Asia Pac. J. Math. 2025 **12**:52

MULTIPLE SOLUTIONS FOR THE DISCRETE $p(k)$-LAPLACIAN PROBLEMS OF KIRCHHOFF TYPE IN TWO-DIMENSIONAL HILBERT SPACE

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Received Apr. 6, 2025

Abstract. In the present paper, we study the existence of at least one weak nontrivial solutions for a discrete nonlinear Dirichlet boundary-value problem of Kirchhoff type in a two-dimensional Hilbert space. We establish three results of the existence of solutions.

2020 Mathematics Subject Classification. 39A70.

Key words and phrases. Kirchhoff type equation; discrete boundary value problem; critical point; weak solution; mountain pass lemma; Palais-Smale condition.

1. Introduction

Difference equations appear in many mathematical models in various fields of research, such as numerical analysis, computer science, mechanical engineering, control systems, artificial or biological neural networks and social sciences, suh as economics. Various methods have been used to deal with the existence of solutions to the discrete boundary value problems. We refer the reader to [1,5,16,21,22] and the references therein. Here, we are interested and investigating nonlinear discrete boundary value problems in two-dimensional Hilbert space. Note that they are few paper deal with this kind of problem. Recently, based on the minimization method in [10], Ibrango et *al.* prove the existence and uniqueness of solutions when the function f depends only on the space variable. Next, when f depends only on the space variable and on the solution u, we prove the existence of at least one solution to the two-dimensional following Dirichlet problem

DOI: 10.28924/APJM/12-52

$$\begin{cases} -\Delta(a(k-1,h-1,\Delta u(k-1,h-1))) = f(k,h,u(k,h)) \\ (k,h) \in \mathbb{N}[1,T_1] \times \mathbb{N}[1,T_2], \\ u(k,h) = 0, \ (k,h) \in \Gamma. \end{cases} \quad (1)$$

A particularly case of problem (1) was studied in [7], where the authors deal with the existence of multiple solutions to the following p-Laplacian problem, based on three critical points theorem established by Ricerri (see [3,4])

$$\Delta_1(\phi_p(\Delta_1(k-1,h))) + \Delta_2(\phi_p(\Delta_2(k,h-1))) + \lambda f(k,h,u(k,h) = 0,$$

for any $(k,h) \in \mathbb{N}[1,T_1] \times \mathbb{N}[1,T_2]$ with $\Delta_1 u(k,h) = u(k+1,h) - u(k,h)$ and $\Delta_2 u(k,h) = u(k,h+1) - u(k,h)$. ϕ_p is the p-Laplacian given by $\phi_p(s) = |s|^{p-2}s$, $1 < p < \infty$ and $f(k,h,.) : \mathbb{R} \longrightarrow \mathbb{R}$, is continuous.

Motivated by the above mentioned, we study the existence and multiplicity of solutions to nonlinear discrete problems of Kirchhoff type namely

$$\begin{cases} -M(A(k-1,h-1,\Delta u(k-1,h-1)))\Delta(a(k-1,h-1,\Delta u(k-1,h-1))) \\ \qquad\qquad = \lambda f(k,h,u(k,h)), \\ (k,h) \in \mathbb{N}[1,T_1] \times \mathbb{N}[1,T_2], \\ u(k,h) = 0, \ (k,h) \in \Gamma \end{cases} \quad (2)$$

where

$$\Gamma = (\{0, T_1+1\} \times \mathbb{N}[0,T_2+1]) \cup (\mathbb{N}[0,T_1+1] \times \{0,T_2+1\})$$

is the boundary of the domain $\mathbb{N}[1,T_1] \times \mathbb{N}[1,T_2]$; Δ is the forward difference operator, $a : \mathbb{N}[1,T_1] \times \mathbb{N}[1,T_2] \times \mathbb{R} \longrightarrow \mathbb{R}$; $f : \mathbb{N}[1,T_1] \times \mathbb{N}[1,T_2] \times \mathbb{R} \longrightarrow \mathbb{R}$ and $M : (0,\infty) \longrightarrow (0,\infty)$ is a non-decreasing continuous function. λ is a positive real number.

Kirchhoff in 1876 (see [13]) suggested a model defined by the following equation

$$\rho \frac{\partial^2 u}{\partial t^2} = \left(T_0 + \frac{Ea}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2}, \quad (3)$$

where $\rho > 0$ is the mass per unit length, T_0 is the base tension, E is the Young modulus, a is the area of cross section and L is the initial length of the string.

Equation (3) takes into account the change of the tension on the string which is caused by the change of its length during the vibration. After that, several physicists also considered such equations for their research in the theory of nonlinear vibrations theoretically or experimentally.

Problem like (2) can be seen as a generaly case of problems studied by Ibrango et *al.* in [11]. In [11], the authors deal with the existence results for weak solutions by using the direct variational method. The goal of the present paper is to stablish the existence of nontrivial solutions for problem (2) by using critical point theory. We firstly, apply the direct variational method and secondly the well known Mountain pass technique known as the Mountain pass theorem due to Ambrosetti and Rabinowitz in order to obtain the existence of at least one nontrivial solution. Third, the use Ekeland's principle.

The rest of this paper is organized in the following way. In Section 2, we give some basic definitions and preliminary facts which will be used throughout the following sections. In section 3, we show that problem (2) admits at least one weak nontrivial solution under suitable hypothesis on the data such as [17].

The last section is devoted to study an extension of problem (2).

2. Assumptions and Preliminary

Define the space

$$\mathcal{H} := \{u : \mathbb{N}[0, T_1 + 1] \times \mathbb{N}[0, T_2 + 1] \longrightarrow \mathbb{R},\ u(k, h) = 0 \quad \forall (k, h) \in \Gamma\}$$

which endowed with the Euclidean norm

$$\|u\| = \left(\sum_{k=1}^{T_1} \sum_{h=1}^{T_2} |u(k, h)|^2 \right)^{\frac{1}{2}}.$$

However, we introduce on the space $\mathcal{H}$ another norm

$$|u|_m = \left(\sum_{k=1}^{T_1} \sum_{h=1}^{T_2} |u(k, h)|^m \right)^{\frac{1}{m}}, \forall m \geq 2.$$

We assume the following conditions on the data.

$$a(k, h, .) : \mathbb{R} \longrightarrow \mathbb{R} \text{ is continuous } \forall (k, h) \in \mathbb{N}[1, T_1] \times \mathbb{N}[1, T_2]$$

and there exists $A : \mathbb{N}[1, T_1] \times \mathbb{N}[1, T_2] \times \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$a(k, h, x) = \frac{\partial}{\partial x} A(k, h, x) \text{ and } A(k, h, 0) = 0 \quad \forall (k, h) \in \mathbb{N}[1, T_1] \times \mathbb{N}[1, T_2]. \tag{4}$$

We also assume that there exist a positive constant C_1 such that

$$|a(k, h, x)| \leq C_1 \left(1 + |x|^{p(k,h)-1}\right) \tag{5}$$

and

$$|x|^{p(k,h)} \leq a(k, h, x)x \leq p(k, h)A(k, h, x), \quad \forall x \in \mathbb{R}. \tag{6}$$

