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UNIQUENESS RESULT FOR SEMI-LINEAR DIFFERENTIAL EQUATIONS WITH STATE-DEPENDENT DELAYS AND ITS APPLICATION

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Abstract. In this paper, we present several results regarding the existence and uniqueness of solutions to certain semi-linear differential equations with state-dependent delays. Our approach relies on Banach's fixed point theorem. To support the theoretical findings, a representative example is provided.

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1. Introduction

In this paper, we investigate the existence and uniqueness of solutions to differential equations with state-dependent delays, defined on the interval $J := [0, T]$, and expressed in the following form:

$$x'(t) = ax(t) + f(t, x(t - \tau(t, x_t))),\ t \in J, \tag{1}$$

with initial condition

$$x(t) = \varphi(t),\ t \in [-r, 0], \tag{2}$$

where $f : J \times \mathbb{R}^n \to \mathbb{R}^n$ is given functions, $a \in \mathbb{R}^n$ is constant and $\tau : [0, T] \times C([-r, 0], \mathbb{R}^n) \to [0, r]$ is given continuous function.

For any function x defined on $[-r, T]$ and any $t \in J$ we denote by x_t the element of $C([-r, 0], E)$ defined by

$$x_t(\theta) = x(t + \theta),\ \theta \in [-r, 0].$$

Here $x_t(\cdot)$ represents the history of the state from time $t - r$, up to the present time t.

In general, a differential equation is a mathematical equation that establishes a relationship between one or more indeterminate functions and their derivatives [17]. Physical quantities are typically represented by functions in applications, their rates of change are represented by derivatives, and a

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relationship between the two is defined by the differential equation. Semi-linear differential equations are a type of differential equation that incorporates both linear and nonlinear components. Semi-linear differential equations are a substantial subject of investigation in the fields of theoretical physics and applied mathematics. Their blended linear and nonlinear characteristics render them both intriguing and difficult to analyze. the comprehension of the methods for solving these equations is essential for applications in a variety of scientific disciplines, including biology [1,3] and engineering [4]. One of the oldest branches of the theory of infinite-dimensional dynamical systems is delay differential equations, which defines the qualitative properties of systems that change over time. We consult the classical monographs on the theory of ordinary delay equations (ODE) [6,11]. the history of functional differential equations (FDEs) can be traced back to the early 20th century, when there was a particular interest in FDEs with constant delays. In recent years, there has been a substantial increase in the theoretical comprehension of FDEs, as a number of researchers have investigated the existence and uniqueness of solutions to a variety of FDE classes [10] and neutral FDEs of fractional order (see [15] and [9]). Functional differential equations are employed in a variety of scientific fields, including control theory [15] and uniform persistence [16]. The application of functional analysis to differential equations is the primary focus of a significant amount of recent research in functional differential equations as [7,8,14].

This paper is structured as follows: Section 2 provides a brief review of essential definitions and preliminary results that will be used in the subsequent sections. In Section 3, we establish one of our main existence results for the solutions of equations (1)-(2), using Banach's fixed point theorem. Section 4 presents an illustrative example to confirm the validity of our findings.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. Let $(\mathbb{R}^n, \|\cdot\|)$ be a Banach space.

$C([-r,T],\mathbb{R}^n)$ is the Banach space of all continuous functions from $[-r,T]$ into $\mathbb{R}^n$ with the norm

$$\|x\|_\infty = \sup_{\theta\in[-r,0]} \sup_{t\in[0,T]} \|x(t+\theta)\|.$$

In a normed space $(X, \|.\|_X)$, the open ball around a point x_0 with radius R is denoted by $B_X(x_0,R)$, i.e., $B_X(x_0,R) := \{x \in X : \|x - x_0\|_X < R\}$, and the corresponding closed ball, by $\bar{B}_X(x_0,R)$.

3. Main Results

In this section we give our main existence result for problem (1)-(2). Before stating and proving this result, we give the definition of its solution.

Definition 3.1. *We say that a continuous function* $x : [-r, T] \to E$ *is a solution of problem* (1)-(2) *if* $x(t) = \varphi(t),\ t \in [-r, 0]$ *and*

$$x(t) = \exp(at)\varphi(0) + \int_0^t \exp(a(t-s))f(s, x(s - \tau(s, x_s)))ds,\ t \in J.$$

Let Set $C_r := C([-r, 0], \mathbb{R}^n)$, $\Omega \in C_r$ be open subset and Let $T > 0$ be finite, or $T = \infty$, in which case $[0, T]$ denotes the interval $[0, \infty)$.

