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NEW COMPOUND POISSON PROCESS: PROPERTIES AND APPLICATION USING SURPLUS MODEL

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Abstract. Because of these advantages, the nonhomogeneous Poisson process has been widely used in many practical applications up to this point. But in terms of application, it also has a lot of limitations. The Poisson modified Lindley process, a groundbreaking counting process model, is created in order to get around these restrictions. It will be demonstrated that this new counting process model is not subject to these restrictions. A few fundamental stochastic properties are obtained. A new idea for positive dependent increments is also developed, and the dependence structure is examined. In this work, some of the attributes will be briefly mentioned. By providing a novel counting process model with an actuarial science application and example, this paper adds a great deal.

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1. Introduction

The most popular counting processes for modeling random recurrent events in many real-world applications are the renewal process and the nonhomogeneous Poisson process (NHPP), which occasionally includes the homogeneous Poisson process (HPP) as an exception. The NHPP lacks the HPP's fixed increments attribute because its rate of occurrence varies over time. In this sense, it is not the HPP. One of the primary characteristics of the NHPP is its ability to produce explicit results in various applications while maintaining the independent increments attribute [9]. The NHPP has been applied widely in practice due to its benefits (see [10–30]). Nevertheless, the NHPP has important application-specific limitations as well. The requirement that the variance and mean of the number of occurrences in $(0, t)$ be equal $Var[N(t)] = E[N(t)]$ is one of the most significant limitations. This limitation means that situations where the observations are either too closely spaced or too widely

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dispersed cannot be used with the NHPP. The fact that it increments independently is a significant disadvantage. Actually, most real-world problems cannot be adequately described by the assumption of increment independence. According to certain shock models, for example, a system is more vulnerable to shocks the more shocks it has previously encountered [11]. The structure of the paper is as follows. A new cumulative process model is defined in Section 2, and the distributions for the number(s) of events in a given time interval(s) are derived. In addition, we determine the mean and variance of the number of events in $(0, t)$ in Section 3 and obtain the stochastic intensity of the new counting process model. In addition, the dependence structure is examined and a new notion for positive dependent increments is defined. In Section 4, the corresponding compound process is briefly discussed, and in Section 5, an actuarial science example is suggested along with a simulation study.

2. Processus Poisson Lindley Modified

The creation of a new counting process model with mathematical properties is one of the work's main goals. We do this by creating a Poisson modified Lindley distribution using the concept. Let Φ follow the modified Lindley distribution [13] with the parameter θ with its probability density function

$$f(\Phi) = \frac{\theta}{1+\theta} e^{(-2\theta\Phi)}[(1+\theta)e^{(\theta\Phi)} + 2\theta\Phi - 1], \Phi \geq 0, \theta \geq 0.$$

The$r$-th moment of the modified Lindley distribution is given by

$$\mu_r' = E\left[\Phi^r\right] = \frac{1}{\theta^r}\left(1 + \frac{r}{2^{r+1}(1+\theta)} r!\right), r = 1, 2....$$

In particular, the first four moments of X are given by

$$\mu_1' = \frac{5+4\theta}{4\theta(1+\theta)}$$

$$\mu_2' = \frac{5+4\theta}{2\theta^2(1+\theta)}$$

$$\mu_3' = \frac{3\left(19+16\theta\right)}{8\theta^3(1+\theta)}$$

$$\mu_3' = \frac{3\left(9+8\theta\right)}{\theta^4(1+\theta)}$$

and the central moments are given by:

$$E\left[\left(\Phi - \mu_r'\right)^k\right] = \sum_{r=0}^{k} (k;r)\, \mu_r' \left(-\mu_1'\right)^{k-r} k = 1, 2...$$

The Poisson modified Lindley distributionis generated by mixing the Poisson distribution with the mean Φ, resulting in the following probability mass function.

$$\begin{aligned} P(x,\theta) &= \int_0^\infty \frac{\Phi^x}{x!} e^{-\Phi} \frac{\theta}{1+\theta} e\left\{-2\theta\Phi\right\}\left[(1+\theta)e\{\theta\Phi\} + 2\theta\Phi - 1\right] \\ &= \frac{\theta}{\theta+1}\frac{1}{x!}\left[(1+\theta)\int_0^\infty \Phi^x e^{-(\theta+1)\Phi} d\Phi + 2\theta \int_0^\infty \Phi^{x+1} e^{-(2\theta+1)\Phi} d\Phi - \int_0^\infty \Phi^x e^{-(2\theta+1)\Phi} d\Phi\right] \end{aligned}$$

$$\begin{aligned} &= \frac{\theta}{\theta+1}\frac{1}{x!}\left[(1+\theta)\frac{\Gamma(x+1)}{(\theta+1)^{x+1}}+2\theta\frac{\Gamma(x+2)}{(2\theta+1)^{x+2}}-\frac{\Gamma(x+1)}{(2\theta+1)^{x+1}}\right] \\ &= \frac{\theta}{\theta+1}\left[\frac{1}{(\theta+1)^{x}}+\frac{2\theta x}{(2\theta+1)^{x+2}}-\frac{1}{(2\theta+1)^{x+2}}\right], x\in\mathbb{N} \end{aligned} \quad (1)$$

To extend this Poisson modified -Lindley distribution to a counting process model with an explicit probability of the number of events, the idea is that in the mixing process in (1), we employ an additional time-dependent term in the mean value of the Poisson distribution.

