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DEGREE SUBTRACTION AND DEGREE SQUARE SUBTRACTION ENERGIES OF VEE GRAPH

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Abstract. A Vee graph is formed by attaching two grid graphs at their endpoints. The graph can be associated with degree-based matrices including degree subtraction and degree square subtraction matrices. This research is devoted to determining the energy of the Vee graph. The first steps in this paper are to present the degree of every vertex and the general formula of the characteristic polynomial of the particular matrix. The result is that the obtained energies are always an even integer and hyperenergetic. Moreover, we highlight the relationship between the energy and its spectral radius: the energy is always twice its spectral radius.

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Key words and phrases. degree subtraction matrix; degree square subtraction matrix; energy of a graph; grid graph; Vee graph.

1. Introduction

Let P_n be the path graph on n vertices. From path graph P_n and P_m, we are able to construct the grid graph of m,n vertices, with the cartesian product between P_n to P_m. Therefore, we have $(P_n \times P_m)$. The Vee graph is built from grid graph $(P_2 \times P_n)$ and $P_2 \times P_{n+1}$ which are attached at their endpoints.

The Vee graph further can be associated with the adjacency matrix. This matrix is square and we can determine the eigenvalues of the graph. The summation of the absolute eigenvalues is the energy of a graph. Gutman [9] pioneered the energy definition in 1978. It has been shown that the energy is not equal to an odd integer [10] and is never equal to its square root [11].

Apart from the adjacency matrix, research on graph matrices continues to expand involving the degree of vertices. Another graph matrix was introduced by [4], it was the degree subtraction (DS) matrix. Furthermore, [6] studied DS-eigenvalues and DS-energy of regular graphs. In 2022, a new graph matrix definition was put forward by Macha and Shinde [5], named the degree square subtraction

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matrix of a graph. Energy studies have been carried out by several authors. Romdhini and Nawawi [16] formulated the degree subtraction energy of commuting graphs for dihedral groups. Romdhini et al. presented the Wiener-Hosoya [17] and Sombor [18] energies and discussed the degree square subtraction energy [19]. The algebraic discussion also can be found in [20]. Therefore, this study aims to analyze the degree subtraction and degree square subtraction energies of Vee graph and its properties.

This paper is organized as follows. Section 2 presents several existing results relevant to our study. In Section 3, we provide the method to determine the characteristic polynomial of a matrix. The degree of every vertex in the Vee graph is presented in Section 4. The degree subtraction energy and the spectral properties of the Vee graph are presented in Section 5, followed by the degree square subtraction energy in 6. An example of computation is shown in Section 7. We summarize the findings of this study in Section 8.

2. Preliminaries

In this part, we begin with the definition of the Vee graph. Let P_n be the path graph on n vertices.

Definition 2.1. A graph obtained from two Grid graphs $(P_2 \times P_n)$ and $(P_2 \times P_{n+1})$ which are attached at the ends is called a Vee graph, and denoted by V_n.

Graph V_n has $4(n+1)$ vertices that labelled as $s_0, s_1, \ldots, s_{2n}$ and $t_0, t_1, \ldots, t_{2n+2}$ and figured by the following:

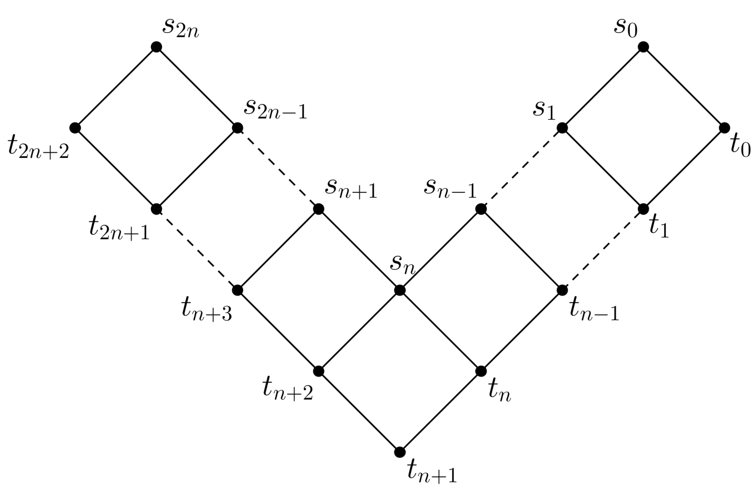

Figure 1. Vee Graph, V_n.