For the function $M:(0,\infty)\longrightarrow(0,\infty)$, we suppose that it is continuous, non-decreasing and there exist positive numbers R_1, R_2 with $R_1\leq R_2$ and $\mu>1$ such that

$$R_1 t^{\mu-1}\leq M(t)\leq R_2 t^{\mu-1}\quad \text{for}\quad t>0. \tag{7}$$

The function $f(k,h,.):\mathbb{R}\longrightarrow\mathbb{R}$ is jointly continuous and there exist the functions $\sigma_1,\sigma_2:\mathbb{N}[1,T_1]\times\mathbb{N}[1,T_2]\longrightarrow(-\infty,0)$; $\phi_1,\phi_2:\mathbb{N}[1,T_1]\times\mathbb{N}[1,T_2]\longrightarrow(0,\infty)$ and a function $\gamma:\mathbb{N}[1,T_1]\times\mathbb{N}[1,T_2]\longrightarrow[2,\infty)$ such that

$$\sigma_1(k,h)+\phi_1(k,h)|x|^{\gamma(k,h)-1}\leq f(k,h,x)\leq\sigma_2(k,h)+\phi_2(k,h)|x|^{\gamma(k,h)-1}. \tag{8}$$

One denotes by

$$\begin{cases}\underline{\sigma_1}=\inf\limits_{k\in\mathbb{N}[1,T_1]}\left(\inf\limits_{h\in\mathbb{N}[1,T_2]}\sigma_1(k,h)\right);\quad \overline{\sigma_1}=\sup\limits_{k\in\mathbb{N}[1,T_1]}\left(\sup\limits_{h\in\mathbb{N}[1,T_2]}\sigma_1(k,h)\right)<0,\\ \underline{\sigma_2}=\inf\limits_{k\in\mathbb{N}[1,T_1]}\left(\inf\limits_{h\in\mathbb{N}[1,T_2]}\sigma_2(k,h)\right);\quad \overline{\sigma_2}=\sup\limits_{k\in\mathbb{N}[1,T_1]}\left(\sup\limits_{h\in\mathbb{N}[1,T_2]}\sigma_2(k,h)\right)<0,\\ 0<\underline{\phi_1}=\inf\limits_{k\in\mathbb{N}[1,T_1]}\left(\inf\limits_{h\in\mathbb{N}[1,T_2]}\phi_1(k,h)\right);\quad \overline{\phi_1}=\sup\limits_{k\in\mathbb{N}[1,T_1]}\left(\sup\limits_{h\in\mathbb{N}[1,T_2]}\phi_1(k,h)\right),\\ 0<\underline{\phi_2}=\inf\limits_{k\in\mathbb{N}[1,T_1]}\left(\inf\limits_{h\in\mathbb{N}[1,T_2]}\phi_2(k,h)\right);\quad \overline{\phi_2}=\sup\limits_{k\in\mathbb{N}[1,T_1]}\left(\sup\limits_{h\in\mathbb{N}[1,T_2]}\phi_2(k,h)\right).\end{cases}$$

and

$$F(k,h,x)=\int_0^x f(k,h,s)\mathrm{d}s,\quad \text{for}\quad (k,h,x)\in\mathbb{N}[1,T_1]\times\mathbb{N}[1,T_2]\times\mathbb{R}. \tag{9}$$

Example 2.1. *We can give the following function satisfies assumptions (4)-(8).*

- $A(k,h,x)=\dfrac{1}{p(k,h)}\left(\left(1+|x|^2\right)^{\frac{p(k,h)}{2}}-1\right)$, *where* $a(k,h,x)=\left(1+|x|^2\right)^{\frac{p(k,h)-2}{2}}x$, $(k,h)\in\mathbb{N}[1,T_1]\times\mathbb{N}[1,T_2],\ x\in\mathbb{R}$.

- $f(k,h,x)=-1+|x|^{\gamma(k,h)-1}$.

- $M(t)=at^{\mu-1}+b$, *a and b two positive constants.*

In this paper, we assume that the function $p:\mathbb{N}[1,T_1]\times\mathbb{N}[1,T_2]\longrightarrow(1,\infty)$ and $\gamma:\mathbb{N}[1,T_1]\times\mathbb{N}[1,T_2]\longrightarrow[2,\infty)$, with

$$p^-=\min_{k\in\mathbb{N}[1,T_1]}\left(\min_{h\in\mathbb{N}[1,T_2]}p(k,h)\right)\quad;\quad p^+=\max_{k\in\mathbb{N}[1,T_1]}\left(\max_{h\in\mathbb{N}[1,T_2]}p(k,h)\right)$$

and

$$\gamma^-=\min_{\{(k,h)\in\mathbb{N}[1,T_1]\times\mathbb{N}[1,T_2]\}}\gamma(k,h)\quad;\quad \gamma^+=\max_{\{(k,h)\in\mathbb{N}[1,T_1]\times\mathbb{N}[1,T_2]\}}\gamma(k,h).$$

To establish our main result, we recall the tools used in [9,10,19].

Lemma 2.1. a) *For any function $u \in \mathcal{H}$ with $\|u\| > 1$, there exist constants C_2, $C_3 > 0$ such that*

$$\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} |\Delta u(k-1,h-1)|^{p(k-1,h-1)} \geq C_2\|u\|^{p^-} - C_3. \tag{10}$$

b) *For any function $u \in \mathcal{H}$ with $\|u\| \leq 1$, there exists constant $C_4 > 0$ such that*

$$\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} |\Delta u(k-1,h-1)|^{p(k-1,h-1)} \geq C_4\|u\|^{p^+}. \tag{11}$$

c) *For any function $u \in \mathcal{H}$ there exist constants $C_5,\ C_6 > 0$ such that*

$$\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} |\Delta u(k-1,h-1)|^{p(k-1,h-1)} \leq C_5\|u\|^{p^+} + C_6. \tag{12}$$

Theorem 2.1. [12] *Let X be reflexive Banach space. If a functional $J \in C^1(X,\mathbb{R})$ is weakly lower semi-continous and coercive, i.e $\lim_{\|u\|\to\infty} J(u) = \infty$, then there exists u_0 such that*

$$J(u_0) = \inf_{u\in X} J(u)$$

and u_0 is also a critical point of J, i.e. $J'(u_0) = 0$. Moreover, if J is strictly convex, then a critical point is unique.

Theorem 2.2. [6] (*Ekeland's principle*) *Let X be a complete metric space and $J : X \longrightarrow \mathbb{R}$ a lower semi-continuous function that is bounded below. Let $\epsilon > 0$ and $\bar{u} \in X$ be given such that*

$$J(\bar{u}) \leq \inf_{u\in X} J(u) + \frac{\epsilon}{2}.$$

Then given $\lambda > 0$ there exists $u_\lambda \in X$ such that

(1) $J(u_\lambda) \leq J(\bar{u})$,

(2) $d(u_\lambda, \bar{u}) < \lambda$,

(3) $J(u_\lambda) < J(u) + \frac{\epsilon}{\lambda} d(u, u_\lambda)$ *for all* $u \neq u_\lambda$.