Assume that the counting process $\{M(t), t \geq 0\}$is orderly. To show that the counting process $\{M(t), t \geq 0\}$ follows the NHPP with its intensity function $\lambda(t)$, we shall utilize the notation $\{M(t), t \geq 0\} \sim PPNH(v(t))$. Additionally, the continuous random variable Φ will be represented as following the modified Lindley distribution with parameter θ using the notation $\Phi \sim LM(\theta)$.

Definition 2.1. *A counting process $\{N(t), t \geq 0\}$ is called a Poisson modified Lindley process with the parameter set $(\lambda(t), \theta), \theta \geq 0, \lambda(t)) \geq 0$, pour $t \geq 0$ if:*

-$\{N(t), t \geq 0\} \mid (\Phi = \phi)$ ∼NHPP$(\Phi\lambda(t))$

-$\Phi \sim ML(\theta)$

Throughout this work, the modified Lindley Poisson process with the set of parameters $(\lambda(t), \theta)$ will be denoted as the (PMLP) and we define

$$\Lambda(t) = \int_0^t \lambda(X)\, dX$$

We will now derive some basic properties of PMLP $(\lambda(t), \theta)$ .First of all, when dealing with a counting process model, one might be interested in the number(s) of events in one or more-time intervals.

Proposition 2.2. *Let $\{N(t), t \geq 0\}$ Poisson Modified Lindley Process $(\lambda(t), \theta)$ then for $t \geq 0$ and $0 = t_0 \leq t_1 \leq t_2 \leq ... \leq t_m$ check the following properties:*

$i)$

$$p(N(t) = n) = \frac{\theta}{1+\theta}\Lambda(t)^n\left[(1+\theta)\frac{1}{(\Lambda(t)+\theta)^{n+1}}+2\theta\frac{n}{(\Lambda(t)+2\theta)^{n+2}}-\frac{1}{(\Lambda(t)+2\theta)^{n+2}}\right] \quad (2)$$

$ii)$

$$\begin{aligned} P(N(t_2) - N(t_1) = n) &= \frac{\theta}{1+\theta}(\Lambda(t_1)-\Lambda(t_2))^n\left[(1+\theta)\frac{1}{((\Lambda(t_2)-\Lambda(t_1))+\theta)^{n+1}}\right. \\ &\left. +2\theta\frac{n}{((\Lambda(t_2)-\Lambda(t_1))+2\theta)^{n+2}}-\frac{1}{((\Lambda(t_2)-\Lambda(t_1))+2\theta)^{n+2}}\right] \end{aligned} \quad (3)$$

$iii)$

$$\begin{aligned} P((N(t_i) - N(t_{i-1})) &= n_i, i = 1, 2, ...m) = \frac{\theta}{1+\theta}\left[\prod_{i=1}^{m}\frac{(\Lambda(t_i)-\lambda(t_{i-1}))^n}{n!}\right]\left(\sum_{i=1}^{m} ni\right)! \\ &\times\left[(1+\theta)\frac{1}{(\sum_{i=1}^{m}(\Lambda(t_i)-\Lambda(t_{i-1}))+\theta)^{\sum_{i=1}^{m} n_{i+1}}}\right. \end{aligned} \quad (4)$$

$$+2\theta\frac{\sum_{i=1}^{m}n_i}{\left(\sum_{i=1}^{m}(\Lambda\left(t_i\right)-\Lambda\left(t_{i-1}\right))+2\theta\right)^{\sum_{i=1}^{m}n_i+2}}\qquad-\frac{1}{\left(\sum_{i=1}^{m}(\Lambda\left(t_i\right)-\Lambda\left(t_{i-1}\right))+2\theta\right)^{\sum_{i=1}^{m}n_i+2}}\Bigg]$$

Proof. According to the definition of $PMLP\left(\lambda\left(t\right),\theta\right)$

$$\begin{aligned}P\left(N\left(t\right)=n\right) &= \int_0^{\infty}\frac{\left(\Phi\Lambda\left(t\right)^{n}e\left\{-\Phi\Lambda\left(t\right)\right\}\right)}{n!}\frac{\theta}{1+\theta}e\left\{-2\theta\Phi\right\}\left[(1+\theta)e\{\theta\Phi\}+2\theta\Phi-1\right]\partial\Phi\\ &= \frac{\theta}{1+\theta}\frac{\Lambda\left(t\right)^{n}}{n!}\left[(1+\theta)\frac{n!}{(\Lambda\left(t\right)+\theta)^{n+1}}+2\theta\frac{(n+1)!}{(\Lambda\left(t\right)+2\theta)^{n+2}}-\frac{n!}{(\Lambda\left(t\right)+2\theta)^{n+2}}\right]\\ &= \frac{\theta}{1+\theta}\Lambda\left(t\right)^{n}\left[(1+\theta)\frac{1}{(\Lambda\left(t\right)+\theta)^{n+1}}+2\theta\frac{n}{(\Lambda\left(t\right)+2\theta)^{n+2}}-\frac{1}{(\Lambda\left(t\right)+2\theta)^{n+2}}\right]\end{aligned}$$