Furthermore, V_n can be associated with graph matrices as given below:

Definition 2.2. [4] The degree subtraction matrix of order $4(n+1)\times 4(n+1)$ associated with V_n is given by $DSt(V_n) = (ds_{pq})$ whose (p,q)-th entry

$$ds_{pq} = \begin{cases} d_{v_p} - d_{v_q}, & \text{if } v_p \neq v_q \\ 0, & \text{if } v_p = v_q. \end{cases}$$

Definition 2.3. [5] The degree square subtraction matrix of order $4(n+1)\times 4(n+1)$ associated with V_n is given by $DSS(V_n) = (dss_{pq})$ whose (p,q)-th entry

$$dss_{pq} = \begin{cases} d^2_{v_p} - d^2_{v_q}, & \text{if } v_p \neq v_q \\ 0, & \text{if } v_p = v_q. \end{cases}$$

The spectrum of $DS(V_n)$, denoted by $Spec_{DS}(V_n)$, is defined as

$$Spec_{DS}(V_n) = \begin{pmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_n \\ k_1 & k_2 & \dots & k_n \end{pmatrix},$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigenvalues (not necessarily distinct) of $DS(V_n)$ with $k_1, k_2, \dots, k_n$ are their respective multiplicities. The degree subtraction energy of V_n is given by

$$E_{DS}(V_n) = \sum_{i=1}^{n} |\lambda_i|,$$

and the degree subtraction spectral radius of V_n is

$$\rho_{DS}(V_n) = max\{|\lambda| : \lambda \in Spec_{DS}(V_n)\}.$$

The above notations can also be applied for $DSS(V_n)$.

Hereafter, a hyperenergetic graph occurs when the energy of a graph with n vertices exceeds the energy of a complete graph with $4n+4$ vertices, K_{4n+4} [23]. Since V_n has $4n+4$, then we have the following definition

Definition 2.4. A $4n4$ vertex graph V_n is hyperenergetic if $E(V_n) > 2(4n+3)$.

3. Characteristic Polynomial

Theorem 3.1. *If* $(4n+4)\times(4n+4)$ *matrix*

$$M = \begin{pmatrix} 0 & aJ_{1\times 5} & bJ_{1\times(4n-2)} \\ aJ_{5\times 1} & c(J-I)_5 & dJ_{5\times(4n-2)} \\ bJ_{1\times(4n-2)} & dJ_{(4n-2)\times 5} & e(J-I)_{4n-2} \end{pmatrix},$$

where a, b, c, d, e *are real numbers, then characteristic polynomial of* M *is*

$$P_M(\mu) = \mu^{4n+2}(\mu^2 + (4n-2)(b^2+5c^2) + 5a^2).$$

Proof.

$$M = \begin{pmatrix} 0 & aJ_{1\times 5} & bJ_{1\times(4n-2)} \\ aJ_{5\times 1} & c(J-I)_5 & dJ_{5\times(4n-2)} \\ bJ_{1\times(4n-2)} & dJ_{(4n-2)\times 5} & e(J-I)_{4n-2} \end{pmatrix},$$

The characteristic polynomial of M is

$$P_M(\mu) = \begin{vmatrix} \mu & -aJ_{1\times 5} & -bJ_{1\times(4n-2)} \\ -aJ_{5\times 1} & (\mu+c)I_5 - cJ_5 & -dJ_{5\times(4n-2)} \\ -bJ_{1\times(4n-2)} & -dJ_{(4n-2)\times 5} & (\mu+e)I_{4n-2} - eJ_{4n-2} \end{vmatrix}. \tag{3.1}$$

The row and column operations apply to Equation 3.1 as follows:

(1) $R_{1+i} \longrightarrow R_{1+i} - R_1$, for $i = 1, 2, 3, 4, 5$.
(2) $R_{7+i} \longrightarrow R_{7+i} - R_7$, for $i = 1, 2, \ldots, 4n-2$.
(3) $C_2 \longrightarrow C_2 + C_3 + C_4 + C_5 + C_6$.
(4) $C_7 \longrightarrow C_7 + C_8 + \ldots + C_{4n+4}$.
(5) $C_1 \longrightarrow C_1 - \frac{b}{\mu} C_7$
(6) $C_2 \longrightarrow C_2 + \frac{5c}{\mu} C_7$

