Asia Pac. J. Math. 2025 **12**:57

CLASSES OF OPERATORS ASSOCIATED WITH (f, g)-ALUTHGE TRANSFORMS

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Received Apr. 6, 2025

Abstract. Let $T \in \mathcal{B}(\mathcal{H})$ be a bounded linear operator on a Hilbert space $\mathcal{H}$ with the polar decomposition $T = U|T|$. The (f, g)-Aluthge transform of the operator T, denoted by $\Delta_{f,g}(T)$, is defined as $\Delta_{f,g}(T) = f(|T|)Ug(|T|)$, where f and g both are non-negative continuous functions on $[0, \infty[$ such that $f(x)g(x) = x$, for all $x \geq 0$. In this paper, firstly, we investigate the relationship between this transform and several classes of operators as quasi-normal, normal, positive, nilpotent and closed range operators. Secondly, we show that under some conditions the (f, g)-Aluthge transform possesses the polar decomposition. Lastly, we provide a characterization of binormal operators from the viewpoint of the polar decomposition and the (f, g)-Aluthge transform.

2020 Mathematics Subject Classification. 47A05; 47B49.
Key words and phrases. (f, g)-Aluthge transform; quasinormal operato; Polar decomposition; binormal operators.

1. Introduction

Let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. In the sequel of this paper,$\mathcal{R}(T)$, $\mathcal{N}(T)$ and T^* stand, respectively, for the range, the null subspace and the adjoint of the operator $T \in \mathcal{B}(\mathcal{H})$. An operator $T \in B(H)$ is said to be self-adjoint if $T = T^*$, normal if $TT^* = T^*T$, positif if $\langle Tx, x\rangle \geq 0$, for all $x \in \mathcal{H}$, quasi-normal if $TT^*T = T^*TT$ and binormal if T^*T and TT^* commute. Remind that an arbitrary operator $T \in \mathcal{B}(\mathcal{H})$ has a unique polar decomposition $T = U|T|$, where $|T| = (T^*T)^{\frac{1}{2}}$ is the modulus of T and U is the associate partial isometry, i.e. $UU^*U = U$ with kernel condition $\mathcal{N}(U) = \mathcal{N}(T) = \mathcal{N}(|T|)$. Then this polar decomposition verifies the following properties :

$P(1)$ $UU^* = P_{\overline{R(T)}} = P_{\overline{R(|T^*|)}}$ and $U^*U = P_{\overline{R(T^*)}} = P_{\overline{R(|T|)}}$, where P_M denotes the orthogonal projection onto the closed subspace M of $\mathcal{H}$.

DOI: 10.28924/APJM/12-57

$P(2)$ $U|T| = |T^*|U$.

$P(3)$ $|T| = U^*|T^*|U$.

$P(4)$ $|T^*| = U|T|U^*$.

$P(5)$ T is quasi-normal if and only if $U|T| = |T|U$.

$P(6)$ T is quasi-normal if and only if $T|T| = |T|T$.

Related to the polar decomposition, the (f,g)-Aluthge transform of an operator T was introduced recently in [9] and defined by

$$\Delta_{f,g}(T) = f(|T|)Ug(|T|),$$

where f and g both are non-negative continuous functions on $[0,\infty[$ such that $f(x)g(x) = x$, for all $x \geq 0$. For the special case $f(x) = x^\lambda$ and $g(x) = x^{1-\lambda}$, where $\lambda \in [0,1]$, we obtain the usual λ-Aluthge transform $\Delta_\lambda(T)$, which was first introduced by Aluthge in the case when $\lambda = \frac{1}{2}$ [1]. After that, many authors began to discuss the properties of (f,g)-Aluthge transform (see [8,9,11]). One of the most important properties of this transform is that T and $\Delta_{f,g}(T)$ have the same spectrum (see [8, Proposition 2.6]). In this paper, we investigate the relationship between this new transform and several classes of operators as quasi-normal, normal, positive and nilpotent operators. As a consequence, We extend various results on λ-Aluthge transform (see [2,10–12]) to (f,g)-Aluthge transforms. Before proceeding, We denote by $\mathcal{C}_+$ the set of all continuous function $f : [0,+\infty[\longrightarrow [0,+\infty[$ whith $f(0) = 0$. Obviously, if $f,g \in \mathcal{C}_+$ such that $f(x)g(x) = x$, for all $x \geq 0$, then $f(T)g(T) = g(T)f(T) = T$, for any positive operator T in $\mathcal{B}(\mathcal{H})$.

The paper is organized as follows. In section 2, firstly, we show that the transform $\Delta_{f,g}(T)$ does not depend of the choice of the partial isometry factor in the polar decomposition of T. Secondly, we study the fixed points of (f,g)-Aluthge transform. Third, we focus on conditions on $\Delta_{f,g}(T)$ under which T is normal or selft-adjoint or positive or nilpotent. In particular, we show that if $f,g \in \mathcal{C}_+$ such that $f(x)g(x) = x$ $(x \geq 0)$, then T is nilpotent of order $d+1$ if and only if $(\Delta_{f,g}(T))^d = 0$, for every $d \in \mathbb{N}^*$. Lastly, we study the closedness of the range of $\Delta_{f,g}(T)$.