According to the definition of PPLM$(\lambda(t),\theta)$ The probability of the number of events in an arbitrary interval $[t_1,t_2]$, $P(N(t_2)-N(t_1)=n)$ can be easily obtained from the proof of property

i) by replacing $\Lambda(t)$ with $\Lambda(t_2)-\Lambda(t_1)$Due to the independent increments property of the NHPP

$$P\left(N\left(t_i\right)-N\left(t_{i-1}\right)\right)=n_i\ i=1,2,...m\mid\Phi=\Phi)$$

$$=\prod_{i=1}^{m}\frac{\left(\Phi\left(\Lambda\left(t_i\right)-\Lambda\left(t_{i-1}\right)\right)\right)^{n_i}}{n_i!}e\left\{-\Phi\left(\Lambda\left(t_i\right)-\Lambda\left(t_{i-1}\right)\right)\right\}$$

$$=\frac{\theta}{1+\theta}\left[\prod_{i=1}^{m}\frac{\left(\Lambda\left(t_i\right)-\Lambda\left(t_{i-1}\right)\right)^{n_i}}{n_i!}\right]*\int_0^{\infty}\left(\Phi^{\sum_{i=1}^{m}n_i}+\Phi^{\sum_{i=1}^{m}n_i+1}\right)\times e\left\{-\Phi\left(\sum_{i=1}^{m}\left(\Lambda\left(t_i\right)-\Lambda\left(t_{i--1}\right)\right)\right)\partial\Phi\right\}$$

$$=\frac{\theta}{1+\theta}\left[\prod_{i=1}^{m}\frac{\left(\Lambda\left(t_i\right)-\Lambda\left(t_{i-1}\right)\right)^{n_i}}{n_i!}\right]\left(\sum_{i=1}^{m}ni\right)!*\left[(1+\theta)\frac{1}{\left(\sum_{i=1}^{m}(\Lambda\left(t_i\right)-\Lambda\left(t_{i-1}\right))+\theta\right)^{\sum_{i=1}^{m}n_{i+1}}}\right.$$

$$\left.+2\theta\frac{\sum_{i=1}^{m}n_i}{\left(\sum_{i=1}^{m}(\Lambda\left(t_i\right)-\Lambda\left(t_{i-1}\right))+2\theta\right)^{\sum_{i=1}^{m}n_i+2}}-\frac{1}{\left(\sum_{i=1}^{m}(\Lambda\left(t_i\right)-\Lambda\left(t_{i-1}\right))+2\theta\right)^{\sum_{i=1}^{m}n_i+2}}\right]$$

For ii) and iii), we use a same idea. □

In most practical applications, the statistical properties of $N(t)$are of great importance, which are given by the following theorem.

In the following, the notation

$$\Psi\left(s\right)=E\left[e^{sN(t)}\right]$$

Denotes the moment-generating function of $N\left(t\right)$.

Proposition 2.3. *Let $\{N\left(t\right),t\geq 0\}$ a Poisson modified Lindley process, then we have the following properties:*

i) The moment-generating function of $N(t)$ is given by:

$$\Psi\left(s\right)=\frac{\theta}{1+\theta}\left[(1+\theta)\frac{1}{\left(\Lambda\left(t\right)-\Lambda\left(t\right)e^{s}+\theta\right)}+\frac{e^{s}\Lambda\left(t\right)-\Lambda\left(t\right)}{\left(\Lambda\left(t\right)-\Lambda\left(t\right)e^{s}+2\theta\right)^{2}}\right],s\leq\ln\left(\frac{\theta+\Lambda\left(t\right)}{\Lambda\left(t\right)}\right)\tag{5}$$

ii) The mean and variance of $N(t)$ are given by:

$$\begin{aligned}E\left[N\left(t\right)\right] &= \frac{(5+4\theta)}{4\theta(1+\theta)}\Lambda\left(t\right)\\ var\left[N\left(t\right)\right] &= \frac{\left(32\theta+15+16\theta^{2}\right)\Lambda\left(t\right)^{2}+\left(5+4\theta\right)\theta\Lambda\left(t\right)}{16\theta^{2}\left(1+\theta\right)^{2}}\end{aligned}\tag{6}$$

According to proposition 2-(ii),

$$var\left[N\left(t\right)\right]-E\left[N\left(t\right)\right]=var\left[N\left(t\right)\right]-E\left[N\left(t\right)\right]=\frac{\Lambda\left(t\right)^{2}\left(16\theta^{2}+32\theta+15\right)-\Lambda\left(t\right)\left(16\theta^{3}+32\theta^{2}+15\theta\right)}{(4\theta)^{2}\left(\theta+1\right)^{2}}\geq 0 \quad (7)$$

Proposition 2.4. *Let* $\{N\left(t\right),t\geq 0\}$ *Let it be the modified Lindley Poisson process (general distribution), then:*

$$var\left[N\left(t\right)\right]\geq E\left[N\left(t\right)\right]. \quad (8)$$

3. Some Mathematical Properties of the PMLP

The PMLP has dependent increments, just like the PPNH. We now derive the stochastic intensity of the PMLP to observe how past events influence future events. Let $\{N(t),t\geq 0\}$ be a marked point process whose history (internal filtration) in $[0,t)$ is denoted by $H_{t-}=\{N(u),0\leq u<t\}$.

