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(μ_1, μ_2)-SEMI WEAKLY GENERALIZED CLOSED SET IN A BIGENERALIZED TOPOLOGICAL SPACE

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Abstract. The aim of this paper is to introduce the concept of (μ_1, μ_2)-semi weakly generalized closed set (or briefly (μ_1, μ_2)-swg closed) in a bigeneralized topological space, defined as A is a $(\mu_1, \mu_2) - swg$-closed set if $cl_{\mu_1}(int_{\mu_2}(A)) \subseteq U$ whenever $A \subseteq U$ and U is μ_2-semi open in X. We also introduced the concept of (μ_1, μ_2)-semi weakly generalized continuous function. Moreover, some properties of $(\mu_1, \mu_2) - swg$-closed set and (μ_1, μ_2)-semi weakly generalized continuous function are obtained and proved. Finally, given corresponding conditions, other related well-known closed sets to $(\mu_1, \mu_2) - swg$ closed set are presented.

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1. Introduction

Dealing with closed sets has become one of the main tools in studying topological spaces. In 1970, Levine [1] introduced the concepts of generalized closed sets in a topological space by comparing the closure of a subset with its open supersets in order to extend many of the important properties of closed sets to a larger family. Significant results pertaining to the theory on separation axioms and continuity can be attributed to the important developments and generalizations of closed sets. Since then, weaker forms of closed sets have been generalized and many interesting results were obtained. In 1990, Arya and Nour [2] defined the generalized semi-open sets and generalized semi-closed sets. Later in 1991, Balachandran [4] introduced the notion of g-continuous functions by using g-closed sets and obtained some of their properties. In 2000, Pushpalatha [17] introduced a new class of closed sets called weakly closed (briefly w-closed) sets and studied their properties. Back in 1969, Kelly [12] introduced already

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the concept of bitopological space. This space is equipped with two arbitrary topologies.

Moreover, in the year 2002, Cs*ászár* [6] introduced the concepts of generalized neighborhood systems and generalized topological spaces. He also introduced the concepts of generalized continuous functions and associated interior and closure operators on generalized neighborhood system. In 2010, Dungthaisong, Boonpok and Viriyapong [10] introduced the concept of bigeneralized topological spaces and studied (m,n)-closed sets and (m,n)-open sets in bigeneralized topological spaces. Also, they introduced the weakly open functions on bigeneralized topological spaces and investigated its properties. Furthermore, in 2011, Duangphui, Boonpok and Viriyapong [8] introduced the notions almost $(\mu,\mu')^{(m,n)}$-continuous and weakly $(\mu,\mu')^{(m,n)}$-continuous functions on bigeneralized topological spaces, its basic properties and characterization. In 2012, Priyadharsini, Chandrika and Parvathi [16] introduced the concepts of $\mu_{(m,n)}$-semi generalized closed sets in bigeneralized topological and $sg_{(m,n)}$-continuous function on bigeneralized topological spaces and investigate some of its properties. In 2015, Laniba and Rara in [14] and the references therein provided a list of definitions of various types of closed sets defined over a generalized topology.

In this paper, the notion of (μ_1,μ_2)-semi weakly generalized closed (or briefly (μ_1,μ_2)-swg closed)set in a bigeneralized topological space and (μ_1,μ_2)-semi weakly generalized continuous (or briefly $swg_{(\mu_1,\mu_2)}$-continuous) functions in bigeneralized topological spaces are introduced and some of their properties are investigated.

2. Preliminaries

The following definitions and properties on generalized topology, μ-open set, μ-closed set, μ-interior and μ-closure of a set are taken from [1,6,7,11,15].

Definition 1. Let X be a nonempty set. A collection μ of subset of X is a **generalized topology (or briefly GT)** on X if it satisfies the two conditions: (i) $\emptyset \in \mu$ and (ii) If $\{G_i : i \in I\} \subseteq \mu$, then $\bigcup_{i\in I} G_i \in \mu$.

If μ is a generalized topology in X, then the pair (X,μ) is called a **generalized topological space (or briefly GT-space)** and the elements of μ are called μ**-open sets**.

Definition 2. Let μ be a GT in X. A subset F of X is said to be a μ**-closed set** if the complement F^c of F is μ-open.

Definition 3. Let (X,μ) be a GT space and let $A \subseteq X$. The μ**-interior** of a set A, denoted by $int_\mu(A)$, is the union of all μ-open sets in X contained in A. That is, $int_\mu(A) = \bigcup\{G : G \text{ is } \mu\text{-open and } G \subseteq A\}$.

Definition 4. Let (X,μ) be a GT-space and let $A \subseteq X$. The μ**-closure** of a set A, denoted by $cl_\mu(A)$, is the intersection of all μ-closed sets in X containing A. That is, $cl_\mu(A) = \bigcap\{F : F \text{ is } \mu\text{-closed and } A \subseteq F\}$.

The following theorems on μ-closure and μ-interior of a set are obtained from [6,7,13].

Theorem 2.1. Let X be a nonempty set and μ be a GT on X. Suppose also that A and B are subsets of X. Then,

(i) $int_\mu(A) \subseteq A$;
(ii) $int_\mu(A)$ is the largest μ-open subset of A;
(iii) A is μ-open if and only if $int_\mu(A) = A$;
(iv) If $A \subseteq B$, then $int_\mu(A) \subseteq int_\mu(B)$;
(v) $int_\mu\big(int_\mu(A)\big) = int_\mu(A)$; and
(vi) $\bigcup_{i\in I} int_\mu(A_i) \subseteq int_\mu\big(\bigcup_{i\in I} A_i\big)$.

