \displaystyle\sum_{j=0}^{k-1}\sum_{i=1}^{n} (-1)^j\, \sigma_{k-1-j}^\infty(u,\mu)\mu_i^{j+1} \quad \text{for } 1 \le k \le n, \\ \sigma_k^\infty(u,\mu) = 0 \quad \text{for} \quad k > n. \end{cases} \tag{1.1}$$

Where $u \in \mathbb{R}$ and $\mu = (\mu_1, ..., \mu_n) \in \mathbb{R}^n$. and $\mu_1, ..., \mu_n$ are the eigenvalues of shape operator A.

The weighted Newton transformations $T_k^\infty(\mu_0, A)$ associated to A are defined inductively by

$$\begin{cases} T_0^\infty(u,A) = I, \\ T_k^\infty(u,A) = \sigma_k^\infty(u,A)I - AT_{k-1}^\infty(u,A) \quad \text{for } k \ge 1. \end{cases} \tag{1.2}$$

Where I stands for the identity on the Lie algebra of vector fields $\varkappa(M)$.

Recall that a weighted manifold $(M^n, \langle,\rangle, dv_f)$, (also known in literature as manifolds with density) is a Riemannian manifold $(M^n, \langle,\rangle)$ endowed with a weighted volume form $dv_f = e^{-f}dv$, where f is a real-valued smooth function on M^n, and dv is the Riemannian volume form associated with the metric $\langle,\rangle$.

In this work we consider a weighted manifold $\overline{M_f}^{n+1}$ that is Ricci soliton. We derive a new integral formulae for hypersurfaces embedded in $\overline{M_f}^{n+1}$. Recall that a Ricci soliton is a Riemannian metric together with a vector field $(M, \langle,\rangle, X)$ that satisfies

$$Ric + \frac{1}{2}L_X\langle,\rangle = \lambda\langle,\rangle \tag{1.3}$$

for some real constant λ.

2. Preliminaries

In this paper, we fix tadopt to the convention introduced in [1] to define and study the weighted symmetric functions and weighted Newton transformations. For more informations see [1,3,8].

Let $\overline{M}_f^{n+1}$ be an $(n+1)-$dimensional connected orientable weighted Riemannian manifold with constant curvature, and $\langle , \rangle$ and $\overline{\nabla}$ will stand for its Riemannian metric and its Levi-Civita connection, respectively.

Let $\psi : M^n \longrightarrow \overline{M}_f^{n+1}$ be a closed oriented hypersurface, and N be the unit normal fields which orient M^n. Denoting by ∇ the Levi-Civita connection of M^n.

The Gauss and Weingarten formulae of the hypersurface are written as:

$$\begin{aligned} \overline{\nabla}_X Y &= \nabla_X Y + \langle AX, Y \rangle .N, \\ AX &= -\left(\overline{\nabla}_X N\right)^{\intercal}. \end{aligned}$$

Where X and Y are tangent vector fields $X, Y \in \varkappa(M^n)$, $\varkappa(M^n)$ is the tangent bundle of M^n, and $A : \varkappa(M^n) \longrightarrow \varkappa(M^n)$ is the shape operator of M^n with respect to the gauss map N.

By the Codazzi equation, we can see that the normal component of the curvature tensor $\overline{R}$ of $\overline{M}_f^{n+1}$ is given in terms of A by

$$\left\langle \overline{R}(U,V)W, N \right\rangle = \left\langle \left(\overline{\nabla}_V A\right) U - \left(\overline{\nabla}_U A\right) V, W \right\rangle,$$

where $U, V, W \in \varkappa(M^n)$. In particular if the ambient space has constant sectional curvature, then we have

$$\left(\overline{\nabla}_V A\right) U = \left(\overline{\nabla}_U A\right) V.$$

Since A is self-adjoint and symmetric, then it is diagonalized. Denoting by $\mu_1, ..., \mu_n$the principal curvatures of M^n, which are the eigenvalues of A.

Following J. Case [8] , we define the weighted elementary symmetric functions $\sigma_k^\infty : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ recursively by:

$$\begin{cases} \sigma_0^\infty = 1, \\ k\sigma_k^\infty = \sigma_{k-1}^\infty \displaystyle\sum_{j=0}^{n} \mu_j + \sum_{i=1}^{k-1} \sum_{j=1}^{n} (-1)^i \, \sigma_{k-1-i}^\infty \mu_j^i \qquad \text{for } k \geq 1. \end{cases}$$

where $\sigma_k^\infty = \sigma_k^\infty(\langle \nabla f, N \rangle, \mu_1, ..., \mu_n)$ and $\mu_1, ..., \mu_n$the principal curvatures of M^n.