The stochastic intensity λ_t of an ordered point process $\{N(t),t\geq 0\}$ is defined as the following limit:

$$\lambda_t=\lim_{\partial t\to 0}\frac{\rho\left(N\left(t,t+\partial t\right)=1\mid H_t-\right)}{\partial t}=\lim_{\partial t\to 0}\frac{E\left[N\left(t,t+\partial t\right)\mid H_t-\right]}{\partial t}$$

where $N\left(t_1,t_2\right)\, t_1\leq t_2$ represents the number of events in $\left[t_1,t_2\right]$.

Theorem 3.1. *The stochastic intensity* λ_t *of the Poisson modified Lindley process* $\left(\lambda\left(t\right),\theta\right)$ *given by::*

$$\lambda_t=\frac{\left(\theta+\Lambda\left(t\right)\right)+\left(N\left(t-\right)+2\right)}{\left(\theta+\Lambda\left(t\right)\right)^{2}\frac{1}{N(t-)+1}+\left(\theta+\Lambda\left(t\right)\right)}\lambda\left(t\right) \quad (9)$$

According to Theorem 1, we can see that the PPLM has a Markov property, meaning that the current state of the process depends only on the last state of the process N_{t-}, and not on the complete history H_{t-}. Moreover, λ_t is increasing in N_{t-}, which implies that the predisposition to the occurrence of a future event is increasing in the number of events that have occurred previously. This implies a kind of property of positive dependent increments. In the following, we analyze the dependency structure in the increments of PPLM. For this, we begin by introducing a concept of multivariable dependence. The random variables $U_1,U_2,,,,,U_m$ are positively dependent in the upper orthant (PUOD) if the inequality

$$p\left(Ui\right)\geq u_i,i=1,2,...m)\geq\prod_{i=1}^{m}p\left(U_i\geq u_i\right),pour\ tout\ u_i i=1,2,..m \quad (10)$$

holds. Intuitively, inequality (3) implies that $U_1,U_2,...,U_m$ are more likely to simultaneously have large values, compared to a vector of independent random variables with the same univariate marginal distributions. We now define a similar concept for multivariate increments in a counting process model.

4. Coumpound of PMLP

The stochastic $\{w(t),t\geq 0\}$ process is called a modified compound Lindley Poisson process if

$$w\left(t\right)=\sum_{i=1}^{N(t)}X_i,t\geq 0.$$

where, $\{N(t), t \geq 0\}$, is a modified Lindley Poisson process and $\{X_i, i \geq 1\}$is a family of independent and identically distributed random variables independent of $\{N(t), t \geq 0\}$,.

Let $M_x(s) = E[e^{sx_i}]$, the MGF of X_i , The following result gives the moment-generating function, the mean, and the variance of $w(t)$.

Theorem 4.1. *The moment-generating function of $w(t)$ denoted by $M_{w(t)}(s)$ is given by:*

$$M_{w(t)}(s) = \frac{\theta}{1+\theta}\left[\frac{(1+\theta)}{(\theta+\Lambda(t)-M_X(s)\Lambda(t))} + \frac{M_X(s)\Lambda(t)-\Lambda(t)}{(2\theta+\Lambda(t)-M_X(s)\Lambda(t))^2}\right] \tag{11}$$

The mean and variance of $w(t)$ are:

$$E[w(t)] = \frac{(5+4\theta)}{4\theta(1+\theta)}E[x]\Lambda(t) \tag{12}$$

$$var[w(t)] = \frac{(16\theta^2+32\theta+15)}{(4\theta)^2(\theta+1)^2}(E[x]\Lambda(t))^2 + \frac{(16\theta^3+32\theta^2+15\theta)}{(4\theta)^2(\theta+1)^2}E[x^2]\Lambda(t)$$

Proof. By conditioning on $N(t)$

$$M_{w(t)}(s) = \sum_{n=0}^{\infty} E\left[e^{sw(t)}\,|_{N(t)=n}\right] p(N(t)=n)$$

$$= \sum_{n=0}^{\infty} E\left[e^{s(x_1+x_2+x_3+\ldots x_n)} \mid N(t)=n\right] p(N(t)=n) = \sum_{n=0}^{\infty} E[e^{s(x_1+x_2+\ldots+x_n)}]p(N(t)=n)$$

$$= \sum_{n=0}^{\infty} (M_x(s))^n p(N(t)=n)$$

□

5. Application in Surplus Model

modeling surplus. Our objective is to replicate an insurance company's surplus procedure. We call this process $(U_t)_{t>0}$, where U_t is the company's surplus at time t. $U_t = u + P_t - W(t)$, where u is the starting capital ($U_0 = u$),P_t is the gain process (premiums received, interest from investments and all other sources of income, etc.), and $W(t)$ is the loss process as defined in (4) (compensation paid, interest from credits, etc.).