Theorem 2.2. Let X be a nonempty set and μ be a GT on X. Suppose also that A and B are subsets of X. Then,

(i) $A \subseteq cl_\mu(A)$;
(ii) $cl_\mu(A)$ is the smallest μ-closed superset of A;
(iii) A is μ-closed if and only if $cl_\mu(A) = A$;
(iv) If $A \subseteq B$, then $cl_\mu(A) \subseteq cl_\mu(B)$;
(v) $cl_\mu\big(cl_\mu(A)\big) = cl_\mu(A)$; and
(vi) $\bigcup_{i\in I} cl_\mu(A_i) \subseteq cl_\mu\big(\bigcup_{i\in I} A_i\big)$.

Theorem 2.3. Let (X, μ) be a generalized topological space. For any subset A of X, the following properties hold: (i) $\big(int_\mu(A)\big)^c = cl_\mu(A^c)$ and (ii) $\big(cl_\mu(A)\big)^c = int_\mu(A^c)$.

Definition 5. [7] Let X be a nonempty set and μ be a generalized topology on X. Then a subset A of X is called μ**-semi open set** if $A \subseteq cl_\mu\big(int_\mu(A)\big)$.

Theorem 2.4. Let X be a nonempty set and μ be a generalized topology on X. Then the collection of μ-semi open sets in X is a generalized topology.

Definition 6. [5] Let X be a nonempty set, and μ_1 and μ_2 be generalized topologies on X. The triple $(X, \mu_1,\ \mu_2)$ is called a **bigeneralized topological space (or briefly BGTS)**.

Remark 1. Let (X, μ_1, μ_2) be a bigeneralized topological space and A be a subset of X. The μ-closure and the μ-interior of A with respect to μ_1 are denoted by $cl_{\mu_1}(A)$ and $int_{\mu_1}(A)$ respectively. The family of all μ_1-closed set is denoted by the symbol μ_1-$C(X)$ and also the family of all μ_1-closed set is denoted by the symbol μ_1-$O(X)$.

Definition 7. [5] A subset A of a bigeneralized topological space (X, μ_1, μ_2) is called (μ_1, μ_2)**-closed set** if $cl_{\mu_1}\big(cl_{\mu_2}(A)\big) = A$. The complement of (μ_1, μ_2)-closed sets is called (μ_1, μ_2)**-open sets**.

Definition 8. [4] Let (X, μ_X) and (Y, μ_Y) be generalized topological spaces. A mapping $f : (X, \mu_X) \to (Y, \mu_Y)$ is said to be **generalized continuous** if $f^{-1}(V)$ is μ_X-open in X for each μ_Y-open V in Y.

Definition 9. [8] Let (X, μ_X^1, μ_X^2) and (Y, μ_Y^1, μ_Y^2) be bigeneralized topological spaces. A mapping $f : (X, \mu_X^1, \mu_X^2) \to (Y, \mu_Y^1, \mu_Y^2)$ is said to be **pairwise continuous** if $f : (X, \mu_X^1) \to (Y, \mu_Y^1)$ and $f : (X, \mu_X^2) \to (Y, \mu_Y^2)$ are generalized continuous.

Definition 10. [9] Let $X \neq \emptyset$. Then the collection of all subsets of X is called the discrete topology.

Definition 11. [3,18,19,22] Let $X \neq \emptyset$ and μ be a generalized topology in X. Then,

(1) A subset A of X is called a semi-open set if $A \subseteq cl(int(A))$. The collection of semi-open sets in X is denoted by $SO(X)$.
(2) A subset A of X is called an α-open set if $A \subseteq int(cl(int(A)))$. The collection of α-open sets in X is denoted by $AO(X)$.
(3) A subset A of X is called a semi-preopen set if $A \subseteq cl(int(cl(A))$. The collection of semi-preopen sets in X is denoted by $SPO(X)$.
(4) A subset A of X is called a b-open set if $A \subseteq cl(int(A)) \cup int(cl(A))$. The collection of b-open sets in X is denoted by $BO(X)$.
(5) A subset A of X is called a preopen set if $A \subseteq int(cl(A))$. The collection of preopen sets in X is denoted by PO(X).

Remark 2. [23] It can be easily shown that $SO(X)$, $AO(X)$, $SPO(X)$, $BO(X)$, and $PO(X)$ are generalized topologies.

Definition 12. [18–21] Let $X \neq \emptyset$ and μ be a topology in X. Then with μ, a subset A of X is a:

(1) Generalized closed (or briefly g-closed) set if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.
(2) Generalized preclosed (or briefly gp-closed) set if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.
(3) Semi-generalized closed (or briefly sg-closed) set if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in X.
(4) Generalized semiclosed (or briefly gs-closed) set if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.
(5) Generalized b-closed (or briefly gb-closed) set if $bcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.
(6) Weakly closed (or briefly w-closed) set if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in X.
(7) Weakly generalized closed (or briefly wg-closed) set if $cl(int(A)) \subseteq U$ whenever $A \subseteq U$ and U is open in X.

(8) Generalized semi-preclosed (or briefly gsp-closed) set $spcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.

(9) Generalized α closed (or briefly g-α-closed) set if $\alpha - cl(int(A)) \subseteq U$ whenever $A \subseteq U$ and U is α-open in X.

(10) α-generalized closed (or briefly αg-closed) set if $\alpha - cl(int(A)) \subseteq U$ whenever $A \subseteq U$ and U is open in X.