Definition 2.1. *Let X be a real Banach space. We say that a functional $J : X \longrightarrow \mathbb{R}$ satisfies the Palais-Smale condition if every sequence $\{u_n\}$ such that $\{T(u_n)\}$ is bounded and $J'(u_n) \longrightarrow 0$ has a convergent subsequence.*

Lemma 2.2. [8] *Let X be a Banach space and $J \in C^1(X,\mathbb{R})$ satisfies the Palais-Smale condition. Assume that there exist u_0, $u_1 \in X$ and a Bounded open neighborhood Ω of u_0 such that $u_1 \notin \bar{\Omega}$ and*

$$\max\{J(u_0), J(u_1)\} < \inf_{u\in\partial\Omega} J(u).$$

Let

$$\Gamma_1 = \{g \in C([0,1], X) : g(0) = u_0,\ g(1) = u_1\}$$

and

$$c = \inf_{g\in\Gamma_1} \max_{x\in[0,1]} J(g(x))$$

Then c is a critical value of J; that is, there exists $u^* \in X$ such that $J'(u^*) = 0$ and $J(u^*) = c$, where $c > \max\{J(u_0), J(u_1)\}$.

3. Existence of Solutions by Direct Variational Method

In this section we are concerned with the applications of Theorem 2.1 in order to get the existence results.

Definition 3.1. *A weak solution for problem (2) is a function $u \in \mathcal{H}$ such that*

$$\begin{cases} M\left(\displaystyle\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} A(k-1,h-1,\Delta u(k-1,h-1))\right)\times \\ \left(\displaystyle\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} a(k-1,h-1,\Delta u(k-1,h-1))\Delta v(k-1,h-1)\right) \\ \qquad = \lambda\displaystyle\sum_{k=1}^{T_1}\sum_{h=1}^{T_2} f(k,h,u(k,h))v(k,h), \end{cases} \tag{13}$$

for all $v \in \mathcal{H}$.

The main result of this paper is given by the following theorem.

Theorem 3.1. *Suppose that $\gamma^+ < \mu p^-$ and $\dfrac{\phi_1}{|\sigma_1|} > \gamma^+$. Then, there exists $\lambda_0 > 0$ such that for any $\lambda > \lambda_0$, the problem (2) has at least one weak nontrivial solution.*

In order to prove Theorem 3.1, we define for each $\lambda > 0$ the functional corresponding to problem (2), $J_\lambda : \mathcal{H} \longrightarrow \mathbb{R}$, by
$J_\lambda(u) = \widehat{M}\left(\displaystyle\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} A(k-1,h-1,\Delta u(k-1,h-1))\right) - \lambda\sum_{k=1}^{T_1}\sum_{h=1}^{T_2} F(k,h,u(k,h))$ where $(k,h) \in \mathbb{N}[1,T_1]\times\mathbb{N}[1,T_2]$, $\widehat{M}(t) = \displaystyle\int_0^t M(s)\mathrm{d}s$ and $F(k,h,x) = \displaystyle\int_0^x f(k,h,s)\mathrm{d}s$. From [20], the functional J_λ

is continuous differentiable in the sense of Gâteaux and J'_λ at u reds

$$\begin{cases} \langle J'_\lambda(u), v\rangle = M\left(\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} A(k-1,h-1,\Delta u(k-1,h-1))\right)\times \\ \left(\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} a(k-1,h-1,\Delta u(k-1,h-1))\Delta v(k-1,h-1)\right) \\ -\lambda\sum_{k=1}^{T_1}\sum_{h=1}^{T_2} f(k,h,u(k,h))v(k,h) \end{cases} \tag{14}$$

for all $v \in \mathcal{H}$.

Assume that $\langle J'_\lambda(u), v\rangle = 0$, which is equivalent to

$$-M(I(u))\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}\left(\Delta a(k-1,\Delta u(k-1)) - \lambda f(k,h,u(k,h))\right)v(k,h) = 0, \tag{15}$$

$\forall\, v \in \mathcal{H}$, with $I(u) = \sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} A(k-1,h-1,\Delta u(k-1,h-1))$.
Therefore, the critical point u to J_λ satisfies the problem (2).

We begin the proof of Theorem 3.1 with some basic properties on functional J_λ.

Proposition 3.1. *Assume that condition (6), (7), (10) are fulfilled with $\mu p^- > \gamma^+$. Then, the functional J_λ is coercive for all $\lambda > 0$.*

Proof. Let $\|u\| > 1$, according to (6), (7) and (10), we have

$$J_\lambda(u) = \widehat{M}\left(\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} A(k-1,h-1,\Delta u(k-1,h-1))\right) - \lambda\sum_{k=1}^{T_1}\sum_{h=1}^{T_2} F(k,h,u(k,h))$$

$$\begin{aligned} J_\lambda(u) &\geq \int_0^{\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} A(k-1,h-1,\Delta u(k-1,h-1))} R_1 s^{\mu-1}\mathrm{d}s - \\ &\quad \lambda\sum_{k=1}^{T_1}\sum_{h=1}^{T_2} F(k,h,u(k,h)) \\ &\geq \frac{R_1}{\mu}\left(\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} A(k-1,h-1,\Delta u(k-1,h-1))\right)^{\mu} - \\ &\quad \lambda\left(\frac{\overline{\phi_2}}{\gamma^-}\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}|u(k,h)|^{\gamma(k,h)} + |\underline{\sigma_2}|\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}|u(k,h)|\right) \\ &\geq \frac{R_1}{\mu(p^+)^{\mu}}\left(\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}|\Delta u(k-1,h-1)|^{p(k-1,h-1)}\right)^{\mu} - \end{aligned}$$

$$\lambda\left(\frac{\overline{\phi_2}}{\gamma^-}\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}|u(k,h)|^{\gamma^+}+\frac{\overline{\phi_2}}{\gamma^-}\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}|u(k,h)|^{\gamma^-}+|\underline{\sigma_2}|\sqrt{T_1\times T_2}\|u\|\right)$$
$$\geq \frac{R_1}{\mu(p^+)^\mu}\left[C_2\|u\|^{p^-}-C_3\right]^\mu-\lambda\left(\frac{\overline{\phi_2}}{\gamma^-}\|u\|^{\gamma^+}+\frac{\overline{\phi_2}}{\gamma^-}\|u\|^{\gamma^-}+|\underline{\sigma_2}|\sqrt{T_1\times T_2}\|u\|\right).$$

Since $\mu p^- > \gamma^+$, we obtain $J_\lambda(u)\to\infty$ as $\|u\|\to\infty$; then the functional $J_\lambda(u)$ is coercive □

Proof. **Proof of Theorem 3.1.** We deduce from [10, 11, 14] that the functional J_λ is of class C^1 and weakly lower semi-continuous. It's also coercive. Let $u_\delta(k,h)\in\mathcal{H}$ a global minimizer of J_λ, a weak solution of problem (2). Now, we show that u_δ is not trivial when $\mu p^- > \gamma^+$ and $\lambda>\lambda_0$.