We can write Equation 3.1 as

$$P_M(\mu) = \begin{vmatrix} \frac{\mu^2+b^2(4n-2)}{\mu} & \frac{-5a\mu-5bc(4n-2)}{\mu} & -aJ_{1\times 4} & -b(4n-2) & -bJ_{1\times(4n-3)} \\ \frac{-bc(4n-2)+a\mu}{\mu} & \frac{\mu^2+5c^2(4n-2)}{\mu} & 0_{1\times 4} & 5c & cJ_{1\times(4n-3)} \\ 0_{4\times 1} & 0_{4\times 1} & \mu I_4 & 0_{4\times 1} & 0_{4\times 4} \\ 0 & 0 & 0_{1\times 4} & \mu & 0 \\ 0J_{(4n-3)\times 1} & 0J_{(4n-3)\times 1} & 0J_{(4n-3)\times 4} & 0J_{(4n-3)\times 1} & \mu I_{4n-3} \end{vmatrix}. \tag{3.2}$$

Then

$$P_M(\mu) = \mu^{4n+2}(\mu^2 + (4n-2)(b^2+5c^2) + 5a^2).$$

□

4. Degree of a Vertex

In this section, we present the degree of a vertex in V_n which is beneficial in the next section.

Theorem 4.1. *Let V_n be the Vee graph, then*

(1) *The degree of s_i in V_n, denoted as $deg(s_i)$, is given by*

$$deg(s_i) = \begin{cases} 2, & \forall i = 0, 2n; \\ 4, & \forall i = n; \\ 3, & otherwise. \end{cases}$$

(2) *The degree of t_j in V_n, denoted as $deg(t_j)$, is given by*

$$deg(t_j) = \begin{cases} 2, & \forall j = 0, n+1, 2n+2; \\ 3, & otherwise. \end{cases}$$

Proof. Given that Vee graph V_n has $4(n+1)$ vertices and $2(3n+2)$ edges. The set of vertices of Vee graph $(V(V_n))$ is

$$V(V_n) = \{s_i | i = 0, 1, 2, \ldots, 2n\} \cup \{t_j | j = 0, 1, 2, \ldots, 2n+2\}.$$

Now we can divide into two cases as follows.

Case 1. Degree of vertices in set $\{s_i | i = 0, 1, 2, \ldots, 2n\}$

- Vertex s_0 has degree 2;
- Vertices $s_1, s_2, \ldots s_{n-1}$ have degree 3;
- Vertex s_n has degree 4;
- Vertices $s_{n+1}, s_{n+2} \ldots, s_{2n-1}$ have degree 3;
- Vertex s_{2n} has degree 2;

The total degree of vertices s_i is

$$\begin{aligned} \text{Total } deg(s_i) &= 2 + (n-1)(3) + 4 + (n-1)(3) + 2 \\ &= 2 + 3n - 3 + 4 + 3n - 3 + 2 \\ &= 6n + 2 \end{aligned}$$

Case 2. Degree of vertices in set $\{t_j | j = 0, 1, 2, \ldots, 2n+2\}$

- Vertex t_0 has degree 2;
- Vertices $t_1, t_2, \ldots t_n$ have degree 3;
- Vertex t_{n+1} has degree 2;
- Vertices $t_{n+2}, t_{n+3} \ldots, t_{2n+1}$ have degree 3;
- Vertex t_{2n+2} has degree 2;

The total degree of vertices s_i is

$$\begin{aligned} \text{Total } deg(t_j) &= 2 + (n)(3) + 2 + (n)(3) + 2 \\ &= 2 + 3n + 2 + 3n + 2 \\ &= 6n + 6 \end{aligned}$$

Based on Case 1 and 2, we get the total degree of all vertices in graph V_n is

$$\begin{aligned} \text{Total } deg(V(V_n)) &= \text{Total } deg(s_i) + \text{Total } deg(t_j) \\ &= (6n+2) + (6n+6) \end{aligned}$$

$$= \quad 12n + 8$$

Now we prove that the total degree of all vertices in graph V_n is equal to twice the number of edges in graph V_n.

$$\begin{aligned} \text{Total } deg(V(V_n)) &= 12n + 8 \\ &= 2(6n + 4) \\ &= 2(2(3n + 2)) \\ &= 2|E(V_n)| \end{aligned}$$