Generally speaking, P_t may be dependent on $(W(u)\ u < t)$. It is true that the premiums can be adjusted to reflect the amount of risk more accurately based on the losses incurred. For instance, each year, the number of incidents in the preceding year is used to reevaluate each driver's price for auto insurance.

Thus, the losses $W(t)$ are not necessarily written as aggregate sums, and if this is the case, the losses X_i are not necessarily independent. However, in this course, we will study only two simplified models:

- Discrete case: we study the model through its increments. We note $Y_t = U_t - U_{t-1}$ the increase of the surplus for the period $(t-1, t]$:

$$Y_t = P_t - P_{t-1} - (W(t) - W(t-1)), t = 1, 2, ...$$

We have $U_t = U_{t-1} + Y_t, t = 1, 2, ...$

- Continuous case: we study the compound Poisson Modified Lindley model $U_t = u + ct - W(t)$, where μ the initial capital, $W(t) = \sum_{i=1}^{N(t)} Xi, t \geq 0$,a compound poisson modified Lindley process where N_t is a Poisson modified Lindley process of intensity λ and c the premium per unit time of the form

$$c = (1+\theta)E[W(t)] = (5+4\theta)\lambda\frac{E[Xi]}{4\theta(1+\theta)}$$

We can show that if

$$c \geq (5+4\theta)\lambda\frac{E[Xi]}{4\theta(1+\theta)}$$

then the business is not destroyed. This value indicates that, in comparison to the compound Poisson process, the modified Lindley process yields more satisfactory results because it has more parameters that can mitigate the compound Poisson process' drawbacks

6. Example and Simulation

We take the continuous-time model is one where the losses are modeled by a compound Poisson modified Lindley process: for $t > 0$

$U_t = u + ct - W(t)$, where $W(t) = \sum_{i=1}^{N(t)} X_i,\ t \geq 0$,Or

- u represents the initial capital,

- c is the premium per unit of time,

- $W(t)$ represents the aggregate losses up to time t, with

the X_i represent the individual losses, they are assumed to be i.i.d. of expectation $E[X_i]$ and independent of $N(t)$.

We have For compound Poisson modified Lindley process:

$$\begin{aligned} E\left[U_t\right] &= u + \left(c - (5+4\theta)\lambda\frac{E[Xi]}{4\theta(1+\theta)}\right)t, \\ Var(U_t) &= t\left(\frac{(16\theta^2+32\theta+15)}{(4\theta)^2\left(\theta+1\right)^2}\lambda^2 E[Xi]^2 + \frac{(16\theta^3+32\theta^2+15\theta)}{(4\theta)^2\left(\theta+1\right)^2}\lambda E\left[Xi^2\right]\right) \end{aligned}$$

Now, if $X_i \rightsquigarrow$ Gamma distribution where $E[X_i] = 0.5, V[X_i] = 1\backslash 2$ and $E[X_i^2] = 3\backslash 4$.

Compound Poisson New XLindley Process (PNXLP)

u	c	λ	θ	T	$E[U_t]$	$V[U_t]$
10	1	0.1	0.2	1	10.625	0.70313
50	5	0.5	1	1	54.625	0.70313
75	10	1	2	1	84.625	0.70313
75	30	1	2	1	104.63	0.70313
75	5	10	2	1	76.25	19.688
100	20	2	5	1	119.7	0.54
100	20	0.2	0.5	2	139.4	1.08
150	50	1	3	2	249.5	0.875
150	10	1	3	2	169.5	0.875
150	1	5	1	2	144.5	39.375

Compound Poisson XLindley Process (PXLP)

u	c	λ	θ	T	$E[U_t]$	$V[U_t]$
10	1	0.1	0.2	1	10.576	0.81486
50	5	0.5	1	1	54.688	0.56641
75	10	1	2	1	84.722	0.49383
75	30	1	2	1	104.72	0.49383
75	5	10	2	1	77.222	11.883
100	20	2	5	1	119.79	0.35059
100	20	0.2	0.5	2	139.42	1.0336
150	50	1	3	2	249.65	0.59397
150	10	1	3	2	169.65	0.59397
150	1	5	1	2	145.75	28.906

Compound Poisson Lindley Process(PLP)

u	c	λ	θ	T	$E[U_t]$	$V[U_t]$
10	1	0.1	0.2	1	10.542	0.19201
50	5	0.5	1	1	54.625	0.39063
75	10	1	2	1	84.667	0.59722
75	30	1	2	1	104.67	0.59722
75	5	10	2	1	76.667	59.722
100	20	2	5	1	119.77	0.75222
100	20	0.2	0.5	2	139.33	0.35111
150	50	1	3	2	249.58	0.70486
150	10	1	3	2	169.58	0.70486
150	1	5	1	2	144.5	78.125