Section 3 deals with the polar decomposition of $\Delta_{f,g}(T))$. At first, we show that under some conditions the (f,g)-Aluthge transform possesses the polar decomposition. Afterwards, we give a characterization of binormal operators via (f,g)-Aluthge. Precisely, we prove that T is binormal if and only if $\Delta_{f,g}(T) = U|\Delta_{f,g}(T)|$, where $T = U|T|$ is the polar decomposition of T and $f,g \in \mathcal{C}_+$ are both increasing functions satisfying $f(x)g(x) = x$ $(x \geq 0)$.

2. Some relationships between an operator and its (f,g)-Aluthge transform

We begin this section with the following two lemmas which will be necessary to prove our main results.

Lemma 2.1. *Let $T = U|T|$ be the polar decomposition of $T \in \mathcal{B}(\mathcal{H})$ and let $f \in \mathcal{C}_+$. Then the following properties hold:*

(i) $Uf(|T|) = f(|T^*|)U$.

(ii) $U^*Uf(|T|) = f(|T|)U^*U = f(|T|)$.

(iii) $UU^*f(|T^*|) = f(|T^*|)UU^* = f(|T^*|)$.

(iv) $U^*f(|T^*|)U = f(|T|) = f(U^*|T^*|U)$.

(v) $Uf(|T|)U^* = f(|T^*|) = f(U|T|U^*)$.

Proof. (i) By $P(2)$, $U|T|^n = |T^*|^n U$, for each $n \in \mathbb{N}^*$. Which implies $UP(|T|) = P(|T^*|)U$, for any polynomial $P(t)$. Since f is non-negative continuous function on $\sigma(|T|) \subset [0, \infty[$ with $f(0) = 0$, so there exist a sequence of polynomial $(P_n)_{n\in\mathbb{N}^*}$ such that $P_n(0) = 0$ for every $n \in \mathbb{N}^*$, and $P_n(t) \longrightarrow f(t)$ uniformly on the interval $[0, \||T|\|]$. Hence,

$$Uf(|T|) = U \lim_{n\to\infty} P_n(|T|) = \lim_{n\to\infty} UP_n(|T|) = \lim_{n\to\infty} P_n(|T^*|)U = f(|T^*|)U,$$

and then, the assertion (i) holds.
(ii) By $P(1)$, we have $U^*U|T| = |T|$. Following the same procedure as (i), we can prove that $U^*Uf(|T|) = f(|T|)$. Hence, by taking the adjoint, we deduce that $f(|T|)U^*U = f(|T|)$.
The proof of (iii) is similar to that of (ii).
(iv) Using (i), (ii) and $P(3)$, we get

$$U^*f(|T^*|)U = U^*Uf(|T|) = f(|T|) = f(U^*|T^*|U).$$

Thus, the property (iv) is satisfied.
(v) is deduced directly from (i), (iii) and $P(4)$.

□

Lemma 2.2. *Let $T \in \mathcal{B}(\mathcal{H})$ be positive and $f, g \in \mathcal{C}_+$ such that $f(x)g(x) = x$, for all $x \geq 0$. Then*

$$\mathcal{N}(T) = \mathcal{N}(f(T)) = \mathcal{N}(g(T)).$$

Proof. The inclusion $\mathcal{N}(f(T)) \subset \mathcal{N}(T)$ is obvious because $T = g(T)f(T)$.
Now, we show the other inclusion. Since T is positive, $\mathcal{N}(T) = \mathcal{N}(T^n)$ for each $n \in \mathbb{N}^*$. On the other hand, since f is a continuous function on $\sigma(T) \subset [0, \infty[$ with $f(0) = 0$, there existe a sequence of polynomial $(P_n)_{n\in\mathbb{N}^*}$ without constant terms such that $P_n(t) \longrightarrow f(t)$ uniformly on the interval

$[0, \|T\|]$. Hence, for all $x \in \mathcal{H}$ and $n \in \mathbb{N}^*$, we have

$$\begin{aligned} x \in \mathcal{N}(T) &\implies Tx = 0 \\ &\implies P_n(T)x = 0 \\ &\implies \lim_{n\to\infty} P_n(T)x = 0 \\ &\implies f(T)x = 0 \\ &\implies x \in \mathcal{N}(f(T)). \end{aligned}$$

Therefore, $\mathcal{N}(T) \subset \mathcal{N}(f(T))$.
By similar way, we can prove that $\mathcal{N}(T) = \mathcal{N}(g(T))$. □

It was shown in [7], that the λ-Aluthge transform does not depend on the partial isometry. This result is also valid for the (f, g)-Aluthge transform.