For $t_0>1$ be a fixed real and $(k_0,h_0)\in\mathbb{N}[0,T_1+1]\times\mathbb{N}[0,T_2+1]$. We define $u_0\in\mathcal{H}$ such that

$$\begin{cases} u_0(k_0,h_0) = t_0 \\ \\ u_0(k,h) = 0, \quad (k,h)\in(\mathbb{N}[0,T_1+1]\times\mathbb{N}[0,T_2+1])\setminus\{(k_0,h_0)\}.\end{cases}$$

$$J_\lambda(u_0)=\widehat{M}\left(\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}A(k-1,h-1,\Delta u_0(k-1,h-1))\right)-\lambda\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}F(k,h,u_0(k,h))$$

According to (6) − (8) and (9), we obtain

$$\begin{aligned}
J_\lambda(u_0) &\leq \int_0^{\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}A(k-1,h-1,\Delta u_0(k-1,h-1))} R_2 s^{\mu-1}\mathrm{d}s - \\
&\quad \lambda\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}F(k,h,u_0(k,h)) \\
&\leq \frac{R_2}{\mu}\left(\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}A(k-1,h-1,\Delta u_0(k-1,h-1))\right)^\mu - \\
&\quad \lambda\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}F(k,h,u_0(k,h)) \\
&\leq \frac{R_2}{\mu}\left(\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}\int_0^{\Delta u_0(k-1,h-1)}|a(k-1,h-1,s)|\mathrm{d}s\right)^\mu \\
&\quad -\lambda\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}F(k,h,u_0(k,h)) \\
&\leq \frac{R_2}{\mu}\left(\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}C_1|\Delta u_0(k-1,h-1)|+\right. \\
&\quad \left.\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}\frac{C_1}{p(k,h)}|\Delta u_0(k-1,h-1)|^{p(k-1,h-1)}\right)^\mu-\lambda\left(-|\underline{\sigma_1}|t_0+\frac{\underline{\phi_1}}{\gamma^+}t_0^{\gamma^-}\right) \\
&\leq \frac{R_2}{\mu}(C_1)^\mu\left(\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}|\Delta u_0(k-1,h-1)|+\right.
\end{aligned}$$

$$\begin{aligned}
& \left.\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}\frac{1}{p^-}|\Delta u_0(k-1,h-1)|^{p(k-1,h-1)}\right)^{\mu}-\lambda\left(-|\underline{\sigma_1}|t_0+\frac{\underline{\phi_1}}{\gamma^+}t_0^{\gamma^-}\right) \\
\leq\ & \frac{R_2}{\mu}(C_1)^{\mu}\left(2t_0+\frac{t_0^{p(k_0,h_0)}+t_0^{p(k_0-1,h_0-1)}}{p^-}\right)^{\mu}-\lambda t_0\left(-|\underline{\sigma_1}|+\frac{\underline{\phi_1}}{\gamma^+}\right) \\
\leq\ & \frac{R_2}{\mu}(4C_1)^{\mu}t_0^{\mu p^+}-\lambda t_0\left(-|\underline{\sigma_1}|+\frac{\underline{\phi_1}}{\gamma^+}\right)
\end{aligned}$$

where

$$\lambda_0=\frac{R_2(4C_1)^{\mu}t_0^{\mu p^+-1}}{\mu\left(-|\underline{\sigma_1}|+\frac{\underline{\phi_1}}{\gamma^+}\right)}.$$

Then, $J_\lambda(u_0)<0$ for any $\lambda\in(\lambda_0,\infty)$. We deduce that $J_\lambda(u_\delta)<0$ for any $\lambda>\lambda_0$, u_δ is a weak nontrivial solution of problem (2). □

4. Existence of Solution by Mountain Pass Lemma

In this section, we deal with the existence of nontrivial weak solutions for the problem (2) by applying Mountain Pass geometry lemma giving by Lemma 2.2.

The main result in this case is the following.

Theorem 4.1. *Assume that condition (6) – (9) and (11) are fulfilled with $\gamma^->\mu p^+$. Then, there exists $\lambda_1>0$ such that for $\lambda<\lambda_1$, the problem (2) has at least one weak nontrivial solution.*

We begin by establishing some basic properties on the functional.

Lemma 4.1. *Assume that condition (4) – (9) and (12) are fulfilled with $\gamma^->\mu p^+$. Then, for any $\lambda>0$, the functional J_λ satisfies the Palais-Smale*
condition.

Proof. Note that $\mathcal{H}$ is finite dimensional Banach space, we only need to show that
$J_\lambda(u_n)\longrightarrow-\infty$ as $\|u_n\|\longrightarrow\infty$. From assumptions (4) – (9) and (12), one has

$$\begin{aligned}
J_\lambda(u_n)\ &=\ \widehat{M}\left(\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}A(k-1,h-1,\Delta u_n(k-1,h-1))\right)- \\
&\quad\ \lambda\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}F(k,h,u_n(k,h)) \\
J_\lambda(u_n)\ &\leq\ \int_0^{\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}A(k-1,h-1,\Delta u_n(k-1,h-1))}R_2 s^{\mu-1}\mathrm{d}s-
\end{aligned}$$

$$
\begin{aligned}
&\lambda \sum_{k=1}^{T_1}\sum_{h=1}^{T_2} F(k,h,u_n(k,h)) \\
\leq\ & \frac{R_2}{\mu}\left(\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} A(k-1,h-1,\Delta u_n(k-1,h-1))\right)^{\mu} - \\
&\qquad \lambda\left(\sum_{k=1}^{T_1}\sum_{h=1}^{T_2} -|\underline{\sigma_1}||u_n(k,h)| + \frac{\underline{\phi_1}}{\gamma^+}\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}|u_n(k,h)|^{\gamma(k,h)}\right) \\
\leq\ & \frac{R_2}{\mu}\left(\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}\int_0^{\Delta u_n(k-1,h-1)} |a(k-1,h-1,s)|\mathrm{d}s\right)^{\mu} - \\
&\qquad \lambda\left(\sum_{k=1}^{T_1}\sum_{h=1}^{T_2} -|\underline{\sigma_1}||u_n(k,h)| + \frac{\underline{\phi_1}}{\gamma^+}\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}|u_n(k,h)|^{\gamma^-}\right) \\
\leq\ & \frac{R_2}{\mu}\left(\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} C_1|\Delta u_n(k-1,h-1)| + \right. \\
&\left.\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}\frac{C_1}{p(k,h)}|\Delta u_n(k-1,h-1)|^{p(k-1,h-1)}\right)^{\mu} - \\
&\lambda\left(-|\underline{\sigma_1}|\sqrt{T_1\times T_2}\|u_n\| + \frac{\underline{\phi_1}}{\gamma^+}\|u_n\|^{\gamma^-}\right) \\
\leq\ & \frac{R_2}{\mu}(C_1)^{\mu}\left(\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}|\Delta u_n(k-1,h-1)| + \right. \\
&\left.\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}\frac{1}{p^-}|\Delta u_n(k-1,h-1)|^{p(k-1,h-1)}\right)^{\mu} - \lambda\left(-|\underline{\sigma_1}|\sqrt{T_1\times T_2}\|u_n\| + \frac{\underline{\phi_1}}{\gamma^+}\|u_n\|^{\gamma^-}\right) \\
\leq\ & \frac{R_2}{\mu}(C_1)^{\mu}\left(2\sqrt{(T_1+1)\times(T_2+1)}\|u_n\| + \frac{C_5}{p^-}\|u_n\|^{p^+} + \frac{C_6}{p^-}\right)^{\mu} - \\
&\qquad \lambda\left(-|\underline{\sigma_1}|\sqrt{T_1\times T_2}\|u_n\| + \frac{\underline{\phi_1}}{\gamma^+}\|u_n\|^{\gamma^-}\right).
\end{aligned}
$$