Compound Poisson Process (PP)

u	c	λ	T	$E[U_t]$	$V[U_t]$
10	1	0.1	1	10.95	0.075
50	5	0.5	1	54.75	0.375
75	10	1	1	84.5	0.75
75	30	1	1	104.5	0.75
75	5	10	1	75.0	7.5
100	20	2	1	119.0	1.5
100	20	0.2	2	139.8	0.3
150	50	1	2	249.0	1.5
150	10	1	2	169.0	1.5
150	1	5	2	147.0	7.5

Compound Poisson Modified Lindley Process

u	c	λ	θ	T	$E[U_t]$	$V[U_t]$
10	1	0.1	0.2	1	10.698	0.41851
50	5	0.5	1	1	54.719	0.43066
75	10	1	2	1	84.729	0.43446
75	30	1	2	1	104.73	0.43446
75	5	10	2	1	77.292	9.9306
100	20	2	5	1	119.79	0.33941
100	20	0.2	0.5	2	139.53	0.66111
150	50	1	3	2	249.65	0.55339
150	10	1	3	2	169.65	0.55339
150	1	5	1	2	146.38	19.688

Table 1. Experance and Variance of U_t using PMLP, PNXLP, PXLP, PLP, and PP

The compound Poisson modified Lindley, Poisson new XLindley, compound Poisson Lindley, and compound Poisson XLindley processes all produce results that are comparable to the compound Poisson process, but Table 1 shows that the compound Poisson XLindley process outperforms the compound Poisson process in terms of parameters. Furthermore, in comparison to the compound Poisson New XLindley and compound Poisson XLindley processes, the compound Poisson Modified Lindley process yields satisfactory results.

Conclusion

In this work, we proposed a Poisson modified Lindley process. This new process's properties are displayed. Furthermore, a simulation study that compares the proposed process with Poisson, Poisson Lindley and Poisson XLindley processes, and Poisson new XLindley process is provided, along with a recommendation to implement this process using the ruin model. The suggested method yields more efficient outcomes than a Poisson, Poisson Lindley, and Poisson XLindley method.

Appendices

Proof of Proposition 2

i) using the form of the generating function of the NHPP

$$\begin{aligned}
\Psi(s) &= \int_0^{\infty} e^{\Phi(e^s\Lambda(t)-\Lambda(t))}\frac{\theta}{1+\theta}e^{-2\theta\Phi}[(1+\theta)e^{\theta\Phi}+2\theta\Phi-1]d\Phi \\
&= \frac{\theta}{(1+\theta)}\left[(1+\theta)\int_0^{\infty} e^{-\Phi(\Lambda(t)+\theta-e^s\Lambda(t))}d\Phi+2\theta\int_0^{\infty}\Phi e^{-\Phi(\Lambda(t)+2\theta-e^s\Lambda(t))}d\Phi-\int_0^{\infty} e^{-\Phi(\Lambda(t)+2\theta-e^s\Lambda(t))}\right] \\
&= \frac{\theta}{(1+\theta)}\left[(1+\theta)\frac{\Gamma(1)}{(\Lambda(t)-e^s\Lambda(t)+\theta)}+2\theta\frac{\Gamma(2)}{(\Lambda(t)-e^s\Lambda(t)+2\theta)^2}-\frac{\Gamma(1)}{(\Lambda(t)-e^s\Lambda(t)+2\theta)}\right] \\
&= .\frac{\theta}{1+\theta}\left[(1+\theta)\frac{1}{(\Lambda(t)-\Lambda(t)e^s+\theta)}+\frac{e^s\Lambda(t)-\Lambda(t)}{(\Lambda(t)-\Lambda(t)e^s+2\theta)^2}\right]
\end{aligned}$$

ii) we can show that:

$$\frac{\partial\Psi(s)}{\partial s}=\frac{\theta}{1+\theta}\left[\frac{(\theta+1)\Lambda(t)e^s}{(\theta+\Lambda(t)-\Lambda(t)e^s)^2}+\frac{\Lambda(t)e^s(2\theta-\Lambda(t)+\Lambda(t)e^s)}{(2\theta+\Lambda(t)-\Lambda(t)e^s)^3}\right]$$

and from there,

$$\begin{aligned}
E[N(t)] &= \frac{\partial\Psi(s)}{\partial s}|_{s=0}=\frac{\theta}{1+\theta}\left[\frac{\Lambda(t)(1+\theta)}{(\theta)^2}-\frac{\Lambda(t)(2\theta)^2}{(2\theta)^4}\right] \\
&= \frac{\Lambda(t)(5+4\theta)}{4\theta(1+\theta)}
\end{aligned}$$