Proposition 2.3. *Let $T = U|T|$ be the polar decomposition of T and let $f, g \in \mathcal{C}_+$ such that $f(x)g(x) = x$, for all $x \geq 0$. If there exists another decomposition $T = V|T|$, then*

$$\Delta_{f,g}(T) = f(|T|)Ug(|T|) = f(|T|)Vg(|T|).$$

Proof. Using the assumptions and Lemma 2.2, we have

$$\mathcal{H} = \mathcal{N}(|T|) \oplus \mathcal{N}(|T|)^{\perp} = \mathcal{N}(g(|T|)) \oplus \mathcal{N}(f(|T|))^{\perp}.$$

In case $x \in \mathcal{N}(g(|T|))$, we obtain

$$\Delta_{f,g}(T)x = f(|T|)Ug(|T|)x = 0 = f(|T|)Vg(|T|)x.$$

So, $f(|T|)Ug(|T|)x = f(|T|)Vg(|T|)x = 0$, on $\mathcal{N}(g(|T|))$.
Now, in case $x \in \mathcal{R}(f(|T|))$, there exists $z \in \mathcal{H}$ such that $x = f(|T|)z$. Then we have

$$\begin{aligned} \Delta_{f,g}(T)x = f(|T|)Ug(|T|)x &= f(|T|)Ug(|T|)f(|T|)z \\ &= f(|T|)U|T|z \\ &= f(|T|)Tz \\ &= f(|T|)V|T|z \\ &= f(|T|)Vg(|T|)f(|T|)z \\ &= f(|T|)Vg(|T|)x. \end{aligned}$$

Hence, $f(|T|)Ug(|T|) = f(|T|)Vg(|T|)$ on $\overline{\mathcal{R}(f(|T|))} = \mathcal{N}(f(|T|))^{\perp}$. Therefore, $f(|T|)Ug(|T|) = f(|T|)Vg(|T|)$ on $\mathcal{H}$.

□

Here, we provide a new characterization of quasi-normal operators as follows:

Proposition 2.4. *Let $T = U|T| \in \mathcal{B}(\mathcal{H})$ be the polar decomposition of T and let f, g be two non-negative continuous functions on $[0, +\infty[$, such that $f(x)g(x) = x$ for all $x \geq 0$. Then the following assertions are equivalent:*

(i) *T is quasi-normal.*

(ii) *$f(|T|)U = Uf(|T|)$ and $g(|T|)U = Ug(|T|)$.*

(iii) *$f(|T|)T = Tf(|T|)$ and $g(|T|)T = Tg(|T|)$.*

Proof. $(i) \Longrightarrow (ii)$. Suppose that T is quasi-normal. By $P(5)$, we have $|T|U = U|T|$. Then

$$|T|^n U = U|T|^n, \text{ for any } n \in \mathbb{N}.$$

Which implies $P(|T|)U = UP(|T|)$, for any polynomial $P(t)$. Since f is a continuous function on $[0, \infty)$, there existe a sequence of polynomial $(P_n)_n$ such that $P_n(t) \longrightarrow f(t)$ uniformly on the interval $[0, \||T|\|]$. Then,

$$Uf(|T|) = U \lim_{n\to\infty} P_n(T) = \lim_{n\to\infty} UP_n(T) = \lim_{n\to\infty} P_n(T)U = f(|T|)U.$$

Hence, $Uf(|T|) = f(|T|)U$.
By similar way we can prove that $g(|T|)U = Ug(|T|)$.
$(ii) \Longrightarrow (iii)$. From (ii), We have

$$f(|T|)T = f(|T|)U|T| = Uf(|T|)|T| = Uf(|T|)g(|T|)f(|T|) = U|T|f(|T|) = Tf(|T|).$$

and

$$g(|T|)T = g(|T|)U|T| = Ug(|T|)|T| = Ug(|T|)f(|T|)g(|T|) = U|T|g(|T|) = Tg(|T|).$$

So, (iii) is proved.
$(iii) \Longrightarrow (i)$. Using the assumption (iii), we obtain

$$\begin{aligned} T|T| &= Tf(|T|)g(|T|) \\ &= f(|T|)Tg(|T|) \\ &= f(|T|)g(|T|)T \\ &= |T|T. \end{aligned}$$

Hence, $T|T| = |T|T$ and so T is quasi-normal, by $P(6)$.

□

In [5, Proposition 1.10] it was proved that quasi-normal operators are exactly the fixed points of the λ-Aluthge transform. However, this is not the case for (f, g)-Aluthge transform as shown by the following example.

Example 2.5. *Consider* $T = \begin{pmatrix} 0 & 0 & 5 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \end{pmatrix} \in \mathbb{C}^3$. *The canonical polar decomposition of T is $T = U|T|$, where*

$$|T| = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix} \quad \text{and} \quad U = T|T|^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \end{pmatrix}.$$

It follows that

$$|T|U = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{5\sqrt{3}}{2} & -\frac{5}{2} & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 & 5 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \end{pmatrix} = U|T|.$$

Therefore, T is not quasi-normal, by $P(5)$. Let $f(x) = e^{(x-3)^2}$ and $g(x) = xe^{-(x-3)^2}$, for $x \geq 0$. Then, f and g are non-negative continuous functions on $[0, \infty[$ such that $f(x)g(x) = x$, for all $x \geq 0$. So, we obtain

$$f(|T|)Ug(|T|) = \begin{pmatrix} 0 & 0 & f(1)g(5) \\ \frac{f(1)g(1)}{2} & \frac{f(1)g(1)\sqrt{3}}{2} & 0 \\ \frac{f(5)g(1)\sqrt{3}}{2} & -\frac{f(5)g(1)}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 5 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \end{pmatrix} = T.$$

Hence, $\Delta_{f,g}(T) = T$, while T is not quasi-normal.