3. Main Results

3.1. (μ_1, μ_2)-Semi Weakly Generalized Closed Set.

Definition 13. *Let X be a nonempty set, and μ_1 and μ_2 be generalized topologies in X. Then, a subset A of X in a bigeneralized topological space (X, μ_1, μ_2) is called (μ_1, μ_2)-**semi weakly generalized closed (or briefly** (μ_1, μ_2)**-swg closed) set** if $cl_{\mu_1}\big(int_{\mu_2}(A)\big) \subseteq U$ whenever $A \subseteq U$ and U is μ_2-semi open in X.*

Example 3.1. Let $\mu_1 = \{\emptyset, \{1,2\}, \{3\}, \{1,2,3\}\}$ and $\mu_2 = \{\emptyset, \{1\}, \{1,2\}, \{1,2,3\}, X\}$, where $X = \{1,2,3,4\}$. Then, we have the μ_1-open sets $\emptyset$, {1,2}, {3} and {1,2,3} whose corresponding μ_1-closed sets are X, {3,4}, {1,2,4} and {4}, respectively. On the other hand, the μ_2-open sets are $\emptyset$, {1}, {1,2}, {1,2,3}, and X, with corresponding μ_2-closed sets X, {2,3,4}, {3,4}, {4} and $\emptyset$, respectively. Now, the μ_2-semi open sets of X are obtained as follows:

(1) When $A = \emptyset, int_{\mu_2}(A) = \emptyset$, which means $cl_{\mu_2}(int_{\mu_2}(A)) = \emptyset \supseteq A$. Hence, $A = \emptyset$ is a μ_2-semi open set. Using a similar fashion, we obtained the following μ_2-semi open sets: $\{1\}, \{1,2\}, \{1,3\}, \{1,4\}\{1,2,3\}, \{1,2,4\}, \{1,3,4\},$ and X.

(2) The other sets do not satisfy the definition of μ_2-semiopen set in Definition 5 which are $\{2\}, \{3\}, \{4\}, \{2,4\}, \{3,4\},$ and $\{2,3,4\}$.

Since we have obtained the set of μ_2-semi open sets, we can now compute the sets of (μ_1, μ_2)-swg closed. That is,

(1) When $A = \emptyset$, $int_{\mu_2}(A) = \emptyset$, $cl_{\mu_1}(int_{\mu_2}(A)) = \{4\}$. Now, $A \subseteq \{1\}$ but $cl_{\mu_1}(int_{\mu_2}(A)) = \{4\} \nsubseteq \{1\}$, where $\{1\}$ is a μ_2-semi open set. Thus, A is not a (μ_1, μ_2)-swg closed. In a similar manner, we can see that the following are not (μ_1, μ_2)-swg closed sets: $\{2\}, \{3\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{1,2,3\}$, and $\{1,3,4\}$

(2) When $A = \{4\}$, $int_{\mu_2}(A) = \emptyset$, $cl_{\mu_1}(int_{\mu_2}(A)) = \{4\}$. Note that $A \subseteq U$ with μ_2-semi open sets such that $U \in \{\{1,4\}, \{1,2,4\}, \{1,3,4\}, X\}$. Now, $cl_{\mu_1}(int_{\mu_2}(A)) = \{4\} \subseteq U$. Thus, A is a (μ_1, μ_2)-swg closed. Using a similar approach, the (μ_1, μ_2)-swg closed sets are {4}, {2,4}, {3,4}, {1,2,4}, {2,3,4} and X.

Theorem 3.1. Let X be a nonempty set, and μ_1 and μ_2 be generalized topologies in X. Then, a subset A of X in a bigeneralized topological space (X, μ_1, μ_2) is called (μ_1, μ_2)-semi weakly generalized open (or briefly (μ_1, μ_2)-swg open) set if and only if $F \subseteq int_{\mu_1}(cl_{\mu_2}(A))$ whenever $F \subseteq A$ and F is μ_2-semi closed in X.

Proof. Let A be a (μ_1, μ_2)-swg open set and $F \subseteq A$ such that F is μ_2-semi closed. Then, A^c is (μ_1, μ_2)-swg closed. That is, $cl_{\mu_1}(int_{\mu_2}(A^c)) \subseteq F^c$. Now, using Theorem 2.3 and elementary properties of set complementation,

$$\begin{aligned}(F^c)^c &\subseteq \left(cl_{\mu_1}(int_{\mu_2}(A^c))\right)^c \\ F &\subseteq \left(cl_{\mu_1}(int_{\mu_2}(A^c))\right)^c \\ &= int_{\mu_1}\left((int_{\mu_2}(A^c))^c\right) \\ &= int_{\mu_1}\left(cl_{\mu_2}((A^c)^c)\right) \\ &= int_{\mu_1}(cl_{\mu_2}(A))\end{aligned}$$

Thus, $F \subseteq int_{\mu_1}(cl_{\mu_2}(A))$ whenever $F \subseteq A$, and F is a μ_2-semi closed set.