Let $\Omega_1 \in C_r$ and $\Omega_2 \in \mathbb{R}^n$ be open subsets of the respective spaces. Let $T > 0$ be finite or $T = \infty$, in which case $[0, T]$ denotes the interval $[0, \infty)$.

We introduce the set

$$\Theta = \{\varphi \in C_r :\ \varphi \in \Omega_1,\ \varphi(-\tau(0, \varphi)) \in \Omega_2\}.$$

Let us introduce the following hypotheses:

(H_1) (i) $f : J \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous.

(ii) There exists a constant $L_1 > 0$ such that

$$\|f(t, u_1)) - f(t, u_2))\| \leq L_1\|u_1 - u_2\|,$$

for every $t \in [0, T]$ and u_1, $u_2 \in \mathbb{R}^n$.

(iii) There exists a constant $L_2 > 0$ such that

$$\|f(t, y_{\tau_1})) - f(t, y_{\tau_2}))\| \leq L_2\|\tau_1 - \tau_2\|,$$

for every $t \in [0, T]$, $\tau_1, \tau_2 \in [-r, 0]$ and $y \in \mathbb{R}^n$.

(H_2) There exists a constant $L_3 > 0$ such that

$$\|\tau(t, \psi) - \tau(t, \bar{\psi})\| \leq L_3\|\psi - \bar{\psi}\|,$$

for all $\psi, \bar{\psi} \in C_r$ and $t \in J$.

Theorem 3.2. *Assume that assumptions* (H_1)*-*(H_2) *hold and let* $\hat{\gamma} \in \Theta$*. Then, there exist* $\delta > 0$ *and* $0 < \alpha \leq T$ *such that for all* $\gamma \in B = \bar{B}_{C_r}(\hat{\gamma}, \delta)$ *the problem* (1)-(2) *has a unique solution on* $[-r, \alpha]$.

Proof: There exists a constant $K > 0$, $M \geq 0$ and $\sigma \in [0, T]$ such that

$$\|f(t, x_{\tau(t, x_t)})\| \leq K$$

and

$$\|\exp(a\sigma)\| \leq M.$$

See details in [9].

We define the following constant

$$\delta = \|\varphi(0)\| + M(\|\varphi(0)\| + \sigma K)$$

and

$$E_0 = \{u \in C([-r,\sigma], E), u(t) = \varphi(t) \text{ if } t \in [-r,0] \text{ and } \sup_{t\in[0,\sigma]} \|u(t) - \varphi(0)\| \le \delta\}.$$

It is clear that E_0 is a closed set of $C([-r,\sigma], E)$. For $t \in [0,\sigma]$ and $u \in E_0$, transform the problem (1)-(2) into a fixed point problem. Consider the operator

$$N : E_0 \to C([-r,\sigma], E)$$

defined by

$$Nx(t) = \begin{cases} \varphi(t), & \text{t} \in [-r,0] \\ \exp(at)\varphi(0) + \int_0^t \exp(a(t-s)) f(s, x(s-\tau(s,x_s)))ds, & \text{t} \in J. \end{cases} \tag{3}$$

Note that a fixed point of N is a solution of (1)-(2). We will show that

$$N(E_0) \subseteq E_0.$$

Let $v \in E_0$ and $t \in [0,\sigma]$. We have

$$\begin{aligned} \|N(v)(t) - \varphi(0)\| &\le \|\exp(at)\varphi(0) - \varphi(0)\| + \int_0^t \|\exp(a(t-s)) f(s, v(s-\tau(s,v_s)))\|ds \\ &\le \exp(a\sigma)\|\varphi(0)\| + \|\varphi(0)\| + \exp(a\sigma)\int_0^\sigma \|f(s, v(s-\tau(s,v_s)))\|ds \\ &\le \|\varphi(0)\| + \exp(a\sigma)(\|\varphi(0)\| + \sigma K) \\ &\le \|\varphi(0)\| + M(\|\varphi(0)\| + \sigma K) \\ &\le \delta. \end{aligned}$$