In the following Theorem, we will show that the fixed points of the (f, g)-Aluthge transform are the quasinormal operators for certain functions f.

Theorem 2.6. *Let $T \in \mathcal{B}(\mathcal{H})$ and let $f, g \in \mathcal{C}_+$ such that $f(x)g(x) = x$, for all $x \geq 0$. If f is increasing, then*

$$T \text{ is quasi-normal} \iff \Delta_{f,g}(T) = T.$$

Proof. $\Longrightarrow$. Note that this implication is true without the increasing condition of f. Since T is quasi-normal, by Proposition 2.4, $f(|T|)U = Uf(|T|)$. Then we have

$$\begin{aligned} \Delta_{f,g}(T) &= f(|T|)Ug(|T|) \\ &= Uf(|T|)g(|T|) \\ &= U|T| \\ &= T. \end{aligned}$$

Therefore, $\Delta_{f,g}(T) = T$.

Conversely, Suppose that $\Delta_{f,g}(T) = T$. Which implies that

$$[f(|T|)U - Uf(|T|)]g(|T|) = 0.$$

So that $f(|T|)U = Uf(|T|)$ on $\overline{\mathcal{R}(g(T))}$. On the other hand, by Lemma 2.2, we get

$$\mathcal{N}(U) = \mathcal{N}(f(|T|)) = \mathcal{N}(g(|T|)).$$

Hence, $f(|T|)U = Uf(|T|) = 0$ on $\mathcal{N}(g(T))$. Consequently, $f(|T|)U = Uf(|T|)$ on $\mathcal{H}$. Since f is increasing it has inverse f^{-1}. Thus, by the continuous functional calculus, we obtain $f^{-1}f(|T|)U = Uf^{-1}f(|T|)$. Which means that $|T|U = U|T|$. Therefore, T is quasi-normal. □

As an application of the previous theorem we state an interesting result as follows:

Corollary 2.7. *Let $T \in \mathcal{B}(\mathcal{H})$ be an invertible operator and let $f, g \in \mathcal{C}_+$ such that $f(x)g(x) = x$, for all $x \geq 0$. If f is increasing, then*

$$\Delta_{f,g}(T) = T \iff T \text{ is normal}.$$

Proof. Since T is an invertible, then

$$T \text{ is quasi-normal} \iff T \text{ is normal}.$$

Therefore, the result is obvious by Theorem 2.6. □

The next Proposition extends Lemma 2.3, obtained in [2] to the case of the (f, g)-Aluthge transform as follows.

Proposition 2.8. . *Let $T \in \mathcal{B}(\mathcal{H})$, $P \in \mathcal{B}(\mathcal{H})$ be an orthogonal projection and let $f, g \in \mathcal{C}_+$ such that $f(x)g(x) = x$, for all $x \geq 0$. If f is increasing, then the following assertions are equivalent:*

(i) $\Delta_{f,g}(TP) = T$.

(ii) $TP = PT = T$ and T is quasi-normal.

Proof. $(i) \Longrightarrow (ii)$. Let $TP = V|TP|$ be the polar decomposition of TP. From the hypothesis $\Delta_{f,g}(TP) = T$, we get

$$f(|TP|)Vg(|TP|) = T \text{ and } g(|TP|)V^*f(|TP|) = T^*.$$

Hence, by Lemma 2.2, we obtain

$$\mathcal{R}(T) \subseteq \mathcal{R}(f(|TP|)) \subseteq \overline{\mathcal{R}(f(|TP|))} = \mathcal{N}(f(|TP|))^\perp = \mathcal{N}(|TP|)^\perp = \mathcal{N}(|TP|^2)^\perp = \overline{\mathcal{R}(|TP|^2)},$$

and

$$\mathcal{R}(T^*) \subseteq \mathcal{R}(g(|TP|)) \subseteq \overline{\mathcal{R}(g(|TP|))} = \mathcal{N}(g(|TP|))^\perp = \mathcal{N}(|TP|)^\perp = \mathcal{N}(|TP|^2)^\perp = \overline{\mathcal{R}(|TP|^2)}.$$

Thus,

$$\mathcal{R}(T) \subseteq \overline{\mathcal{R}(|TP|^2)} \text{ and } \mathcal{R}(T^*) \subseteq \overline{\mathcal{R}(|TP|^2)}. \tag{2.1}$$

On the other hand, since $|TP|^2 = PT^*TP = P|T|^2P$, we have

$$\overline{\mathcal{R}(|TP|^2)} \subseteq \overline{\mathcal{R}(P)} = \mathcal{R}(P).$$

Using (2.1), it follows that

$$\mathcal{R}(T) \subset \mathcal{R}(P) \text{ and } \mathcal{R}(T^*) \subset \mathcal{R}(P).$$

Which implies that $T = PT$ and $T^* = PT^* = (TP)^*$. By taking the adjoint, we deduce that $PT = TP = T$. Thus,

$$T = \Delta_{f,g}(TP) = \Delta_{f,g}(T).$$

Moreover, since f is increasing, T is quasi-normal, by Theorem 2.6.
$(ii) \Longrightarrow (i)$ is deduced directly from Theorem 2.6. □

Obviously, every self-adjoint operator is quasi-normal but the converse is not true in general. Next, we give certain conditions under which a quasi-normal operator becomes self-adjoint.