Conversely, let $F \subseteq A$, and F be a μ_2-semi closed set in X such that $F \subseteq int_{\mu_1}(cl_{\mu_2}(A))$. By taking the complement of both sides, we have $\left(int_{\mu_1}(cl_{\mu_2}(A))\right)^c \subseteq F^c$ whenever $A^c \subseteq F^c$, and F^c is μ_2-semi open in X. Again, by Theorem 2.3,

$$\begin{aligned}\left(int_{\mu_1}(cl_{\mu_2}(A))\right)^c &= cl_{\mu_1}\left((cl_{\mu_2}(A))^c\right) \\ &= cl_{\mu_1}(int_{\mu_2}(A^c))\end{aligned}$$

So, $cl_{\mu_1}(int_{\mu_2}(A^c)) \subseteq F^c$ whenever $A^c \subseteq F^c$, and F^c is a μ_2-semi open in X. This means that A^c is (μ_1, μ_2)-swg closed set. Therefore, A is a (μ_1, μ_2)-swg open set. □

Remark 3. The Family of all (μ_1, μ_2)-swg closed (resp. (μ_1, μ_2)-swg open) sets in a bigeneralized topological space (X, μ_1, μ_2) is denoted by (μ_1, μ_2)-$swgC(X)$ (resp. (μ_1, μ_2)-$swgO(X)$).

Remark 4. Let (X, μ_1, μ_2) be a BGT-space. Then (μ_1, μ_2)-$swgC(X)$ is not necessarily equal to (μ_2, μ_1)-$swgC(X)$.

Example 3.2. Suppose $X = \{1, 2, 3, 4\}$ and consider the two generalized topologies

$$\mu_1 = \{\emptyset, \{1,2\}, \{3\}, \{1,2,3\}\} \quad \text{and} \quad \mu_2 = \{\emptyset, \{1\}, \{1,2\}, \{1,2,3\}, X\}.$$

Then,

$$(\mu_1, \mu_2) - swgC(X) = \{\{4\}, \{2,4\}, \{3,4\}, \{1,2,4\}, \{2,3,4\}, X\}, \text{ and}$$

$$(\mu_2, \mu_1) - swgC(X) = \{\ \emptyset, \{1\}, \{2\}, \{4\}, \{1,4\}, \{2,4\}, \{3,4\}, \{1,3,4\}, \{2,3,4\}, X\ \}.$$

Thus, (μ_1, μ_2)-$swgC(X) \neq (\mu_2, \mu_1)$-$swgC(X)$.

Definition 14. Let X be a nonempty set, and μ_1 and μ_2 be generalized topologies in X. Then, a subset A of X in a bigeneralized topological space (X, μ_1, μ_2) is called a **pairwise** (μ_1, μ_2)**-semi weakly generalized closed (or briefly pairwise** (μ_1, μ_2)**-swg closed) set** if A is (μ_1, μ_2)-swg closed and (μ_2, μ_1)-swg closed. The complement of a pairwise (μ_1, μ_2)-swg closed set is called **pairwise** (μ_1, μ_2)**-swg open**.

Remark 5. The (μ_1, μ_2)-$swgC(X)$ does not necessarily form a generalized topology.

Example 3.3. Consider the Example 3.1, $\emptyset \notin (\mu_1, \mu_2)$-$swgC(X)$. Thus, (μ_1, μ_2)-$swgC(X)$ is not a generalized topology.

Theorem 3.2. If A is μ_1-closed set and $A \subseteq U$ where U is μ_2-semi open, then A is (μ_1, μ_2)-swg closed set.

Proof. Let A be a μ_1-closed set such that $A \subseteq U$, where U is a μ_2-semi open set in X. Then, by Theorem 2.1 (i), $int_{\mu_2}(A) \subseteq A$. Consequently, $cl_{\mu_1}(int_{\mu_2}(A)) \subseteq cl_{\mu_1}(A)$ by using Theorem 2.2 (iv). Note that since A is μ_1-closed, Theorem 2.3 (iii) implies $cl_{\mu_1}(A) = A$. In effect, $cl_{\mu_1}(int_{\mu_2}(A)) \subseteq A \subseteq U$. Therefore, A is a (μ_1, μ_2)-swg closed set. □

Remark 6. The Converse of Theorem 3.2 is not necessarily true.

Example 3.4. In Example 3.1, the set {2,4} is a (μ_1, μ_2)-swg closed set and $\{2, 4\} \subseteq \{1, 2, 4\}$, where {1,2,4} is a μ_2-semi open set. However, {2,4} is not μ_1-closed set. Whence, the assertion.

Theorem 3.3. If A is a (μ_1, μ_2)-swg closed set such that $B \subseteq A$, then B is a (μ_1, μ_2)-swg closed set.

Proof. Let A be a (μ_1, μ_2)-swg closed set, then $cl_{\mu_1}(int_{\mu_2}(A)) \subseteq U$ whenever $A \subseteq U$, and U is a μ_2-semi open set. Since $B \subseteq A$, then $B \subseteq U$. By Theorem 2.1 (iv), $int_{\mu_2}(B) \subseteq int_{\mu_2}(A)$. Thus, by Theorem 2.2 (iv), $cl_{\mu_1}(int_{\mu_2}(B)) \subseteq cl_{\mu_1}(int_{\mu_2}(A))$. That is, $cl_{\mu_1}(int_{\mu_2}(B)) \subseteq U$ whenever $B \subseteq U$ and U is a μ_1-semi open set. Therefore, B is a (μ_1, μ_2)-swg closed set. □

Theorem 3.4. If $cl_{\mu_1}(int_{\mu_2}(A)) \subseteq A$ such that $A \subseteq U$ where U is μ_2-semi open set, then A is (μ_1, μ_2)-semi weakly generalized closed set.