Since $\gamma^- > \mu p^+$ and $\|u_n\| \longrightarrow \infty$, we obtain $J_\lambda(u_n) \longrightarrow -\infty$. □

Proof. **Proof of Theorem 4.1**. Set $\Omega := \{u \in \mathcal{H} : \|u\| \leq \theta\}$, with $\theta \in (0,1)$. Recall that

$$J_\lambda(u) = \widehat{M}\left(\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} A(k-1,h-1,\Delta u(k-1,h-1))\right) - \lambda\sum_{k=1}^{T_1}\sum_{h=1}^{T_2} F(k,h,u(k,h)).$$

For $u \in \Omega$, from (6) $-$ (9) and (11), it follows that

$$
\begin{aligned}
J_\lambda(u) \quad \geq\quad & \frac{R_1}{\mu(p^+)^{\mu}}\left(\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}|\Delta u(k-1,h-1)|^{p(k-1,h-1)}\right)^{\mu} - \\
& \lambda\left(\frac{\overline{\phi_2}}{\gamma^-}\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}|u(k,h)|^{\gamma^-} + |\underline{\sigma_2}|\sqrt{T_1\times T_2}\left(\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}|u(k,h)|^2\right)^{\frac{1}{2}}\right)
\end{aligned}
$$

$$\geq \frac{R_1}{\mu}\left(\frac{C_4}{(p^+)}\right)^{\mu}\|u\|^{\mu p^+} - \lambda\left(\frac{\overline{\phi_2}}{\gamma^-}\|u\|^{\gamma^-} + |\underline{\sigma_2}|\sqrt{T_1\times T_2}\|u\|\right).$$

For $u \in \partial\Omega$, one has

$$\begin{aligned} J_\lambda(u) &\geq \frac{R_1}{\mu}\left(\frac{C_4}{(p^+)}\right)^{\mu}\theta^{\mu p^+} - \lambda\left(\frac{\overline{\phi_2}}{\gamma^-}\theta^{\gamma^-} + |\underline{\sigma_2}|\sqrt{T_1\times T_2}\theta\right) \\ &\geq \frac{R_1}{\mu}\left(\frac{C_4}{(p^+)}\right)^{\mu}\theta^{\mu p^+} - \lambda\theta\left(\frac{\overline{\phi_2}}{\gamma^-} + |\underline{\sigma_2}|\sqrt{T_1\times T_2}\right). \end{aligned}$$

So for every $\lambda \in (0,\lambda_1)$; $J_\lambda(u) > 0$ for all $u \in \partial\Omega$ with

$$\lambda_1 = \frac{\frac{R_1}{\mu}\left(\frac{C_4}{(p^+)}\right)^{\mu}\theta^{\mu p^+ - 1}}{\frac{\overline{\phi_2}}{\gamma^-} + |\underline{\sigma_2}|\sqrt{T_1\times T_2}}. \tag{16}$$

For $u \in \mathcal{H}$ such that $u(k,h) > 1$, for $(k,h) \in \mathbb{N}[1,T_1+1]\times\mathbb{N}[1,T_2+1]$; from (4) – (6) and 8, we obtain

$$\begin{aligned} J_\lambda(u) &\leq \frac{R_2}{\mu}(C_1)^{\mu}\left(\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}|\Delta u(k-1,h-1)| + \right. &(17)\\ &\left.\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}\frac{1}{p^-}|\Delta u(k-1,h-1)|^{p(k-1,h-1)}\right)^{\mu} - \\ &\lambda\left(\sum_{k=1}^{T_1}\sum_{h=1}^{T_2} -|\underline{\sigma_1}||u(k,h)| + \frac{\underline{\phi_1}}{\gamma^+}\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}|u(k,h)|^{\gamma(k,h)}\right). &(18) \end{aligned}$$

Consider $u_t \in \mathcal{H}$ defined in the following way

$$\begin{cases} u_t(k,h) = t \quad \text{for} \quad (k,h) \in (\mathbb{N}[0,T_1+1]\times\mathbb{N}[0,T_2+1])\setminus\{\Gamma\} \\ u_t(k,h) = 0 \quad \text{in} \quad \Gamma. \end{cases} \tag{19}$$

Using (17) and (19), there exist integers N_1, N_2, N_3 and N such that

$$\begin{aligned} J_\lambda(u) &\leq \frac{R_2}{\mu}(C_1)^{\mu}\left((N_1+N_2)t + \frac{1}{p^-}(N_1 t^{p^+} + N_2 t^{p^+})\right)^{\mu} - &(20)\\ &\lambda N_3\left(-|\underline{\sigma_1}|t + \frac{\underline{\phi_1}}{\gamma^+}t^{\gamma^-}\right) \\ &\leq \frac{R_2}{\mu}(C_1)^{\mu}\left(2(N_1+N_2)t^{p^+}\right)^{\mu} - \lambda N_3\left(-|\underline{\sigma_1}|t + \frac{\underline{\phi_1}}{\gamma^+}t^{\gamma^-}\right) &(21)\\ &\leq \frac{R_2}{\mu}(4NC_1)^{\mu}t^{\mu p^+} - \lambda N_3\left(-|\underline{\sigma_1}|t + \frac{\underline{\phi_1}}{\gamma^+}t^{\gamma^-}\right) &(22) \end{aligned}$$

where $N = \max\{N_1, N_2\}$. Since $\gamma^- > \mu p^+$, then $\lim_{t\to\infty} J_\lambda(u) = -\infty$. Thus, there exists t_1 such that for $u_{t_1} \in \mathcal{H}\setminus\{\Omega\}$, $J_\lambda(u_{t_1}) < \min_{u\in\partial\Omega} J_\lambda(u)$. According to Lemma 2.2 the problem (2) has at least one weak nontrivial solution. □

5. Existence of solution by Ekeland's principle

In this section, existence of nontrivial weak solutions are obtained by using Theorem 2.2.

The main result in this case is the following.