In particular,

$$\sigma_1^\infty = H_f = \sigma_1 + \left\langle \overline{\nabla} f, N \right\rangle,$$

is the weighted mean curvature introduced by Gromov [10].

For the non weighted case, we have that $\nabla f = 0$, and $\sigma_k^\infty = \sigma_k$ are the classical elementary symmetric functions defined and studied in [17].

The weighted Newton transformations $T_k^\infty = T_k^\infty(\langle \nabla f, N\rangle, \mu_1, ..., \mu_n)$ are defined inductively from A by:

$$\begin{cases} T_0^\infty = I \\ T_k^\infty = \sigma_k^\infty I - AT_{k-1}^\infty \quad \text{for } k \geq 1 \end{cases}$$

or equivalently

$$T_k^\infty = \sum_{j=0}^{k} (-1)^j \, \sigma_{k-j}^\infty A^j$$

In [8], J. Case showed that T_k^∞ satisfies the following properties:

$$trace(AT_k^\infty) = (k+1)\sigma_{k+1}^\infty - \langle \nabla f, N\rangle \, \sigma_k^\infty. \tag{2.1}$$

For $i \in \{1, ..., n\}$ we have

$$\sigma_{k,i}^\infty = \sigma_k^\infty - \mu_i \sigma_{k-1,i}^\infty,$$

where

$$\sigma_{k,i}^\infty = \sigma_k^\infty \left(u, \mu_1, ..., \mu_{i-1}, \mu_{i+1}, ..., \mu_n\right).$$

The eigenvalues of T_k^∞ are given by $\sigma_{k,i}^\infty$.

We can also proof that for $k \geq 1$, we have

$$trace(T_k^\infty) = (n-k)\, \sigma_k^\infty + \left\langle \overline{\nabla} f, N\right\rangle \sigma_{k-1}^\infty. \tag{2.2}$$

Definition 1. *The weighted divergence of the weighted Newton transformation T_k^∞ is defined as:*

$$\operatorname{div}_f T_k^\infty = e^f \operatorname{div}\left(e^{-f} T_k^\infty\right).$$

An easy computation shows that

$$\operatorname{div}_f T_k^\infty = \operatorname{div} T_k^\infty - T_k^\infty \left(\nabla f\right)$$

Where

$$\operatorname{div} T_k^\infty = \text{trace}\left(\nabla T_k^\infty\right) = \sum_{i=0}^{n} \nabla_{e_i} \left(T_k^\infty\right)(e_i)$$

and $\{e_1, ..., e_n\}$ is a local orthonormal frame of the tangent space of M^n :

$$\operatorname{div} T_k^\infty = T_{k-1}^\infty \circ \nabla u + \sum_{j=1}^{k} \frac{u^j}{j!} \operatorname{div} T_{k-j},$$

We finish this part by the following proposition whose proof can found in [2].

Proposition 2. *The weighted divergence of T_k^∞ is given by*

$$\operatorname{div}_f T_0^\infty = -\nabla f,$$

and for $1 \leq k \leq n$:

$$\operatorname{div}_f T_k^\infty = T_{k-1}^\infty \circ \nabla \langle \nabla f, N \rangle + \sum_{j=1}^{k} \frac{\langle \nabla f, N \rangle^j}{j!} \operatorname{div} T_{k-1}^\infty - T_k^\infty (\nabla f).$$

In particular, if $\overline{M}^{n+1}$ has constant sectional curvature, then we have

$$\operatorname{div}_f T_k^\infty = \sum_{j=0}^{k-1} \left[(-1)^j \sigma_{k-1-j}^\infty A^j \circ \nabla u \right] - T_k^\infty (\nabla f). \tag{2.3}$$

3. Main Results

Let M be a Ricci soliton. That is a Riemannian manifold whose metric toghether with a vector field X satisfies:

$$Ric + \frac{1}{2} L_X \langle , \rangle = \lambda \langle , \rangle. \tag{3.1}$$

for some real constant λ.

Here Ric denotes the Ricci tensor of the metric $\langle , \rangle$ on M and L_X is the Lie derivative in the direction of X.

If $X = \nabla f$ for some smooth function $f : M \longrightarrow \mathbb{R}$, we say that $(M, \langle , \rangle, \nabla f)$ is a gradient Ricci soliton with potential f. In this situation, the soliton equation reads:

$$Ric + Hess(f) = \lambda \langle , \rangle \tag{3.2}$$

Where $Hess(f)$ is the hessian of f.