So, we can conclude that the Theorem 4.1 holds for graph V_n. □

5. Degree Subtraction Energy

Theorem 5.1. *Let V_n be the Vee graph. Then the characteristic polynomial of V_n associated with the degree subtraction matrix is*

$$P_{DS(V_n)}(\mu) = \mu^{4n+2}(\mu^2 + 24n + 8).$$

Proof. Based on Theorem 4.1 and Definition 2.2, we can construct the degree subtraction matrix of V_n as follows:

$$\begin{array}{c|ccccccccccccccccccc}
 & s_n & s_0 & s_{2n} & t_0 & t_{n+1} & t_{2n+2} & s_1 & \cdots & s_{n-1} & s_{n+1} & \cdots & s_{2n-1} & t_1 & \cdots & t_n & t_{n+2} & \cdots & t_{2n+1} \\ \hline
s_n & 0 & 2 & 2 & 2 & 2 & 2 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
s_0 & -2 & 0 & 0 & 0 & 0 & 0 & -1 & \cdots & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 \\
s_{2n} & -2 & 0 & 0 & 0 & 0 & 0 & -1 & \cdots & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 \\
t_0 & -2 & 0 & 0 & 0 & 0 & 0 & -1 & \cdots & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 \\
t_{n+1} & -2 & 0 & 0 & 0 & 0 & 0 & -1 & \cdots & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 \\
t_{2n+2} & -2 & 0 & 0 & 0 & 0 & 0 & -1 & \cdots & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 \\
s_1 & -1 & 1 & 1 & 1 & 1 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
s_{n-1} & -1 & 1 & 1 & 1 & 1 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
s_{n+1} & -1 & 1 & 1 & 1 & 1 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
s_{2n-1} & -1 & 1 & 1 & 1 & 1 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
t_1 & -1 & 1 & 1 & 1 & 1 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
t_n & -1 & 1 & 1 & 1 & 1 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
t_{n+2} & -1 & 1 & 1 & 1 & 1 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
t_{2n+1} & -1 & 1 & 1 & 1 & 1 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0
\end{array}.$$

We simplify the matrix to the block matrices as follows:

$$DS(V_n) = \begin{pmatrix} 0 & 2J_{1\times 5} & J_{1\times(4n-2)} \\ -2J_{5\times 1} & 0_5 & -J_{5\times(4n-2)} \\ -J_{1\times(4n-2)} & J_{(4n-2)\times 5} & 0_{4n-2} \end{pmatrix},$$

The characteristic polynomial of $DS(V_n)$ is

$$P_{DS(V_n)}(\mu) = \begin{vmatrix} \mu & -2J_{1\times 5} & -J_{1\times(4n-2)} \\ 2J_{5\times 1} & \mu I_5 & J_{5\times(4n-2)} \\ J_{1\times(4n-2)} & -J_{(4n-2)\times 5} & \mu I_{4n-2} \end{vmatrix}.$$

By Theorem 3.1 with $a = 2$ and $b = c = 1$, then we have

$$P_{DS(V_n)}(\mu) = \mu^{4n+2}(\mu^2 + 24n + 8).$$

□

Theorem 5.2. *Let V_n be the Vee graph, then the DS-spectral radius for V_n is*

$$\rho_{DS}(V_n) = 2\sqrt{6n+2}.$$

Proof. The formula of $P_{DS(V_n)}(\mu)$ of Theorem 5.1 result the eigenvalues for V_n. We have $\mu_1 = 0$ of multiplicity $4n+2$, $\mu_{2,3} = \pm 2i\sqrt{6n+2}$ of multiplicity 1, respectively. Hence, the DS-spectrum for V_n is as follows

$$Spec_{DS}(V_n) = \left\{ \left(2i\sqrt{6n+2}\right)^1, (0)^{4n+2}, \left(-2i\sqrt{6n+2}\right)^1 \right\}.$$

Now for $i = 1, 2, 3$, the maximum of $|\lambda_i|$ is the DS-spectral radius of V_n,

$$\rho_{DS}(V_n) = 2\sqrt{6n+2}.$$

□

Theorem 5.3. *Let V_n be the Vee graph, then the DS-energy for V_n is*

$$E_{DS}(V_n) = 4\sqrt{6n+2}.$$

Proof. From the DS-spectrum in Theorem 5.2, we can calculate the DS-energy for V_n. By the definition of energy, we obtain

$$\begin{aligned} E_{DS}(V_n) =& (4n+2)|0| + (1)\left|2i\sqrt{6n+2}\right| + (1)\left|-2i\sqrt{6n+2}\right| \\ =& 4\sqrt{6n+2}. \end{aligned}$$

□

6. Degree Square Subtraction Energy

In this part, we present the degree square subtraction matrix of the Vee graph.