Proposition 2.9. *. Let $T \in \mathcal{B}(\mathcal{H})$ be a quasi-normal operator and let $f, g \in \mathcal{C}_+$ such that $f(x)g(x) = x$, for all $x \geq 0$. If $\Delta_{f,g}(T^*) = T$, then T is self-adjoint.*

Proof. Let $T = U|T|$ be the polar decomposition of T. Since T is quasi-normal, by using Proposition 2.4, we have $f(|T|)U = Uf(|T|)$ and $g(|T|)U = Ug(|T|)$. By taking the adjoint, we get

$$U^*f(|T|) = f(|T|)U^* \text{ and } U^*g(|T|) = g(|T|)U^*.$$

Since $T^* = U^*|T^*|$ is the polar decomposition of T^*, it follows that

$$\begin{aligned}
\Delta_{f,g}(T^*) &= f(|T^*|)U^*g(|T^*|) \\
&= Uf(|T|)U^*U^*Ug(|T|)U^* \quad \text{by Lemma 2.1 } (v) \\
&= Uf(|T|)U^*g(|T|)U^* \quad \text{by Lemma 2.1 } (ii) \\
&= Uf(|T|)U^*U^*g(|T|) \\
&= U(U^*)^2 f(|T|)g(|T|) \\
&= U(U^*)^2|T|.
\end{aligned}$$

Thus, from the assumption $\Delta_{f,g}(T^*) = T$, we obtain $U(U^*)^2|T| = U|T|$. Multiplying this equality by U^* on the left side, we get

$$U^*U(U^*)^2|T| = (U^*)^2|T| = |T|.$$

So $(U^*)^2|T|$ is self-djoint. Moreover, since T is quasi-normal, then we have

$$|T| = (U^*)^2|T| = |T|U^2 = U|T|U.$$

Thus,

$$T = U|T| = |T|U = U^*U|T|U = U^*|T| = |T|U^* = T^*.$$

Therefore, T is self-adjoint. □

Let $T = U|T|$ be the polar decomposition of $T \in \mathcal{B}(\mathcal{H})$ and $\Delta(T) = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ be its Aluthge transform. In [10], the authors showed that if U is unitary and $T = \alpha\Delta(T)$ for some complex number α, then T is normal. In the following three results, we discuss the similar situation of (f, g)-Aluthge transforms.

Proposition 2.10. *Let f and g be two increasing functions in $\mathcal{C}_+$ such that $f(x)g(x) = x$, for all $x \geq 0$ and let $T = U|T|$ be the polar decomposition of $T \in \mathcal{B}(\mathcal{H})$. If $T = \alpha\Delta_{f,g}(T)$ for some complex number α, then*

$$Uf(|T|) = \alpha f(|T|)U \ \text{ and } \ \alpha \geq 1.$$

Proof. Let $T = U|T|$ be the polar decomposition of T. Then, we have

$$\begin{aligned} T = \alpha\Delta_{f,g}(T) &\iff U|T| = \alpha f(|T|)Ug(|T|) \\ &\iff Uf(|T|)g(|T|) = \alpha f(|T|)Ug(|T|) \\ &\iff [Uf(|T|) - \alpha f(|T|)U]g(|T|) = 0, \end{aligned}$$

and thus $Uf(|T|) = \alpha f(|T|)U$ on $\overline{\mathcal{R}(g(|T|))} = \mathcal{N}(g(|T|))^{\perp}$. Since by Lemma 2.2, $\mathcal{N}(f(|T|)) = \mathcal{N}(U) = \mathcal{N}(g(|T|))$, then , it is clear that $Uf(|T|) = \alpha f(|T|)U = 0$ on $\mathcal{N}(g(|T|))$. Hence, $Uf(|T|) = \alpha f(|T|)U$ on $\mathcal{H}$.
Multiplying this equality by U^* on the left side and using Lemma 2.1 (ii), we get

$$f(|T|) = \alpha U^* f(|T|)U.$$

Hence $\alpha > 0$, because $f(|T|)$ and $U^*f(|T|)U$ are positive. Moreover, since f and g are increasing, then we have

$$\begin{aligned} \|T\| &= |\alpha|\|\Delta_{f,g}(T)\| \\ &\leq \alpha\|f(|T|)\|\|U\|\|g(|T|)\| \\ &= \alpha\|f(|T|)\|\|g(|T|)\| \qquad \text{since } \|U\| = 1 \\ &= \alpha f(\||T|\|)g(\||T|\|) \\ &= \alpha f(\|T\|)g(\|T\|) \qquad \text{since } \||T|\| = \|T\| \\ &= \alpha\|T\|. \end{aligned}$$

Thus, $\alpha \geq 1$. □

We say that $T \in \mathcal{B}(\mathcal{H})$ is normaloid if and only if $r(T) = \|T\|$, where $r(T)$ denotes the spectral radius of T. In [11, Corollary 9] the authors showed that the inequality $\|\Delta_{f,g}(T)\| \geq \|T\|$ holds, for any normaloid operator T in $\mathcal{B}(\mathcal{H})$. Next, We use this result to prove the following corollary.