Theorem 5.1. *Assume that condition (5) and (7) − (9) are fulfilled. Then, for any* $\lambda \in (0, \lambda_1)$ *such that* $\mu p^- > \gamma^-$ *and* $\dfrac{\phi_1}{|\underline{\sigma_1}|} > \gamma^+ a_0$*, the problem (2) has at least one weak nontrivial solution.*

Proof. For $\lambda \in (0, \lambda_1)$. Using the proof of Theorem 4.1, for $u \in \partial\Omega$, we obtain $J_\lambda(u) > 0$. From Weierstrass theorem, one has

$$\min_{u\in\partial\Omega} J_\lambda(u) > 0. \tag{23}$$

Taking $u(k,h) \in (0,\theta)$, according to (5) and (7) − (9), we have

$$\begin{aligned} J_\lambda(u) \;\le\; & \frac{R_2}{\mu}(C_1)^\mu \left(\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} |\Delta u(k-1,h-1)| + \right. \\ & \left. \sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} \frac{1}{p^-}|\Delta u(k-1,h-1)|^{p(k-1,h-1)}\right)^\mu - \lambda \left(\sum_{k=1}^{T_1}\sum_{h=1}^{T_2} -|\underline{\sigma_1}||u(k,h)| + \right. \\ & \left. \frac{\phi_1}{\gamma^+}\sum_{k=1}^{T_1}\sum_{h=1}^{T_2} |u(k,h)|^{\gamma(k,h)}\right). \end{aligned}$$

For $s \in (0,\theta)$, assume that

$$s < \sqrt[(\mu p^- - \gamma^-)]{\frac{\lambda\left(-|\underline{\sigma_1}|a_0 + \dfrac{\phi_1}{\gamma^+}\right)}{\dfrac{R_2}{\mu}(2C_1)^\mu\left(a_1 + \dfrac{1}{p^-}\right)^\mu}}.$$

Let $(k_0,h_0) \in (\mathbb{N}[1,T_1+1] \times \mathbb{N}[1,T_2+1]) \setminus \{\Gamma\}$ such that $\gamma(k_0,h_0) = \gamma^-$.

We choose $u_0 \in \mathcal{H}$ be a function such that $u_0(k_0,h_0) = s$ and $u_0(k,h) = 0$ for any $(k,h) \in (\mathbb{N}[1,T_1+1] \times \mathbb{N}[1,T_2+1]) \setminus \{(k_0,h_0)\}$. We get

$$\begin{aligned} J_\lambda(u_0) \;&\le\; \frac{R_2}{\mu}(C_1)^\mu\left(2s + \frac{s^{p(k_0,h_0)} + s^{p(k_0-1,h_0-1)}}{p^-}\right)^\mu - \lambda\left(-|\underline{\sigma_1}|s + \frac{\phi_1}{\gamma^+}s^{\gamma^-}\right) \\ &\le\; \frac{R_2}{\mu}(C_1)^\mu\left(2s + \frac{2s^{p^-}}{p^-}\right)^\mu - \lambda\left(-|\underline{\sigma_1}|s + \frac{\phi_1}{\gamma^+}s^{\gamma^-}\right) \\ &\le\; \frac{R_2}{\mu}(2C_1)^\mu\left(s + \frac{s^{p^-}}{p^-}\right)^\mu - \lambda\left(-|\underline{\sigma_1}|s + \frac{\phi_1}{\gamma^+}s^{\gamma^-}\right). \end{aligned} \tag{24}$$

We can find two constants a_1, $a_0 > 1$ such that $a_1 s^{p^-} \ge s$ and $a_0 s^{\gamma^-} \ge s$. Then, inequality (24) gives

$$J_\lambda(u_0) \;\le\; \frac{R_2}{\mu}(2C_1)^\mu s^{\mu p^-}\left(a_1 + \frac{1}{p^-}\right)^\mu - \lambda\left(-|\underline{\sigma_1}|a_0 + \frac{\phi_1}{\gamma^+}\right)s^{\gamma^-} \tag{25}$$

$$\le\; \frac{R_2}{\mu}(2C_1)^\mu s^{\mu p^-}\left(a_1 + \frac{1}{p^-}\right)^\mu - \lambda\left(-|\underline{\sigma_1}|a_0 + \frac{\phi_1}{\gamma^+}\right)s^{\gamma^-} < 0. \tag{26}$$

Thus, $J_\lambda(u_0) < 0$ for $u_0 \in int(\Omega)$. Therefore,

$$-\infty < \inf_{u \in int(\Omega)} J_\lambda(u) < 0.$$

So, we have

$$\inf_{u \in int(\Omega)} J_\lambda(u) < \inf_{u \in \partial\Omega} J_\lambda(u).$$

Using the proof in [15], it follows that

$$\inf_{u \in \partial\Omega} J_\lambda(u) - \inf_{u \in int(\Omega)} J_\lambda(u) > \epsilon > 0.$$

Applying Ekeland's variational principle to the functional $J_\lambda : \Omega \longrightarrow \mathbb{R}$,

we find $u_\epsilon \in \Omega$ such that

$$J_\lambda(u_\epsilon) < \inf_{u \in \Omega} J_\lambda(u) + \epsilon$$

and

$$J_\lambda(u_\epsilon) < J_\lambda(u) + \epsilon\|u - u_\epsilon\| \qquad \text{for} \qquad u \neq u_\epsilon.$$

Since

$$J_\lambda(u_\epsilon) < \inf_{u \in \Omega} J_\lambda(u) + \epsilon \leq \inf_{u \in int(\Omega)} J_\lambda(u) + \epsilon < \inf_{u \in \partial\Omega} J_\lambda(u)$$

we deduce $u_\epsilon \in int(\Omega)$. Now, we define $H_\lambda : \Omega \longrightarrow \mathbb{R}$ by

$$H_\lambda(u) = J_\lambda(u) + \epsilon\|u - u_\epsilon\| \qquad \text{for} \qquad u \neq u_\epsilon. \tag{27}$$

We have u_ϵ as a minimum of the functional H_λ and therefore $\forall u \in \Omega$

$$H_\lambda(u) \geq H_\lambda(u_\epsilon). \tag{28}$$

Taking $u = u_\epsilon + tv$, $v \in \Omega$ and $t > 0$ in the relation (27). From (28) and letting $t \longrightarrow 0$, it follows that

$$\langle J'_\lambda(u_\epsilon), v\rangle + \epsilon\|v\| \geq 0 \tag{29}$$

and

$$\|J'_\lambda(u_\epsilon)\| \leq \epsilon. \tag{30}$$

Thus, there exists a sequence $\{y_n\} \subset int(\Omega)$, (see [18]) such that

$$J_\lambda(y_n) \longrightarrow \inf_{u \in \Omega} J_\lambda(u) \quad \text{and} \quad J'_\lambda(y_n) \longrightarrow 0.$$

Since $\{y_n\}$ is bounded in $\mathcal{H}$ there exists $y_0 \in \mathcal{H}$ such that, up to a subsequence $\{y_n\}$ converge to $y_0 \in \mathcal{H}$. Thus,

$$J_\lambda(y_0) = \inf_{u \in \Omega} J_\lambda(u) \quad \text{and} \quad J'_\lambda(y_0) = 0.$$

y_0 is one weak nontrivial solution for the problem (2). □

6. An Extension

In this section we are concerned the study of the general boundary value problems namely

$$\begin{cases} -M(A(k-1,h-1,\Delta u(k-1,h-1)))\Delta(a(k-1,h-1,\Delta u(k-1,h-1))) \\ +|u(k,h)|^{r(k,h)-2}u(k,h) = \lambda f(k,h,u(k,h)), (k,h) \in \mathbb{N}[1,T_1] \times \mathbb{N}[1,T_2], \\ u(k,h) = 0, \ (k,h) \in \Gamma, \end{cases} \tag{31}$$

where $r : \mathbb{N}[1,T_1] \times \mathbb{N}[1,T_2] \longrightarrow (2,\infty)$.