Clearly, equations (3.1) and (3.2) can be considered as perturbations of the Einstein equation:

$$Ric = \lambda \langle , \rangle$$

and reduce to this latter in case where X or ∇f are Killing vector fields.

Taking the trace, equations (3.1) and (3.2) becomes respectively:

$$S + \operatorname{div} X = n\lambda.$$

And

$$S + \Delta f = n\lambda.$$

The left hand part of equation (3.1) is the so called Baky-Emery Ricci curvature associated to the weighted manifold $\left(M, \langle , \rangle, e^{-f} dv_g\right)$.

The assertion that $(M, \langle , \rangle , \nabla f)$ is a gradient Ricci soliton amounts to saying that the Bakry-Emery Ricci tensor Ric_f of the weighted Riemannian manifold $\left(M, \langle , \rangle , e^{-f} dv_g\right)$ is constant. Thus Ric_f is the most natural geometric object associated to a gradient Ricci soliton.

Both equations (3.1) and (3.2) can be considered as perturbations of the Einstein equation

$$Ric = \lambda \langle , \rangle$$

and reduce to this latter in case where X or ∇f are Killing vector fields.

Let $\overline{M_f}^{n+1}$ a Ricci soliton, and $\varphi : M^n \longrightarrow \overline{M_f}^{n+1}$ be a closed oriented hypersurface. Denoting by N the unit normal fields which orient M^n, and ∇ the Levi-Civita connection of M^n.

Before to compute our integral formulae, we need the following result.

Proposition 3. *Let g be a smooth function on M^n, and X be a Ricci soliton on M^n, then we have*

$$\operatorname{div}_f \left(T_k^{\infty} X^{\top}\right) = \left\langle \operatorname{div}\left(T_k^{\infty}\right), X^{\top}\right\rangle + \lambda tr T_k^{\infty} + \langle X, N\rangle\, tr\left(A T_k^{\infty}\right) - \sum_{i=0}^{n} Ric\left(T_k^{\infty}\left(e_i\right), e_i\right) - \left\langle \nabla f, T_k^{\infty}(X^{\top})\right\rangle . \quad (3.3)$$

Where trT_k^{∞} and $tr\left(AT_k^{\infty}\right)$ are given by (2.1) and (2.2), N is a unit vector field normal to M^n and T_k^{∞} is the weighted Newton transformations.

Proof. We have

$$\begin{aligned}
\operatorname{div}_f \left(T_k^{\infty} X^{\top}\right) &= e^f \operatorname{div}\left(e^{-f}.T_k^{\infty}(X^{\top})\right)) \\
&= \operatorname{div}\left(T_k^{\infty}(X^{\top})\right)) + e^f \left\langle \nabla e^{-f}, T_k^{\infty}(X^{\top})\right\rangle \\
&= \operatorname{div}\left(T_k^{\infty} X^{\top}\right) - \left\langle \nabla f, T_k^{\infty}(X^{\top})\right\rangle
\end{aligned}$$

Where $\operatorname{div}\left(T_k^{\infty}(X^{\top})\right)$ is the classical (non weighted) divergence of $T_k^{\infty}(X^{\top})$.

On the other hand, we have

$$\begin{aligned}
\operatorname{div}\left(T_k^{\infty} X^{\top}\right) &= \sum_{i=1}^{n} \left\langle \nabla_{e_i}\left(T_k^{\infty} X^{\top}\right), e_i\right\rangle \\
&= \sum_{i=1}^{n} \left\langle \left(\nabla_{e_i} T_k^{\infty}\right)(X^{\top}), e_i\right\rangle + \sum_{i=1}^{n} \left\langle \left(T_k^{\infty}\right)(\nabla_{e_i} X^{\top}), e_i\right\rangle \\
&= \sum_{i=1}^{n} \left\langle X^{\top}, \left(\nabla_{e_i} T_k^{\infty}\right) e_i\right\rangle + \sum_{i=1}^{n} \left\langle (\nabla_{e_i} X^{\top}), T_k^{\infty} e_i\right\rangle \\
&= \left\langle \operatorname{div} T_k^{\infty}, X\right\rangle + \sum_{i=1}^{n} \left\langle (\nabla_{e_i} X^{\top}), T_k^{\infty} e_i\right\rangle
\end{aligned}$$

Where $\{e_1, ..., e_n\}$ be an orthonormal basis of $T_p M^n$.