Corollary 2.11. *Let f, g be two increasing functions in $\mathcal{C}_+$ such that $f(x)g(x) = x$, for all $x \geq 0$. If $T \in \mathcal{B}(\mathcal{H})$ is a non-zero normaloid operator such that $T = \alpha\Delta_{f,g}(T)$, for some complex number α, then T is quasi-normal.*

Proof. Suppose that $T = \alpha\Delta_{f,g}(T)$. From Proposition 2.10, we obtain $\alpha \geq 1$. On the other hand since T is normaloid and by using [11, Corollary 9], we have

$$\|T\| = \|\alpha\Delta_{f,g}(T)\| \geq \alpha\|T\|.$$

It follows that $\alpha \leq 1$ and so $\alpha = 1$. which means that $T = \Delta_{f,g}(T)$. Therefore, T is quasi-normal by Theorem 2.6. □

Proposition 2.12. *Let f, g be two increasing functions in $\mathcal{C}_+$ such that $f(x)g(x) = x$, for all $x \geq 0$, and let $T = U|T|$ be the polar decomposition of $T \in \mathcal{B}(\mathcal{H})$ with U unitary. If $T = \alpha\Delta_{f,g}(T)$ for some complex number α, then T is normal.*

Proof. Using Proposition 2.10, we have $Uf(|T|) = \alpha f(|T|)U$. Since $f(|T|)$ is positive, by [10, Proposition 2.10], we deduce that $\alpha = 1$. This implies that $T = \Delta_{f,g}(T)$. Therefore, from Theorem 2.6, T is quasi-normal. So T is normal because U is unitaire. □

Now, we present some relationships between a positive operator and its (f, g)-Aluthge transform.

Theorem 2.13. *Let $T \in \mathcal{B}(\mathcal{H})$ be an invertible operator. Then the following assertions are equivalent:*

(i) *T is positive.*

(ii) *$\Delta_{f,g}(T)$ is positive, for every $f, g \in \mathcal{C}_+$ which satisfying $f(x)g(x) = x$, for all $x \geq 0$.*

(iii) *$\Delta_{f,g}(T)$ is positive, for some $f, g \in \mathcal{C}_+$ which satisfying $f(x)g(x) = x$, for all $x \geq 0$.*

Proof. $(i) \Rightarrow (ii)$. Let $T = U|T|$ be the polar decomposition T. Since T is positive and invertible, it follows that

$$\begin{aligned} U &= T|T|^{-1} \\ &= TT^{-1} \\ &= I. \end{aligned}$$

Thus, $\Delta_{f,g}(T) = f(|T|)g(|T|) = |T|$, and so $\Delta_{f,g}(T)$ is positive .

$(ii) \Rightarrow (iii)$. Trivial.

$(iii) \Rightarrow (i)$. Assume that $\Delta_{f,g}(T)$ is positive. Since T is invertible, then $|T|$ is invertible and by the continuous functional calculus, $f(|T|)$ and $g(|T|)$ are also invertible. We put $A = (g(|T|))^{-1}f(|T|)$.

Then A is the product of two commuting and positive operators so $A > 0$. As a consequence, we have

$$\begin{aligned} AU &= g(|T|))^{-1}f(|T|)U \\ &= (g(|T|))^{-1}(f(|T|)Ug(|T|))(g(|T|))^{-1} \\ &= (g(|T|))^{-1}\Delta_{f,g}(T)(g(|T|))^{-1}. \end{aligned}$$

Hence, $AU = (g(|T|))^{-1}\Delta_{f,g}(T)(g(|T|))^{-1}$ is positive. Which means that $AU = U^*A$. By multiplying this equation on the left by U, we get

$$\begin{aligned} UAU = UU^*A &\implies UAU = A \quad \text{since } U \text{ is unitary} \\ &\implies (AU)^2 = A^2 \\ &\implies AU = A \quad \text{since } AU \text{ and } A \text{ are positive} \\ &\implies U = I \quad \text{since } A \text{ is invertible.} \end{aligned}$$

That implies $T = |T|$ and so T is positive. □

The following theorem shows that the (f, g)-Aluthge transform of a nilpotent operator is nilpotent too. This theorem was proved by Jung, Ko and Pearcy in [6], for λ-Aluthge transforms.