Proof. Suppose that $cl_{\mu_1}(int_{\mu_2}(A)) \subseteq A$ where $A \subseteq U$, and U is a μ_2-semi open set. By Theorem 2.2 (iv), $cl_{\mu_1}(cl_{\mu_1}(int_{\mu_2}(A))) \subseteq cl_{\mu_1}(A)$. Thus, by Theorem 2.2 (v), $cl_{\mu_1}(int_{\mu_2}(A)) \subseteq cl_{\mu_1}(A)$, by Theorem 2.2 (iii), we can say that $cl_{\mu_1}(A) = A$. This means that A is μ_1-closed. By Theorem 3.2 A is (μ_1, μ_2)-swg closed set. □

Theorem 3.5. Let (X, μ_1, μ_2) be a BGT-space. Then X is a (μ_1, μ_2)-swg closed set.

Proof. We first show that X is μ_2-semi open. Suppose $A = X$, and by Theorem 2.1 (ii), there exists a largest μ_2-open set G such that $int_{\mu_2}(A) = G$. Consequently, by Theorem 2.2 (iv), $cl_{\mu_2}(int_{\mu_2}(A)) =$

$cl_{\mu_2}(G)$. In effect, $cl_{\mu_2}\big(int_{\mu_2}(A)\big) = X$. It follows that $A \subseteq cl_{\mu_2}\big(int_{\mu_2}(A)\big) = X$. Hence, X is a μ_2-semi open set. Now, by Theorem 2.1 (i) and Theorem 2.2 (iv.), $int_{\mu_2}(X) \subseteq X$, and $cl_{\mu_1}\big(int_{\mu_2}(X)\big) \subseteq cl_{\mu_1}(X)$, respectively. Theorem 2.2 (iii) implies $cl_{\mu_1}(X) = X$. That is, $cl_{\mu_1}\big(int_{\mu_2}(X)\big) \subseteq X$. Therefore, X is a (μ_1, μ_2)-swg closed set. □

Corollary 3.5.1. Let (X, μ_1, μ_2) be a BGT-space. Then $\emptyset$ is a (μ_1, μ_2)-swg open set.

Theorem 3.6. If A is a (μ_1, μ_2)-closed set, and $A \subseteq U$ where U is a μ_2-semi open set. Then, A is a (μ_1, μ_2)-swg closed set.

Proof. Let A be a (μ_1, μ_2)-closed set, then $cl_{\mu_1}\big(cl_{\mu_2}(A)\big) = A$. This means that A is a μ_1-closed set. By Theorem 3.2, A is a (μ_1, μ_2)-swg closed set. □

Theorem 3.7. If for each $i \in I$, A_i is (μ_1, μ_2)-swg closed set in X, then $\bigcap\limits_{i\in I} A_i$ is (μ_1, μ_2)-swg closed set in X.

Proof. Suppose that for each $i \in I$, A_i is a $(\mu_1, \mu_2) - swg$ closed set. Then $cl_{\mu_1}\big(int_{\mu_2}(A_i)\big) \subseteq U$ whenever $A_i \subseteq U$ and U is μ_2-semi open for each $i \in I$. Now, by an elementary property of set operations, $\bigcap\limits_{i\in I} A_i \subseteq A_i$. Also, by Theorem 2.1 (iv), $int_{\mu_2}(\bigcap\limits_{i\in I} A_i) \subseteq int_{\mu_2}(A_i)$. Following Theorem 2.2 (iv), we have

$$cl_{\mu_1}\Big(int_{\mu_2}\Big(\bigcap_{i\in I} A_i\Big)\Big) \subseteq cl_{\mu_1}\big(int_{\mu_2}(A_i)\big).$$

In effect,

$$cl_{\mu_1}\Big(int_{\mu_2}\Big(\bigcap_{i\in I} A_i\Big)\Big) \subseteq U, \text{ whenever } \bigcap_{i\in I} A_i \subseteq U \text{ and } U \text{ is } \mu_2 - \text{semi open in } X.$$

Therefore, $\bigcap\limits_{i\in I} A_i$ is (μ_1, μ_2)-swg closed set in X. □

Remark 7. If A and B are both (μ_1, μ_2)-swg closed sets in X, then $A \cup B$ need not be (μ_1, μ_2)-swg closed set in X.

Corollary 3.7.1. The union of any (μ_1, μ_2)-swg open sets is (μ_1, μ_2)-swg open.

Proof. Suppose that for each $i \in I$, A_i is a (μ_1, μ_2)-swg open set. Then A_i^c is (μ_1, μ_2)-swg closed set for each $i \in I$. By Theorem 3.7, $\bigcap A_i^c$ is a (μ_1, μ_2)-swg closed. By De Morgan's Law, $\bigcap\limits_{i\in I} A_i^c = \Big(\bigcup\limits_{i\in I} A_i\Big)^c$ is a (μ_1, μ_2)-swg closed set. Therefore, $\bigcup\limits_{i\in I} A_i$ is (μ_1, μ_2)-swg open set. □

Theorem 3.8. The (μ_1, μ_2)-$swgO(X)$ is a generalized topology.

Proof. By Corollary 3.5.1 , $\emptyset \in (\mu_1, \mu_2)$-$swgO(X)$. Moreover, by Corollary 3.7.1, the arbitrary union of any (μ_1, μ_2)-swg open sets is a (μ_1, μ_2)-swg open set. Therefore, (μ_1, μ_2)-$swgO(X)$ is a generalized topology. □

Theorem 3.9. Let μ_1 and μ_2 be generalized topologies in X. If $\mu_1 \subseteq \mu_2$ then μ_2-$sO(X) \subseteq \mu_1$-$sO(X)$.