If $\{e_1, ..., e_n\}$ is an orthonormal basis of $T_p M$ that diagonalizes A, it diagonalizes also T_k^{∞}, and we have:

$$T_k^{\infty} e_i = \sigma_{k,i}^{\infty} e_i.$$

Therefore

$$\begin{aligned} 2\left\langle \nabla_{e_i} X^{\top}, T_k^{\infty} e_i \right\rangle &= \langle \nabla_{e_i} X, T_k^{\infty} e_i \rangle + \left\langle \nabla_{e_i} X, \sigma_{k,i}^{\infty} e_i \right\rangle - 2\left\langle \nabla_{e_i} X^{\perp}, T_k^{\infty} e_i \right\rangle \\ &= \langle \nabla_{e_i} X, T_k^{\infty} e_i \rangle + \left\langle \nabla_{\sigma_{k,i}^{\infty} e_i} X, e_i \right\rangle - 2\langle X, N\rangle \langle \nabla_{e_i} N, T_k^{\infty} e_i \rangle \\ &= \langle \nabla_{e_i} X, T_k^{\infty} e_i \rangle + \left\langle \nabla_{T_k^{\infty} e_i} X, e_i \right\rangle + 2\langle X, N\rangle \langle A e_i, T_k^{\infty} e_i \rangle \\ &= L_X \langle T_k^{\infty} e_{i, e_i} \rangle + 2\langle X, N\rangle \langle (A T_k^{\infty}) e_i, e_i \rangle \\ &= 2\lambda \langle T_k^{\infty} e_i, e_i \rangle - 2Ric(T_k^{\infty} e_i, e_i) + 2\langle X, N\rangle \langle (A T_k^{\infty}) e_i, e_i \rangle \end{aligned}$$

Hence

$$\operatorname{div}\left(T_k^{\infty} X^{\top}\right) = \langle \operatorname{div} T_k^{\infty}, X\rangle + \lambda tr T_k^{\infty} + \langle X, N\rangle\, tr\,(A T_k^{\infty}) - \sum_{i=0}^{n} Ric\,(T_k^{\infty}(e_i), e_i)$$

Finally

$$\operatorname{div}_f\left(T_k^{\infty} X^{\top}\right) = \langle \operatorname{div} T_k^{\infty}, X\rangle + \lambda tr T_k^{\infty} + \langle X, N\rangle\, tr\,(A T_k^{\infty}) - \sum_{i=0}^{n} Ric\,(T_k^{\infty}(e_i), e_i) - \left\langle \nabla f, T_k^{\infty}(X^{\top}) \right\rangle.$$

This ends the proof. □

Integrate the two side of (3.3) and applying the weighted version of the divergence theorem, we have:

Theorem 4. *Let $\overline{M_f}^{n+1}$ be a Ricci soliton and $\varphi : M^n \longrightarrow \overline{M_f}^{n+1}$ be a closed oriented hypersurface. Denoting by N the unit normal fields which orient M^n. Then we have:*

$$\int_{M^n} \left(\langle \operatorname{div} T_k^{\infty}, X\rangle + \lambda tr T_k^{\infty} + \langle X, N\rangle\, tr\,(A T_k^{\infty}) - \sum_{i=0}^{n} Ric\,(T_k^{\infty}(e_i), e_i) - \langle \nabla f, T_k^{\infty}(X^{\top}) \rangle \right) dv_f = 0 \tag{3.4}$$

3.1. **Some consequences.** If $\overline{M}^{n+1}$ is (Non weighted) Riemannian manifold equiped with a conformal vector field X, then $T_k^{\infty} = T_k$ is the classical Newton transformations, and

$$\int_{M^n} \left(\langle \operatorname{div} T_k^{\infty}, X\rangle + \lambda (n-r)\, \sigma_r + \langle X, N\rangle (r+1)\, \sigma_{r+1} \right) dv_f = 0.$$

If in addition $\overline{M}^{n+1}$ is a space form, then T_k is divergence free, and we recover the well known r^{th} Minkowski formula:

$$\int_{M^n} \left(\lambda (n-r)\, \sigma_r + \langle X, N\rangle (r+1)\, \sigma_{r+1} \right) dv = 0.$$

For hypersurfaces with constant higher order mean curvature, we have the following lemma.

Lemma 5. *Let $\varphi : M^n \longrightarrow \overline{M}^{n+1}$ be a closed hypersuraces in a compact oriented manifold $\overline{M}^{n+1}$. Suppose that there exist a conformal vector field X on $\overline{M}^{n+1}$. If σ_r is constant, then*

$$|\sigma_r| \leq \frac{(r+1)}{(n-r)\,\lambda vol(M^n)} \int_{M^n} |\sigma_{r+1}|\, dv$$

For $k = 0$, equation (3.4) becomes

$$\int\limits_{M^n} \left(n\lambda + \langle X, N \rangle \sigma_1 - S - \left\langle \nabla f, X^\top \right\rangle \right) dv_f = 0$$

where S denotes the scalar curvature of M^n.