Moreover, it can also be shown that:

$$\frac{\partial\Psi(s)}{\partial s^2} = \frac{\theta}{1+\theta}\Lambda(t)e^s\left[(\theta+1)\frac{\theta+\Lambda(t)+\Lambda(t)e^s}{(\theta+\Lambda(t)-\Lambda(t)e^s)^3}\right.$$

$$+\frac{1}{\left(2\theta+\Lambda\left(t\right)-\Lambda\left(t\right)e^{s}\right)^{4}}\left(\Lambda\left(t\right)^{3}e^{3s}-\Lambda\left(t\right)^{3}e^{s}+8\theta\Lambda\left(t\right)^{2}e^{2s}+4\theta^{2}\Lambda\left(t\right)e^{s}\right)\Bigg]$$

$$E\left[N\left(t\right)^{2}\right]=\frac{\partial^{2}\Psi\left(s\right)}{\partial s^{2}}\mid_{s=0}=\left[\frac{\left(8\theta+10\right)\Lambda\left(t\right)^{2}+\left(5+4\theta\right)\theta\Lambda\left(t\right)}{4\theta^{2}\left(1+\theta\right)}\right]$$

Thus

$$\begin{aligned} var\left[N\left(t\right)\right] &= E\left[N\left(t\right)^{2}\right]-\left(E\left[N\left(t\right)\right]\right)^{2}=\left[\frac{\left(8\theta+10\right)\Lambda\left(t\right)^{2}+\left(5+4\theta\right)\theta\Lambda\left(t\right)}{4\theta^{2}\left(1+\theta\right)}-\frac{\Lambda\left(t\right)^{2}\left(5+4\theta\right)^{2}}{16\theta^{2}(1+\theta)^{2}}\right] \\ &= \frac{\left(32\theta+15+16\theta^{2}\right)\Lambda\left(t\right)^{2}+\left(5+4\theta\right)\theta\Lambda\left(t\right)}{16\theta^{2}\left(1+\theta\right)^{2}}. \end{aligned}$$

Proof of Proposition 3

we have

$$var\left[N\left(t\right)\right]=E\left[var\left[N\left(t\right)\mid\Phi\right]\right]+var\left[E\left[N\left(t\right)\mid\Phi\right]\right]$$

Like

$$\left\{N\left(t\right),t\geq 0\right\}\mid\left(\Phi=\phi\right)\sim NHPP\left(\Phi\lambda\left(t\right)\right)$$

$$var\left[N\left(t\right)\mid\Phi\right]=E\left[N\left(t\right)\mid\Phi\right]$$

And

$$var\left[N\left(t\right)\right]=E\left[E\left[N\left(t\right)\mid\Phi\right]\right]+var\left[E\left[N\left(t\right)\mid\Phi\right]\right]=E\left[N\left(t\right)\right]+var\left[E\left[N\left(t\right)\mid\Phi\right]\right]\geq E\left[N\left(t\right)\right].$$

Proof of Theorem 1

we have

$$\lambda\left(t\right)=\lim_{\partial t\to 0}\frac{\rho\left(N\left(t,t+\partial t\right)=1\mid H_{t}\right)}{\partial t}=E\left(\Phi\mid H_{t}-\right)\left[\lim_{\partial t\to 0}\frac{P\left(N\left(t,t+\partial t\right)=1\mid\Phi:H_{t}\right)}{\partial t}\right]$$

where $E\left(\Phi\mid H_{t}-\right)[.]$ represents the expectation with respect to the conditional distribution of $\left(\Phi\mid H_{t}\right)$ and

$$\lim_{\partial t\to 0}\frac{p\left(N\left(t,t+\partial t\right)=1\mid\Phi:H_{t}-\right)}{\partial t}=\Phi\lambda\left(t\right)$$

thus

$$\lambda_{t}=E\left(\Phi\mid H_{t}-\right)\left[\Phi\lambda\left(t\right)\right]$$

Note that H_{t-}can be defined in terms of the number of events in $(0,t)$ denoted N(- and the sequential arrival times of the events, i.e., $0\leq T_{1}\leq T_{2}\leq -\leq T_{N(t-)}<t$ and the conditional probability mass function of $\left(H_{t-}\mid\Phi\right)=\left(T_{1},T_{2},...T_{N(t-)}\mid\Phi\right)$ is given by

$$\left\{\prod_{j=1}^{n}\left(\phi\lambda\left(t_{j}\right)\right)\right\}\exp\left\{-\phi\int_{0}^{t}\lambda\left(x\right)\partial x\right\}0\leq t_{1}\leq t_{2}\leq...\leq t_{n}\leq t,n=0,1,2,...$$

where:$\prod_{j=1}^{n}(.) = 1$ for $n = 0$, t_j represents the realisation of T_j and n represents that of $N(t)$. Thus, the conditional joint distribution of $(\Phi \mid H_t = h_t-)$ where $h_t = (t_1, t_2, ...t_n, n)$ is the achievement of H_{t-} is given by:

$$\left(\phi^n \left[\prod_{i=1}^{n} \lambda(t_i)\right] \exp\left\{-\phi \int_0^t \lambda(x)\, dx\right\} f(\phi)\right) \times \left(\int_0^\infty \phi^n \left[\prod_{i=1}^{n} \lambda(t_i)\right] \exp\left\{-\phi \int_0^t \lambda(x)\, dx\right\} f(\phi)\, d\phi\right)^{-1}$$