Proof. Let $A \subseteq X$ be a μ_2-semi open. Since $\mu_1 \subseteq \mu_2$, then $int_{\mu_1}(A) \subseteq int_{\mu_2}(A)$. It follows that $cl_{\mu_2}\big(int_{\mu_2}(A)\big) \subseteq cl_{\mu_1}\big(int_{\mu_1}(A)\big)$. Since A is a μ_2-semi open set, then $A \subseteq cl_{\mu_2}\big(int_{\mu_2}(A)\big)$. It follows that $A \subseteq cl_{\mu_1}\big(int_{\mu_1}(A)\big)$. In effect, A is a μ_1-semi open set. Therefore, μ_2-$sO(X) \subseteq \mu_1$-$sO(X)$. □

Theorem 3.10. Let μ_1 and μ_2 be generalized topologies in X. If $\mu_1 \subseteq \mu_2$, then (μ_1, μ_2)-$swgC(X) \subseteq (\mu_2, \mu_1)$-$swgC(X)$.

Proof. Let A be a (μ_1, μ_2)-$swgC(X)$. Then, $cl_{\mu_1}\big(int_{\mu_2}(A)\big) \subseteq U_2$ whenever $A \subseteq U_2$ where U_2 is a μ_2-semi open set. Since $\mu_1 \subseteq \mu_2$, we have μ_1-$C(X) \subseteq \mu_2$-$C(X)$. By Theorem 3.9, μ_2-$sO(X) \subseteq \mu_1$-$sO(X)$. This means that $A \subseteq U_1$, where U_1 is μ_1-semi open. By Theorem 3.2, A is a μ_1-closed set. But μ_1-$C(X) \subseteq \mu_2$-$C(X)$, it follows that A is a μ_2-closed set. In effect, A is (μ_2, μ_1)-$swgC(X)$. Therefore, (μ_1, μ_2)-$swgC(X) \subseteq (\mu_2, \mu_1)$-$swgC(X)$. □

3.2. (μ_1, μ_2)**-Semi Weakly Generalized Continuous Functions.** This section will introduce the concept on $swg_{(\mu_1,\mu_2)}$-continuous functions in bigeneralized topological spaces, and investigate some of their properties.

Definition 15. Let (X, μ_1, μ_2) be a bigeneralized topological space, and (Y, μ) be a generalized topological space. A function $f : (X, \mu_1, \mu_2) \to (Y, \mu)$ is called a (μ_1, μ_2)**-semi weakly generalized continuous (or briefly** $swg_{(\mu_1,\mu_2)}$**-continuous)** if $f^{-1}(F)$ is (μ_1, μ_2)-swg closed in X for every μ-closed F in Y.

Theorem 3.11. If $f : (X, \mu_1, \mu_2) \to (Y, \mu)$ is $swg_{(\mu_1,\mu_2)}$-continuous, then $f^{-1}(F)$ is (μ_1, μ_2)-swg open in X for every μ-open F of Y.

Proof. Suppose that $f : (X, \mu_1, \mu_2) \to (Y, \mu)$ is $swg_{(\mu_1,\mu_2)}$-continuous such that F is a μ-open set in Y. Then, F^c is a μ-closed set in Y. Since f is $swg_{(\mu_1,\mu_2)}$-continuous, $f^{-1}(F^c)$ is (μ_1, μ_2)-swg closed in X. But, $f^{-1}(F^c) = (f^{-1}(F))^c$. This means that $(f^{-1}(F))^c$ is (μ_1, μ_2)-swg closed in X. It follows that $f^{-1}(F)$ is (μ_1, μ_2)-swg-open in X. Therefore, $f^{-1}(F)$ is a (μ_1, μ_2)-swg open set in X for every μ-open set F in Y. □

Definition 16. Let (X, μ_1, μ_2) be a bigeneralized topological space, and (Y, μ) be a generalized topological space. A function $f : (X, \mu_1, \mu_2) \to (Y, \mu)$ is said to be **pairwise** (μ_1, μ_2)**-semi weakly generalized continuous (or briefly pairwise** $swg_{(\mu_1,\mu_2)}$**-continuous)** if f is $swg_{(\mu_1,\mu_2)}$-continuous and $swg_{(\mu_2,\mu_1)}$-continuous.

Theorem 3.12. If a function $f : (X, \mu_1, \mu_2) \to (Y, \mu)$ is an injective function, then the following properties are equivalent:

(i) f is $swg_{(\mu_1,\mu_2)}$-continuous;

(ii) For each $x \in X$, and for every μ-open set V containing $f(x)$, there exists a (μ_1, μ_2)-swg open set U containing x such that $f(U) \subseteq V$;

(iii) $f\big(cl_{\mu_1}(A)\big) \subseteq cl_\mu\big(f(A)\big)$ for every subset A of X; and

(iv) $cl_{\mu_1}\big(f^{-1}(B)\big) \subseteq f^{-1}\big(cl_\mu(B)\big)$ for every subset B of Y.

Proof. (i) $\Rightarrow$ (ii) Let $x \in X$ and V be a μ-open subset of Y containing $f(x)$. By Theorem 3.12 (i), $f^{-1}(V)$ is (μ_1, μ_2)-swg open set in X containing x, this means that $U \subseteq f^{-1}(V)$, $f(U) \subseteq f(f^{-1}(V))$ but, $f(f^{-1}(V)) \subseteq V$. Therefore $f(U) \subseteq V$.