Theorem 6.1. *Let V_n be the Vee graph. Then the characteristic polynomial of V_n associated with the degree square subtraction matrix is*

$$P_{DSS(V_n)}(\mu) = \mu^{4n+2}(\mu^2 + 696n + 372).$$

Proof. Based on Theorem 4.1 and Definition 2.3, we can construct the degree subtraction matrix of V_n, $DSS(V_n)$ as follows:

$$\begin{array}{c|ccccccccccccccccccc|}
 & s_n & s_0 & s_{2n} & t_0 & t_{n+1} & t_{2n+2} & s_1 & \dots & s_{n-1} & s_{n+1} & \dots & s_{2n-1} & t_1 & \dots & t_n & t_{n+2} & \dots & t_{2n+1} \\
s_n & 0 & 12 & 12 & 12 & 12 & 12 & 7 & \dots & 7 & 7 & \dots & 7 & 7 & \dots & 7 & 7 & \dots & 7 \\
s_0 & -12 & 0 & 0 & 0 & 0 & 0 & -5 & \dots & -5 & -5 & \dots & -5 & -5 & \dots & -5 & -5 & \dots & -5 \\
s_{2n} & -12 & 0 & 0 & 0 & 0 & 0 & -5 & \dots & -5 & -5 & \dots & -5 & -5 & \dots & -5 & -5 & \dots & -5 \\
t_0 & -12 & 0 & 0 & 0 & 0 & 0 & -5 & \dots & -5 & -5 & \dots & -5 & -5 & \dots & -5 & -5 & \dots & -5 \\
t_{n+1} & -12 & 0 & 0 & 0 & 0 & 0 & -5 & \dots & -5 & -5 & \dots & -5 & -5 & \dots & -5 & -5 & \dots & -5 \\
t_{2n+2} & -12 & 0 & 0 & 0 & 0 & 0 & -5 & \dots & -5 & -5 & \dots & -5 & -5 & \dots & -5 & -5 & \dots & -5 \\
s_1 & -7 & 5 & 5 & 5 & 5 & 5 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots \\
s_{n-1} & -7 & 5 & 5 & 5 & 5 & 5 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\
s_{n+1} & -7 & 5 & 5 & 5 & 5 & 5 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots \\
s_{2n-1} & -7 & 5 & 5 & 5 & 5 & 5 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\
t_1 & -7 & 5 & 5 & 5 & 5 & 5 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots \\
t_n & -7 & 5 & 5 & 5 & 5 & 5 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\
t_{n+2} & -7 & 5 & 5 & 5 & 5 & 5 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots \\
t_{2n+1} & -7 & 5 & 5 & 5 & 5 & 5 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0
\end{array}.$$

We simplify the matrix to the block matrices as follows:

$$DSS(V_n) = \begin{pmatrix} 0 & 12J_{1\times5} & 7J_{1\times(4n-2)} \\ -12J_{5\times1} & 0_5 & -5J_{5\times(4n-2)} \\ -7J_{1\times(4n-2)} & 5J_{(4n-2)\times5} & 0_{4n-2} \end{pmatrix},$$

The characteristic polynomial of $DSS(V_n)$ is

$$P_{DSS(V_n)}(\mu) = \begin{vmatrix} \mu & -12J_{1\times5} & -7J_{1\times(4n-2)} \\ 12J_{5\times1} & \mu I_5 & 5J_{5\times(4n-2)} \\ 7J_{1\times(4n-2)} & -5J_{(4n-2)\times5} & \mu I_{4n-2} \end{vmatrix}.$$

By Theorem 3.1 with $a = 12$, $b = 7$, and $c = 5$, then we have

$$P_{DSS(V_n)}(\mu) = \mu^{4n+2}(\mu^2 + 696n + 372).$$

□

Theorem 6.2. *Let V_n be the Vee graph, then the DSS-spectral radius for V_n is*

$$\rho_{DSS}(V_n) = 2\sqrt{6n-2}.$$

Proof. The formula of $P_{DSS(V_n)}(\mu)$ of Theorem 6.1 result the eigenvalues for V_n. We have $\mu_1 = 0$ of multiplicity $4n+2$, $\mu_{2,3} = \pm 2i\sqrt{174n+93}$ of multiplicity 1, respectively. Hence, the DSS-spectrum for V_n is as follows

$$Spec_{DSS}(V_n) = \left\{ \left(2i\sqrt{174n+93}\right)^1, (0)^{4n+2}, \left(-2i\sqrt{174n+93}\right)^1 \right\}.$$