Hence,

$$N(E_0) \subseteq E_0.$$

On the other hand, let $v, w \in E_0$. Then for $t \in [0,\sigma]$, we have

$$\begin{aligned} \|N(v)(t) - N(w)(t)\| &\le \int_0^t \|\exp(a(t-s))\| \|f(s, v(s-\tau(s,v_s))) - f(s, w(s-\tau(s,w_s)))\|ds \\ &\le M\int_0^t \|f(s, v(s-\tau(s,v_s))) - f(s, w(s-\tau(s,v_s)))\|ds \\ &\quad + M\int_0^t \|f(s, w(s-\tau(s,v_s))) - f(s, w(s-\tau(s,w_s)))\|ds \\ &\le ML_1\int_0^t \|v(s-\tau(s,v_s)) - w(s-\tau(s,v_s))\|ds \\ &\quad + ML_2\int_0^t \|\tau(s,v_s) - \tau(s,w_s)\|ds \\ &\le ML_1\int_0^t \|v(s-\tau(s,v_s)) - w(s-\tau(s,v_s))\|ds \end{aligned}$$

$$
\begin{aligned}
& +ML_2L_3\int_0^t \|v_s - w_s\| ds \\
\leq \quad & ML_1\int_0^t \sup_{r\in[0,\sigma]} \|v(r) - w(r)\| ds \\
& +ML_2L_3\int_0^t \sup_{s\in[0,\sigma]} \|v_s - w_s\| ds \\
\leq \quad & M\sigma(L_1 + L_2L_3)\|v - w\|_\infty.
\end{aligned}
$$

Consequently

$$\|N(v) - N(w)\|_\infty \leq M\sigma(L_1 + L_2L_3)\|v - w\|_\infty.$$

Since $M\sigma(L_1 + L_2L_3) < 1$, N is a contraction. By the Banach fixed point theorem [12], we conclude that N has a unique fixed point in E_0 and the problem (1)-(2) has a unique solution on $[-r, \sigma]$.

4. Illustrative Example

Let $t \geq 0$, consider

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} -0.03 & 0 \\ 0 & 0.03 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} f_1(t, x_1(t - \tau(t, x_t))) \\ f_2(t, x_2(t - \tau(t, x_t))) \end{pmatrix} \tag{4}$$

and

$$x(t) = (x_1(t), x_1(t)) = \varphi(t) = (\sin(t), \sin(t)), t \in [-\pi, 0], \tag{5}$$

where $f_i(t, x_i(t - \tau(t, x_t))) = \sin(x_i(t - |\sin(x_1(t) + x_2(t))|), i = 1, 2.$
$\tau(t, x_t) = -|\sin(x_1(t) + x_2(t))|$ and $a = \begin{pmatrix} -0.03 & 0 \\ 0 & 0.04 \end{pmatrix}$.

Evidently, $L_1 = 2$, $L_3 = 1$, $K = 1$. We have $\|f(t, x_{\tau_1})) - f(t, x_{\tau_2}))\| = 0$, we can take $L_2 = 0.5$ and $\sigma = \frac{\pi}{10} \in J$.
We have

$$M = e^{(0.04\times\frac{\pi}{10})} \geq \|e^{a\sigma}\| = \|e^{(a\sigma)}\| = \max\{e^{(-0.03\frac{\pi}{10})}, e^{(0.04\frac{\pi}{10})}\}$$

and

$$\delta = M(\|\varphi(0)\| + \sigma K) = \frac{\pi}{10}e^{(0.04\times\frac{\pi}{10})}.$$

Let

$$\Theta = \{\varphi \in C_r : \ \varphi \in \Omega_1, \ \varphi(-\tau(0, \varphi)) \in \Omega_2\}$$

with $\Omega_1 = B_{C_r}(0, 1.5)$ and $\Omega_2 = B_{\mathbb{R}^2}(0, 2)$.

Let $\hat{\gamma} = \varphi(0) = (0, 0) \in \Theta$, Since $M\sigma(L_1 + L_2L_3) = e^{(0.04*\frac{\pi}{10})} \times \frac{\pi}{10} \times 3 < 1$, from theorem 3.2, for all $\gamma \in B = \bar{B}_{C_r}(\hat{\gamma}, \delta)$ the problem (4)-(5) has a unique solution on $[-r, \sigma]$.

Conclusions

In this paper, we studied the solution and application of the proven theorem, focusing on the existence and uniqueness of solutions for delay differential equations with state-dependent delays. Through detailed analysis and practical implementation, we demonstrated how these results contribute to a deeper understanding of such equations. The significance of these findings extends beyond reinforcing existing theories, as they also open new avenues for future research and potential applications in the field of state-dependent delays. Future work may focus on extending these results to more complex cases and exploring their implications in various applied contexts.

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