(ii) $\Rightarrow$ (iii) Let A be a subset of X, and $f(x) \notin cl_\mu\big(f(A)\big)$. This means that there exists a μ-open set V containing $f(x)$, and $f(A)$ subset of any μ-closed set. It follows that $V \cap f(A) = \emptyset$. Now, by Theorem 3.12 (ii), there exists a (μ_1, μ_2)-swg open set such that $f(x) \in f(U) \subseteq V$. Hence $f(U) \cap f(A) = \emptyset$ implies $U \cap A = \emptyset$. This means that $x \in U$ and $x \notin A$. Consequently, by Theorem 2.2 (i), $A \subseteq cl_{\mu_1}(A)$. In effect, $x \notin cl_{\mu_1}(A)$. Hence, $f(x) \notin f\big(cl_{\mu_1}(A)\big)$. Therefore, $f\big(cl_{\mu_1}(A)\big) \subseteq cl_\mu\big(f(A)\big)$.

(iii) $\Rightarrow$ (iv) Let B be a subset of Y. Note that since the function is injective, $A = f^{-1}(B)$. Now, by Theorem 3.12 (iii), $f\big(cl_{\mu_1}(A)\big) \subseteq cl_\mu\big(f(A)\big)$. So, $f\big(cl_{\mu_1}(f^{-1}(B))\big) \subseteq cl_\mu(B)$ implying $f^{-1}\Big(f\big(cl_{\mu_1}(f^{-1}(B))\big)\Big) \subseteq f^{-1}\big(cl_\mu(B)\big)$. By definition of f, the proof is complete.

(iv) $\Rightarrow$ (i) Let F be a μ-closed subset of Y and U be a μ_2-semi open subset of X such that $f^{-1}(F) \subseteq U$. Since $cl_\mu(F) = F$ and by Theorem 3.12(iv), $cl_{\mu_1}\big(f^{-1}(F)\big) \subseteq f^{-1}\big(cl_\mu(F)\big) = f^{-1}(F)$. Now, using Theorem 2.1 (i) followed by Theorem 2.2 (iv), $cl_{\mu_1}\Big(int_{\mu_2}\big(f^{-1}(F)\big)\Big) \subseteq cl_{\mu_1}\big(f^{-1}(F)\big)$. Thus, $cl_{\mu_1}\Big(int_{\mu_2}\big(f^{-1}(F)\big)\Big) \subseteq f^{-1}(F) \subseteq U$. Therefore, the assertion.

□

3.3. (μ_1, μ_2)**-Semi Weakly Generalized Closed sets in relation to the other well known closed sets.** This section will present some of the well known closed sets defined in literature that are related to the $(\mu_1, \mu_2) - swg$-closed sets with the corresponding conditions. We imposed that by construction, μ_1 and μ_2 are generalized topological spaces.

(1) **Generalized closed (or briefly $g-$closed set)**

If $\mu_1 = SO(X)$ where SO(X) is the collection of μ_2-semi open sets in X, and μ_2 is the discrete topology in X. Then it follows that every $(\mu_1, \mu_2) - swg$-closed set is a $g-$closed set with respect to μ_1. That is,

$$cl_{\mu_1}(int_{\mu_2}(A)) = cl_{\mu_1}(A) \subseteq U$$

whenever $A \subseteq U$ and U is μ_1-open in X.

(2) **Semi-generalized closed (or briefly $sg-$closed set)**

If $\mu_1 = SO(X)$ where $SO(X)$ is the collection of μ_2-semi open sets in X, and μ_2 is the discrete topology in X, then every $(\mu_1, \mu_2) - swg$ closed set is $sg-$closed since

$$cl_{\mu_1}(int_{\mu_2}(A)) = cl_{\mu_1}(A) = cl_{SO(X)}(A) = scl(A) \subseteq U$$

whenever $A \subseteq U$ and U is μ_1-semi open in X.

(3) **Generalized semi-closed (or briefly $gs-$closed set)**

If $\mu_1 = SO(X)$ where $SO(X)$ is the collection of μ_2-semi-open sets in X, and μ_2 is the discrete topology in X, then every $(\mu_1, \mu_2) - swg$ closed set is $sg-$closed since

$$cl_{\mu_1}(int_{\mu_2}(A)) = cl_{\mu_1}(A) = cl_{SO(X)}(A) = scl(A) \subseteq U$$

whenever $A \subseteq U$ and U is μ_1-open in X.

(4) **Generalized b-closed (or briefly $gb-$closed set)**

If $\mu_1 = BO(X)$ is the collection of μ_2-b-open sets in X, and μ_2 is the discrete topology, then with μ_2-semi open U, every $(\mu_1, \mu_2) - swg$ closed set is $gb-$closed since

$$cl_{\mu_1}(int_{\mu_2}(A)) = cl_{\mu_1}(A) = cl_{BO(X)}(A) = bcl(A) \subseteq U$$

whenever $A \subseteq U$ and U is μ_1-open in X. It holds since every μ_2-semi open set U is μ_1-open in X.

(5) **Weakly closed (or briefly $w-$closed set)**

If $\mu_1 = SO(X)$ is the collection of semi open sets in X, and μ_2 is the discrete topology, then every $(\mu_1, \mu_2) - swg$ closed set is $w-$closed since

$$cl_{\mu_1}(int_{\mu_2}(A)) = cl_{\mu_1}(A) = cl_{\mu_1}(A) \subseteq U$$

whenever $A \subseteq U$ and U is μ_1-semi open in X.