Now for $i = 1, 2, 3$, the maximum of $|\lambda_i|$ is the DSS-spectral radius of V_n,

$$\rho_{DSS}(V_n) = 2\sqrt{174n+93}.$$

□

Theorem 6.3. *Let V_n be the Vee graph, then the DSS-energy for V_n is*

$$E_{DSS}(V_n) = 4\sqrt{174n+93}.$$

Proof. From the DSS-spectrum in Theorem 6.2, we can calculate the DSS-energy for V_n. By the definition of energy, we obtain

$$\begin{aligned} E_{DSS}(V_n) =& (4n+2)|0| + (1)\left|2i\sqrt{174n+93}\right| + (1)\left|-2i\sqrt{174n+93}\right| \\ =& 4\sqrt{174n+93}. \end{aligned}$$

□

7. Example

Let us take $n = 1$, then we have V_1 with 8 vertices as seen in Figure 2.

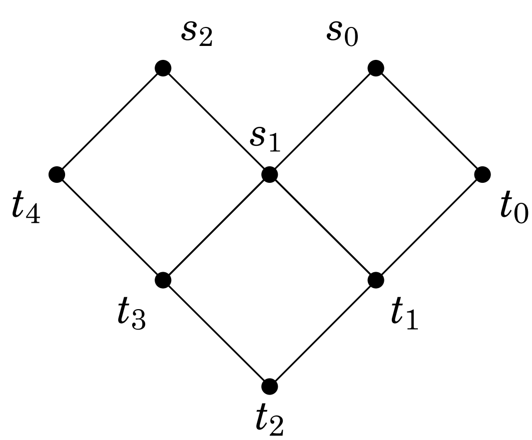

Figure 2. Vee Graph, V_1

The DS and DSS-matrices of V_1 are as follows, respectively.

$$DS(V_1) = \begin{pmatrix} 0 & 2 & 2 & 2 & 2 & 2 & 1 & 1 \\ -2 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ -2 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ -2 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ -2 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ -2 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ -1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

and

$$DSS(V_1) = \begin{pmatrix} 0 & 12 & 12 & 12 & 12 & 12 & 7 & 7 \\ -12 & 0 & 0 & 0 & 0 & 0 & -5 & -5 \\ -12 & 0 & 0 & 0 & 0 & 0 & -5 & -5 \\ -12 & 0 & 0 & 0 & 0 & 0 & -5 & -5 \\ -12 & 0 & 0 & 0 & 0 & 0 & -5 & -5 \\ -12 & 0 & 0 & 0 & 0 & 0 & -5 & -5 \\ -7 & 5 & 5 & 5 & 5 & 5 & 0 & 0 \\ -7 & 5 & 5 & 5 & 5 & 5 & 0 & 0 \end{pmatrix}.$$

The characteristic polynomials of both matrices are

$$P_{DS(V_1)}(\mu) = \mu^{4n+2}(\mu^2 + 32) \text{ and } P_{DSS(V_1)}(\mu) = \mu^{4n+2}(\mu^2 + 1068).$$

It is confirmed by Maple that the spectrum of V_n is

$$Spec_{DS}(V_1) = \left\{ \left(4i\sqrt{2}\right)^1, (0)^{4n+2}, \left(-4i\sqrt{2}\right)^1 \right\}, \text{ and}$$

$$Spec_{DSS}(V_1) = \left\{ \left(2i\sqrt{267}\right)^1, (0)^{4n+2}, \left(-2i\sqrt{267}\right)^1 \right\}.$$

Then the spectral radius of V_1 regarding both matrices is as follows.

$$\rho_{DS}(V_1) = 4\sqrt{2} \text{ and } \rho_{DSS}(V_1) = 2\sqrt{267}.$$

Eventually, we can write the energy of V_1 is

$$E_{DS}(V_1) = 8\sqrt{2} = 2 \cdot \rho_{DS}(V_1) \text{ and } E_{DSS}(V_1) = 4\sqrt{267} = 2 \cdot \rho_{DSS}(V_1).$$

8. Discussions

Based on Theorem 5.2, 5.3, 6.2, and 6.3, we get the following facts:

Corollary 8.1. *Let V_n be the Vee graph, then*

(1) $E_{DS}(V_n) = 2 \cdot \rho_{DS}(V_n)$,
(2) $E_{DSS}(V_n) = 2 \cdot \rho_{DSS}(V_n)$.

Corollary 8.2. *Let V_n be the Vee graph, then the DS and DSS-energies of V_n are always an even integer.*

Corollary 8.3. *Let V_n be the Vee graph, then V_n is hyperenergetic corresponding to DS and DSS-matrices.*

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