We denote by

$$r^- = \min_{\{(k,h)\in\mathbb{N}[1,T_1]\times\mathbb{N}[1,T_2]\}} r(k,h) \quad \text{and} \quad r^+ = \max_{\{(k,h)\in\mathbb{N}[1,T_1]\times\mathbb{N}[1,T_2]\}} r(k,h).$$

Definition 6.1. *A function $u \in \mathcal{H}$ is a solution of problem (31) if for any $v \in \mathcal{H}$,*

$$\begin{cases} M\left(\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} A(k-1,h-1,\Delta u(k-1,h-1))\right) \times \\ \sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} a(k-1,h-1,\Delta u(k-1,h-1))\Delta v(k-1,h-1) + \\ \sum_{k=1}^{T_1}\sum_{h=1}^{T_2} |u(k,h)|^{r(k,h)-2}u(k,h)v(k,h) = \lambda \sum_{k=1}^{T_1}\sum_{h=1}^{T_2} f(k,h,u(k,h))v(k,h). \end{cases}$$

6.1. The main results and their proofs.

Theorem 6.1. *Assume that condition (6) − (9) are fulfilled. Then, for $\mu p^- > \gamma^+$ and $\dfrac{\phi_1}{|\sigma_1|} > \gamma^+$, there exists $\lambda_2 \in (0,\infty)$ such that for all $\lambda > \lambda_2$, the problem (31) has at least one weak nontrivial solution.*

Proof. We define energy functional corresponding to problem (31) by

$$\begin{aligned} J_\lambda(u) &= \widehat{M}\left(\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} A(k-1,h-1,\Delta u(k-1,h-1))\right) + \\ &\quad \sum_{k=1}^{T_1}\sum_{h=1}^{T_2} \frac{1}{r(k,h)}|u(k,h)|^{r(k,h)} - \lambda \sum_{k=1}^{T_1}\sum_{h=1}^{T_2} F(k,h,u(k,h)). \end{aligned}$$

From [10], the functional J_λ is weakly lower semi-continuous and is of class C^1 and

$$\begin{cases} \langle J'_\lambda(u), v\rangle = M\left(\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} A(k-1,h-1,\Delta u(k-1,h-1))\right) \times \\ \sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} a(k-1,h-1,\Delta u(k-1,h-1))\Delta v(k-1,h-1) + \\ \sum_{k=1}^{T_1}\sum_{h=1}^{T_2} |u(k,h)|^{r(k,h)-2}u(k,h)v(k,h) - \lambda \sum_{k=1}^{T_1}\sum_{h=1}^{T_2} f(k,h,u(k,h))v(k,h). \end{cases} \tag{32}$$

Since

$$\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}\frac{1}{r(k,h)}|u(k,h)|^{r(k,h)}\geq 0$$

it follows that

$$J_\lambda(u)\geq \widehat{M}\left(\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}A(k-1,h-1,\Delta u(k-1,h-1))\right)-\lambda\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}F(k,h,u(k,h)).$$

According to proposition 3.1, J_λ is coercive.

Let $\tilde{u}$ be a global minimizer of J_λ. For $t_0>1$ be a fixed real and $(k_0,h_0)\in\mathbb{N}[0,T_1+1]\times\mathbb{N}[0,T_2+1]$. We define $u_0\in\mathcal{H}$ in the following way

$$\begin{cases} u_0(k_0,h_0) &= t_0 \\ \\ u_0(k,h) &= 0, \quad (k,h)\in(\mathbb{N}[0,T_1+1]\times\mathbb{N}[0,T_2+1])\setminus\{(k_0,h_0)\}.\end{cases}$$

$$J_\lambda(u_0)=\widehat{M}\left(\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}A(k-1,h-1,\Delta u_0(k-1,h-1))\right)-\lambda\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}F(k,h,u_0(k,h)).$$

We have

$$J_\lambda(u_0)\leq\frac{R_2}{\mu}(4C_1)^\mu t_0^{\mu p^+}+\frac{t_0^{r^+}}{r^-}-\lambda t_0\left(-|\underline{\sigma_1}|+\frac{\underline{\underline{\phi_1}}}{\gamma^+}\right) \tag{33}$$

where

$$\lambda_2=\frac{\frac{R_2}{\mu}(4C_1)^\mu t_0^{\mu p^+-1}+\frac{t_0^{r^+-1}}{r^-}}{-|\underline{\sigma_1}|+\frac{\underline{\underline{\phi_1}}}{\gamma^+}}. \tag{34}$$

It follows that $J_\lambda(\tilde{u})<0$ for any $\lambda>\lambda_2$, $\tilde{u}$ is a weak nontrivial solution of problem (31). □

Lemma 6.1. *Assume that condition (4) – (6) and (8) – (9) are fulfilled with $\gamma^->\max\{p^+,r^+\}$. Then, for any $\lambda>0$, the functional J_λ satisfies the Palais-Smale condition.*

Proof. We follows the results in the proof of Lemma 4.1, to obtain

$$\begin{aligned} J_\lambda(u_n) &\leq \frac{R_2}{\mu}(C_1)^\mu\left(\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}|\Delta u_n(k-1,h-1)|+\right. \\ &\quad \left.\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}\frac{1}{p^-}|\Delta u_n(k-1,h-1)|^{p(k-1,h-1)}\right)^\mu+\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}\frac{1}{r(k,h)}|u_n(k,h)|^{r(k,h)}- \\ &\quad \lambda\left(-|\underline{\sigma_1}|\sqrt{T_1\times T_2}\|u_n\|+\frac{\underline{\underline{\phi_1}}}{\gamma^+}\|u_n\|^{\gamma^-}\right) \\ &\leq \frac{R_2}{\mu}(C_1)^\mu\left(2\sqrt{(T_1+1)\times(T_2+1)}\|u_n\|+\frac{C_5}{p^-}\|u_n\|^{p^+}+\frac{1}{r^-}\|u_n\|^{r^+}+\frac{C_6}{p^-}\right)^\mu- \\ &\quad \lambda\left(-|\underline{\sigma_1}|\sqrt{T_1\times T_2}\|u_n\|+\frac{\underline{\underline{\phi_1}}}{\gamma^+}\|u_n\|^{\gamma^-}\right).\end{aligned}$$

Since $\gamma^->\max\{p^+,r^+\}$, then $J_\lambda(u_n)\longrightarrow-\infty$ as $\|u_n\|\longrightarrow\infty$. □

Theorem 6.2. *Assume that condition (6), (8), (9) and (11) are fulfilled with $\gamma^- > \max\{\mu p^+, r^+\}$. Then, for $\lambda \in (0, \lambda_1)$, the problem (31) has at least one weak nontrivial solution.*