(6) **Weakly generalized closed (or briefly $wg-$closed set)**

If $\mu_1 = SO(X)$ is the collection of μ_2-semi open sets in X, and μ_2 is a generalized topological space, then every $(\mu_1, \mu_2) - swg$ closed set is $wg-$closed since $cl_{\mu_1}(int_{\mu_2}(A)) \subseteq U$ whenever $A \subseteq U$ and U is μ_1-open in X.

(7) **Generalized preclosed (or briefly gp-closed set)**

If $\mu_1 = PO(X)$ is the collection of μ_2-pre open sets in X, and μ_2 is the discrete topological space, then with μ_2-semi open U, every $(\mu_1, \mu_2) - swg$ closed set is $wg-$closed since

$$cl_{\mu_1}(int_{\mu_2}(A)) = cl_{\mu_1}(A) = cl_{PO(X)}(A) = pcl(A) \subseteq U$$

whenever $A \subseteq U$ and U is μ_1-open in X. It follows from the fact that

$$U \subseteq cl_{\mu_2}(int_{\mu_2}(U)) = int_{\mu_2}(cl_{\mu_2}(U)) = int_{\mu_1}(cl_{\mu_1}(U)).$$

(8) **Generalized semi-preclosed (or briefly gsp-closed set)**

If $\mu_1 = SPO(X)$ is the collection of μ_2-semi pre-open sets in X, and μ_2 is the discrete topological space, then with μ_2-semi open U, every $(\mu_1, \mu_2) - swg$ closed set is $gsp-$closed since

$$cl_{\mu_1}(int_{\mu_2}(A)) = cl_{\mu_1}(A) = cl_{SPO(X)}(A) = spcl(A) \subseteq U$$

whenever $A \subseteq U$ and U is μ_1-open in X. It holds since

$$\begin{aligned} U &\subseteq cl_{\mu_2}(int_{\mu_2}(U)) = U \\ &= cl_{\mu_2}(int_{\mu_2}(cl_{\mu_2}(U))) = cl_{\mu_1}(int_{\mu_1}(cl_{\mu_1}(U))). \end{aligned}$$

(9) **Generalized α-closed (or briefly $g\alpha$-closed set)**

If $\mu_1 = AO(X)$ is the collection of μ_2-α-open sets in X, and μ_2 is any topological space, then with μ_2-semi open U, every $(\mu_1, \mu_2) - swg$ closed set is $g\alpha-$closed since

$$cl_{\mu_1}(int_{\mu_2}(A)) = cl_{\mu_1}(A) = cl_{AO(X)}(A) = \alpha cl(int(A)) \subseteq U$$

whenever $A \subseteq U$ and U is μ_1-α-open in X. The assertion holds from the fact that

$$\begin{aligned} U &\subseteq cl_{\mu_2}(int_{\mu_2}(U)) \\ &\subseteq int_{\mu_2}(cl_{\mu_2}(int_{\mu_2}(U))) = int_{\mu_1}(cl_{\mu_1}(int_{\mu_1}(U))). \end{aligned}$$

(10) **α-generalized closed (or briefly αg-closed set)**

If $\mu_1 = AO(X)$ is the collection of μ_2-α-open sets in X, and μ_2 is the discrete topological space, then with μ_2-semi open U, every $(\mu_1, \mu_2) - swg$ closed set is $g\alpha-$closed since

$$cl_{\mu_1}(int_{\mu_2}(A)) = cl_{\mu_1}(A) = cl_{AO(X)}(A) = \alpha cl(A) \subseteq U$$

whenever $A \subseteq U$ and U is μ_1-open in X. Similar argument for U is used as that of $g\alpha$-closed set.

4. Conclusion

This study introduced and explored the concept of (μ_1, μ_2)-semi weakly generalized closed ($(\mu_1, \mu_2)-swg$ closed) sets within the framework of bi-generalized topological spaces. The fundamental properties of these sets were thoroughly examined, along with their interrelationships with existing classes of closed sets, such as generalized closed, semi-generalized closed, and weakly closed sets. The investigation revealed that $(\mu_1, \mu_2) - swg$ closed sets provide a broader generalization of previously known closed sets, offering new insights into topological structures characterized by two distinct generalized topologies. A major result established in this work is that the family of $(\mu_1, \mu_2) - swg$ closed sets does not necessarily form a generalized topology, as it does not always satisfy the property of being closed under arbitrary unions. However, it retains key closure properties under finite intersections, demonstrating its robustness as a generalized closure operator. The study also characterized $(\mu_1, \mu_2) - swg$ open sets and provided necessary conditions for their existence, further reinforcing the structural significance of these sets in bi-generalized topological spaces.

On the one hand, this research introduced the notion of (μ_1, μ_2)-semi weakly generalized continuous ($swg_{(\mu_1,\mu_2)}$-continuous) functions and examined their properties. It was shown that these functions preserve $(\mu_1, \mu_2) - swg$ closed sets under pre-image operations, ensuring that the fundamental aspects of continuity extend naturally to this new class of closed sets. Furthermore, necessary and sufficient conditions for $swg_{(\mu_1,\mu_2)}$-continuity were established using closure and interior properties, providing a deeper understanding of how these functions behave under different topological settings. For interested researchers, furthering the study on $(\mu_1, \mu_2) - swg$ closed sets have the potential to contribute to the study of separation axioms, compactness, and connectedness in generalized topological spaces. Future investigations could explore the relationships between $(\mu_1, \mu_2) - swg$ closed sets and other known generalized structures, as well as their applications in analysis and applied topology. The extension of these concepts to multi-topological spaces and their impact on broader mathematical fields remains a promising direction for future study.

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