$$= \left(\phi^n \exp\left\{-\phi \int_0^t \lambda(x)\, dx\right\} f(\phi)\right) \times \left(\int_0^\infty \phi^n \exp\left\{-\phi \int_0^t \lambda(x)\, \partial x\right\} f(\phi)\, d\phi\right)^{-1}, \phi \geq 0$$

so,

$$\lambda_t = E(\Phi \mid H_{t-})[\Phi\lambda(t)] = \frac{\int_0^\infty \phi^{n+1} \exp\left\{-\phi \int_0^t \lambda(x)\, \partial x\right\} f(\phi)\, \partial\phi}{\int_0^\infty \phi^n \exp\left\{-\phi \int_0^t \lambda(x)\, \partial x\right\} f(\phi)\, \partial\phi} \lambda(t)$$

$$\int_0^\infty \phi^n \exp\left\{-\phi \int_0^t \lambda(x)\, \partial x\right\} f(\phi)\, \partial\phi = \int_0^\infty \phi^n e^{-\phi\Lambda(t)} \left\{\frac{\theta}{1+\theta} e^{-2\theta\phi}[(1+\theta)e^{\theta\phi} + 2\theta\phi - 1] d\phi\right\}$$

$$\begin{aligned} &= \frac{\theta}{(1+\theta)} \left[(1+\theta) \int_0^\infty \phi^n e^{-\phi(\Lambda(t)+\theta)} + 2\theta \int_0^\infty \phi^{n+1} e^{-\phi(\Lambda(t)+2\theta)} - \int_0^\infty \phi^n e^{-\phi(\Lambda(t)+2\theta)}\right] \\ &= \frac{\theta}{\theta+1} \left[(1+\theta)\frac{\Gamma(n+1)}{(\theta+\Lambda(t))^{n+1}} + 2\theta \frac{\Gamma(n+2)}{(2\theta+\Lambda(t))^{n+2}} - \frac{\Gamma(n+1)}{(\Lambda(t)+2\theta)^{n+1}}\right] \end{aligned}$$

A same step for calculate $\int_0^\infty \phi^{n+1} \exp\left\{-\phi \int_0^t \lambda(x)\, \partial x\right\} f(\phi)\, \partial\phi$, which

$$\int_0^\infty \phi^{n+1} \exp\left\{-\phi \int_0^t \lambda(x)\, \partial x\right\} f(\phi)\, \partial\phi \frac{\theta}{\theta+1} \left[(1+\theta)\frac{\Gamma(n+2)}{(\theta+\Lambda(t))^{n+2}} + 2\theta \frac{\Gamma(n+3)}{(2\theta+\Lambda(t))^{n+3}} - \frac{\Gamma(n+2)}{(\Lambda(t)+2\theta)^{n+2}}\right]$$

Consequently,

$$\lambda_t = \frac{(\theta+\Lambda(t)) + (N(t-)+2)}{(\theta+\Lambda(t))^2 \frac{1}{N(t-)+1} + (\theta+\Lambda(t))} \lambda(t).$$

Different Compound Process

For compound Poisson XLindley process (see Sakri et al. 2023 [29])

$$\begin{aligned} E[U_t] &= u + \left(c - (\theta(2+\theta)+2)\lambda \frac{E[X_i]}{\theta(1+\theta)^2}\right) t, \\ Var(U_t) &= tVar[W(t)] = t\left(\frac{(\theta^4 + 4\theta^3 + 8\theta^2 + 8\theta + 4)}{\theta^2(\theta+1)^4} \lambda^2 E[Xi]^2 + \frac{(\theta(2+\theta)+2))}{\theta(\theta+1)^2} \lambda E\left[X^2\right]\right). \end{aligned}$$

For compound Poisson new XLindley process(see Benatmane et al. 2024 [4])

$$\begin{aligned} E[U_t] &= u + \left(c - 3\lambda \frac{E[X_i]}{2\theta}\right) t, \\ Var(U_t) &= tVar[W(t)] = t\left(\frac{9(E[X_i]\lambda)^2 + 6\theta E\left[X_i^2\right]\lambda}{4\theta^2}\right). \end{aligned}$$

For compound Poisson Lindley process (see Cha (2019) [12])

$$E[U_t] = u + \left(c - (\theta+2)\lambda\frac{E[X_i]}{\theta(\theta+1)}\right)t,$$

$$Var(U_t) = tVar[W(t)] = t\left(\frac{\theta+2}{\theta(\theta+1)}(E[X_i^2]\lambda^2 + \frac{(\theta^2+4\theta+2)}{\theta^2(\theta+1)^2}(E[X_i]\lambda)^2\right).$$

For compound Poisson process (see Last and Penrose (2017) [30])

$$E[U_t] = u + (c - \lambda 0.5)t, \;\; Var(U_t) = tVar[W(t)] = t\left((E[X_i^2]\lambda\right).$$

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- Razika Grine: Investigation; formal analysis; methodology, writing—original draft; simulation; software; interpretation of results; writing—review and editing
- Imen Grabsia: Writing—original draft ; formal analysis

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