Theorem 2.14. *Let $T \in \mathcal{B}(\mathcal{H})$ and let f and g be as in Theorem 2.13. Then for every $d \in \mathbb{N}^*$, we have*

$$T^{d+1} = 0 \iff (\Delta_{f,g}(T))^d = 0.$$

Proof. Let $T = U|T|$ be the polar decomposition of T and $d \in \mathbb{N}^*$. Then, it is easy to see the following equalities:

$$\begin{aligned} T^{d+1} = (U|T|)^{d+1} &= (Ug(|T|)f(|T|))^{d+1} \\ &= Ug(|T|)(f(|T|)Ug(|T|))^d f(|T|) \\ &= Ug(|T|)(\Delta_{f,g}(T))^d f(|T|). \end{aligned} \tag{2.2}$$

Thus, $(\Delta_{f,g}(T))^d = 0$ implies that $T^{d+1} = 0$. Conversely, we have

$$\begin{aligned} T^{d+1} = 0 &\implies Ug(|T|)(\Delta_{f,g}(T))^d f(|T|) = 0 \quad \text{by (2.2)} \\ &\implies U^*Ug(|T|)(\Delta_{f,g}(T))^d f(|T|) = 0 \\ &\implies g(|T|)(\Delta_{f,g}(T))^d f(|T|) = 0 \quad \text{by Lemma 2.1}(ii) \\ &\implies f(|T|)g(|T|)(\Delta_{f,g}(T))^d f(|T|)g(|T|) = 0 \\ &\implies |T|(\Delta_{f,g}(T))^d|T| = 0. \end{aligned}$$

Hence, for all $x \in \mathcal{H}$, it follows that

$$\langle |T|(\Delta_{f,g}(T))^d|T|x, x\rangle = \langle (\Delta_{f,g}(T))^d|T|x, |T|x\rangle = 0.$$

Thus, $(\Delta_{f,g}(T))^d = 0$ on $\mathcal{R}(|T|)$. Moreover, from Lemma 2.2, we have

$$\mathcal{N}(|T|) = \mathcal{N}(g(|T|)) \subset \mathcal{N}(\Delta_{f,g}(T)),$$

which gives, $(\Delta_{f,g}(T))^d = 0$ on $\mathcal{N}(|T|) = 0$. Therefore, $(\Delta_{f,g}(T))^d = 0$ on $\mathcal{H}$.

□

In what follows of this section, we study the closedness of the range of $\Delta_{f,g}(T)$.

Proposition 2.15. *Let $T \in \mathcal{B}(\mathcal{H})$ be positive and let $f, g \in \mathcal{C}_+$ such that $f(x)g(x) = x$, for all $x \geq 0$. Then the following assertions are equivalent.*

(i) $\mathcal{R}(T)$ is closed,

(ii) $\mathcal{R}(f(T))$ is closed,

(iii) $\mathcal{R}(g(T))$ is closed.

In any case, $\mathcal{R}(T) = \mathcal{R}(f(T)) = \mathcal{R}(g(T))$.

In order to prove Proposition 2.15, we need to recall the reduced minimum modulus that measures the closedness of the range of an operator.

Lemma 2.16. [3] *Let $T \in \mathcal{B}(\mathcal{H})$. Then the reduced minimum modulus of T is defined by:*

$$\gamma(T) := \begin{cases} \inf\{\|Tx\|; \|x\| = 1,\, x \in \mathcal{N}(T)^{\perp}\} & if\ T \neq 0 \\ +\infty & if\ T = 0. \end{cases}$$

Thus, $\gamma(T) > 0$ if and only if T has a closed range .

Proof. (Proposition 2.15)
$(i) \Rightarrow (ii)$. Assume that $\mathcal{R}(T)$ is closed and $\mathcal{R}(f(T))$ is not closed. By Lemma 2.16, $\gamma(f(T)) = 0$. So, there exists a sequence of unit vectors $x_n \in \mathcal{N}(f(T))^{\perp}$ such that $f(T)x_n \longrightarrow 0$. From Lemma 2.2, $x_n \in \mathcal{N}(T)^{\perp}$ and $Tx_n = g(T)f(T)x_n \longrightarrow 0$. This contradict the fact that $\mathcal{R}(T)$ is closed.
$(ii) \Rightarrow (i)$. Suppose that $\mathcal{R}(f(T))$ is closed and $\mathcal{R}(T)$ is not closed. Thus, $\gamma(T) = 0$. So, we can choose a sequence of unit vectors $x_n \in \mathcal{N}(T)^{\perp}$ such that $Tx_n \to 0$. Which means that $f(T)g(T)x_n \to 0$. By using again Lemma 2.2, $x_n \in \mathcal{N}(T)^{\perp} = \mathcal{N}(g(T))^{\perp}$. So, there exists $\alpha > 0$ such that $\|g(T)x_n\| \geq \alpha$ for all n. We put $y_n = \dfrac{g(T)x_n}{\|g(T)x_n\|}$. Then clearly, $\|y_n\| = 1$ and $f(T)y_n \to 0$. Moreover

$$y_n \in \mathcal{R}(g(T)) \subset \overline{\mathcal{R}(g(T))} = \mathcal{N}(g(T))^{\perp} = \mathcal{N}(f(T))^{\perp},$$

For all n. This contradicts the fact that $\mathcal{R}(f(T))$ is closed.
With similar arguments we prove the equivalence $(i) \iff (iii)$.