Proof. Let $\Omega := \{u \in \mathcal{H} : \|u\| \leq \theta\}$ with $\theta \in (0,1)$. Recall that,

$$J_\lambda(u) = \widehat{M}\left(\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} A(k-1,h-1,\Delta u(k-1,h-1))\right) - \lambda \sum_{k=1}^{T_1}\sum_{h=1}^{T_2} F(k,h,u(k,h)).$$

Taking $u \in \Omega$, from (6) − (8), (9) and (11), one has

$$\begin{aligned} J_\lambda(u) \quad &\geq \quad \frac{R_1}{\mu(p^+)^\mu}\left(\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}|\Delta u(k-1,h-1)|^{p(k-1,h-1)}\right)^\mu \\ &\quad -\lambda\left(\frac{\overline{\phi_2}}{\gamma^-}\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}|u(k,h)|^{\gamma^-} + |\underline{\sigma_2}|\sqrt{T_1\times T_2}\left(\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}|u(k,h)|^2\right)^{\frac{1}{2}}\right). \end{aligned}$$

For $u \in \partial\Omega$, we obtain

$$J_\lambda(u) \quad \geq \quad \frac{R_1}{\mu(p^+)^\mu}C_4\theta^{\mu p^+} - \lambda\theta\left(\frac{\overline{\phi_2}}{\gamma^-} + |\underline{\sigma_2}|\sqrt{T_1\times T_2}\right). \tag{35}$$

So, for any $\lambda < \lambda_1$; $J_\lambda(u) > 0$ for all $u \in \partial\Omega$. For $u \in \mathcal{H}$ such that $u(k,h) > 1$, for $(k,h) \in \mathbb{N}[1, T_1+1] \times \mathbb{N}[1, T_2+1]$. From (4) − (6) and (8), we get

$$\begin{aligned} J_\lambda(u) \quad &\leq \quad \frac{R_2}{\mu}(C_1)^\mu\left(\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}|\Delta u(k-1,h-1)| + \right. \\ &\left.\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}\frac{1}{p^-}|\Delta u(k-1,h-1)|^{p(k-1,h-1)}\right)^\mu + \sum_{k=1}^{T_1}\sum_{h=1}^{T_2}\frac{1}{r(k,h)}|u(k,h)|^{r(k,h)} \\ &-\lambda\left(\sum_{k=1}^{T_1}\sum_{h=1}^{T_2} -|\underline{\sigma_1}||u(k,h)|\right. \\ &\left. + \frac{\underline{\phi_1}}{\gamma^+}\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}|u(k,h)|^{\gamma(k,h)}\right). \end{aligned} \tag{36}$$

Define $u_t \in \mathcal{H}$ such that

$$\begin{cases} u_t(k,h) \quad = \quad t \quad \text{for } (k,h) \in (\mathbb{N}[0,T_1+1]\times\mathbb{N}[0,T_2+1]) \setminus \{\Gamma\}, \\ \\ u_t(k,h) \quad = \quad 0 \quad \text{in } \Gamma. \end{cases} \tag{37}$$

We can find integers N_1, N_2, N_3 and N such that

$$J_\lambda(u) \quad \leq \quad \frac{R_2}{\mu}(4NC_1)^\mu t^{\mu p^+} + N_3 t^{r^+} - \lambda N_3\left(-|\underline{\sigma_1}|t + \frac{\underline{\phi_1}}{\gamma^+}t^{\gamma^-}\right) \tag{38}$$

with $N = \max\{N_1, N_2\}$. Since $\gamma^- > \max\{\mu p^+, r^+\}$, then $\lim_{t\to\infty} J_\lambda(u) = -\infty$. Thus, there exists t_0 such that for $u_{t_0} \in \mathcal{H} \setminus \{\Omega\}$, $J_\lambda(u_{t_0}) < \min_{u\in\partial\Omega} J_\lambda(u)$. According to Lemma 2.2, the problem (31) has at least one weak nontrivial solution. □

Next, Apply Ekeland's variational principle with $\min\{\mu p^-, r^-\} > \gamma^-$, we will use the result of case $\mu p^- > \gamma^-$ and $\frac{\phi_1}{|\underline{\sigma_1}|} > \gamma^+ a_0$. It well know that for $\lambda \in (0, \lambda_1)$, one has

$$\min_{u\in\partial\Omega} J_\lambda(u) > 0. \tag{39}$$

Let $s \in (0, \theta)$ and assume that

$$s < \sqrt[\left(\min\{\mu p^-, r^-\}-\gamma^-\right)]{\frac{\lambda\left(-|\underline{\sigma_1}|a_0 + \frac{\phi_1}{\gamma^+}\right)}{\frac{R_2}{\mu}(2C_1)^\mu\left(\left(a_2+\frac{1}{p^-}\right)^\mu + \frac{\mu}{R_2(2C_1)^\mu}\right)}}. \tag{40}$$

Taking $(k_0, h_0) \in \mathbb{N}[0, T_1 + 1] \times \mathbb{N}[0, T_2 + 1] \setminus \{\Gamma\}$ such that $\gamma(k_0, h_0) = \gamma^-$.

Let $u_0 \in \mathcal{H}$ be a function such that $u_0(k_0, h_0) = s$ and $u_0(k, h) = 0$ for any $(k, h) \in (\mathbb{N}[1, T_1 + 1] \times \mathbb{N}[1, T_2 + 1]) \setminus \{(k_0, h_0)\}$. From inequality (36), one has

$$J_\lambda(u_0) \leq \frac{R_2}{\mu}(2C_1)^\mu\left(s + \frac{s^{p^-}}{p^-}\right)^\mu + s^{r^-} - \lambda\left(-|\underline{\sigma_1}|s + \frac{\phi_1}{\gamma^+}s^{\gamma^-}\right). \tag{41}$$

We can find two constants $a_2, a_0 > 1$ such that $a_2 s^{p^-} \geq s$ and $a_0 s^{\gamma^-} \geq s$. Thus, we obtain

$$\begin{aligned} J_\lambda(u_0) &\leq \frac{R_2}{\mu}(2C_1)^\mu s^{\mu p^-}\left(a_2 + \frac{1}{p^-}\right)^\mu + s^{r^-} - \lambda\left(-|\underline{\sigma_1}|a_0 + \frac{\phi_1}{\gamma^+}\right)s^{\gamma^-} \\ &\leq \frac{R_2}{\mu}(2C_1)^\mu s^{\min\{\mu p^-, r^-\}}\left(\left(a_2 + \frac{1}{p^-}\right)^\mu + \frac{\mu}{R_2(2C_1)^\mu}\right) - \lambda\left(-|\underline{\sigma_1}|a_0 + \frac{\phi_1}{\gamma^+}\right)s^{\gamma^-}. \end{aligned}$$

Thus, $J_\lambda(u_0) < 0$, for $u_0 \in int(\Omega)$.

In the sequel, we follows the results in the proof of Theorem 5.1, to show that the problem (31) has at least one weak nontrivial solution.

Authors' Contributions. All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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