For the gradient Ricci soliton, we have $X = \nabla f$ and we obtain:

Proposition 6. *Under the hypothesis of the above theorem, we have for any gradient Ricci soliton:*

$$\mathcal{F}\left(\langle,\rangle, f\right) = n\lambda vol_f M^n + \int\limits_{M^n} \langle \nabla f, N \rangle \sigma_1 dv_f.$$

Where $\mathcal{F}\left(\langle,\rangle, f\right)$ is the Perelman $\mathcal{F}-$functional defined by [16]:

$$\mathcal{F}\left(\langle,\rangle, f\right) = \int\limits_{M^n} \left(S + |\nabla f|^2\right) dv_f.$$

Authors' Contributions. All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] M. Abdelmalek, Generalized Hsiung–Minkowski Formulae on Manifolds with Density, Arab. J. Math. 10 (2021), 253–260. https://doi.org/10.1007/s40065-021-00322-4.

[2] K. Mokhtari, M. Abdelmalek, Generalized r-Minkowski Formula on Weighted Manifolds, Palestine J. Math. Accepted.

[3] M. Abdelmalek, M. Benalili, Some Integral Formulae on Weighted Manifolds, J. Geom. 112 (2021), 30. https://doi.org/10.1007/s00022-021-00592-5.

[4] H. Alencar, A.G. Colares, Integral Formulas for the r-Mean Curvature Linearized Operator of a Hypersurface, Ann. Glob. Anal. Geom. 16 (1998), 203–220. https://doi.org/10.1023/A:1006555603714.

[5] L.J. Alías, J.H.S. De Lira, J.M. Malacarne, Constant Higher-Order Mean Curvature Hypersurfaces in Riemannian Spaces, J. Inst. Math. Jussieu 5 (2006), 527–562. https://doi.org/10.1017/S1474748006000077.

[6] D. Bakry, M. Émery, Diffusions Hypercontractives, in: J. Azéma, M. Yor (Eds.), Séminaire de Probabilités XIX 1983/84, Springer, Berlin, Heidelberg, 1985: pp. 177–206. https://doi.org/10.1007/BFb0075847.

[7] I. Bivens, Integral Formulas and Hyperspheres in a Simply Connected Space Form, Proc. Amer. Math. Soc. 88 (1983), 113–118. https://doi.org/10.1090/S0002-9939-1983-0691289-2.

[8] J.S. Case, A Notion of the Weighted σ_k-Curvature for Manifolds with Density, Adv. Math. 295 (2016), 150–194. https://doi.org/10.1016/j.aim.2016.03.010.

[9] K. Castro, C. Rosales, Free Boundary Stable Hypersurfaces in Manifolds with Density and Rigidity Results, J. Geom. Phys. 79 (2014), 14–28. https://doi.org/10.1016/j.geomphys.2014.01.013.

[10] M. Gromov, Isoperimetry of Waists and Concentration of Maps, Geom. Funct. Anal. 13 (2003), 178–215.

[11] C.C. Hsiung, Some Integral Formulas for Closed Hypersurfaces, Math. Scand. 2 (1954), 286–294.

[12] Y. Katsurada, Generalized Minkowski Formulas for Closed Hypersuperfaces in Riemann Space, Ann. Mat. Pura Appl. 57 (1962), 283–293. https://doi.org/10.1007/BF02417744.

[13] K.-K. Kwong, An Extension of Hsiung–Minkowski Formulas and Some Applications, J. Geom. Anal. 26 (2016), 1–23. https://doi.org/10.1007/s12220-014-9536-8.

[14] H. Minkowski, Volumen und Oberfläche, Math. Ann. 57 (1903), 447–495.

[15] F. Morgan, Manifolds With Density, Notices Amer. Math. Soc. 52 (2005), 853–858.

[16] G. Perelman, The Entropy Formula for the Ricci Flow and Its Geometric Applications, arXiv:math/0211159 [math.DG] (2002). https://doi.org/10.48550/ARXIV.MATH/0211159.

[17] R.C. Reilly, Variational Properties of Functions of the Mean Curvatures for Hypersurfaces in Space Forms, J. Differ. Geom. 8 (1973), 465–477. https://doi.org/10.4310/jdg/1214431802.

[18] G. Wei, W. Wylie, Comparison Geometry for the Bakry-Emery Ricci Tensor, J. Differ. Geom. 83 (2009), 337–405. https://doi.org/10.4310/jdg/1261495336.