□

Theorem 2.17. *Let $T \in \mathcal{B}(\mathcal{H})$ and let $f, g \in \mathcal{C}_+$ such that $f(x)g(x) = x$, for all $x \geq 0$. If $\mathcal{N}(T) \subseteq \mathcal{N}(T^*)$, then*

$$\mathcal{R}(T) \ \ \textit{is closed} \quad \Longleftrightarrow \quad \mathcal{R}(\Delta_{f,g}(T)) \ \ \textit{is closed}.$$

Proof. First, recall that the closedness of any one of the following sets implies the closedness of the remaining three sets:

$$\mathcal{R}(T), \mathcal{R}(T^*), \mathcal{R}(|T|) \text{ and } \mathcal{R}(|T^*|).$$

If $\mathcal{R}(T)$ is closed, then $\mathcal{R}(T) = \mathcal{R}(|T^*|)$ and $\mathcal{R}(T^*) = \mathcal{R}(|T|)$.

$\Longrightarrow$. By taking the orthogonal complements in the relation $\mathcal{N}(T) \subseteq \mathcal{N}(T^*)$ and since $\mathcal{R}(T)$ is closed, we get that $\mathcal{R}(T) \subseteq \mathcal{R}(T^*)$. This implies that $P_{\mathcal{R}(T^*)}P_{\mathcal{R}(T)} = P_{\mathcal{R}(T)}$. Therefore, we have

$$\begin{aligned}
\mathcal{R}(T) \ \text{ is closed} \quad &\Longrightarrow \quad \mathcal{R}(P_{\mathcal{R}(T^*)}P_{\mathcal{R}(T)}) \ \text{ is closed} \\
&\Longrightarrow \quad \mathcal{R}(P_{\mathcal{R}(T)}P_{\mathcal{R}(T^*)}) \ \text{ is closed} \\
&\Longrightarrow \quad P_{\mathcal{R}(T)}\mathcal{R}(P_{\mathcal{R}(T^*)}) \ \text{ is closed} \\
&\Longrightarrow \quad P_{\mathcal{R}(T)}\mathcal{R}(|T|) \ \text{ is closed} \\
&\Longrightarrow \quad P_{\mathcal{R}(T)}\mathcal{R}(f(|T|)) \ \text{ is closed} \quad \text{ by Proposition 2.15} \\
&\Longrightarrow \quad \mathcal{R}(f(|T|)P_{\mathcal{R}(T)}) \ \text{ is closed} \\
&\Longrightarrow \quad f(|T|)\mathcal{R}(P_{\mathcal{R}(T)}) \ \text{ is closed} \\
&\Longrightarrow \quad f(|T|)\mathcal{R}(|T^*|) \ \text{ is closed} \\
&\Longrightarrow \quad f(|T|)\mathcal{R}(g(|T^*|f(|T^*|)) \ \text{ is closed} \\
&\Longrightarrow \quad f(|T|)g(|T^*|)\mathcal{R}(f(|T^*|)) \ \text{ is closed} \\
&\Longrightarrow \quad f(|T|)g(|T^*|)\mathcal{R}(|T^*|) \ \text{ is closed} \\
&\Longrightarrow \quad f(|T|)g(|T^*|)\mathcal{R}(U) \ \text{ is closed} \\
&\Longrightarrow \quad f(|T|)\mathcal{R}(g(|T^*|)U) \ \text{ is closed} \\
&\Longrightarrow \quad f(|T|)\mathcal{R}(Ug(|T|)) \ \text{ is closed} \quad \text{ by Lemma 2.1}, (i) \\
&\Longrightarrow \quad f(|T|)\mathcal{R}(Ug(|T|)) \ \text{ is closed} \\
&\Longrightarrow \quad \mathcal{R}(\Delta_{f,g}(T)) \ \text{ is closed} .
\end{aligned}$$

$\Longleftarrow$ Suppose that $\mathcal{R}(\Delta_{f,g}(T))$ is closed and $\mathcal{R}(T)$ is not closed. Then $\mathcal{R}(|T|)$ is not closed. It follows from Proposition 2.15 that $\mathcal{R}(g(T))$ is nonclosed and so there exists a sequence of unit vectors $x_n \in \mathcal{N}(g(|T|))^{\perp}$ such that $g(|T|)x_n \to 0$. This implies that $\Delta_{f,g}(T)x_n = f(|T|)Ug(|T|)x_n \to 0$. Now, we show that $x_n \in \mathcal{N}(\Delta_{f,g}(T))^{\perp}$, for all n. It is enough to prove that $\mathcal{N}(\Delta_{f,g}(T)) \subset \mathcal{N}(g(|T|))$. Let $x \in \mathcal{N}(\Delta_{f,g}(T))$. Then $f(|T|)Ug(|T|)x = 0$, which means that

$$Ug(|T|)x \in \mathcal{N}(f(|T|)) \cap \mathcal{R}(U) = \mathcal{N}(|T|) \cap \overline{\mathcal{R}(T)} \subset \mathcal{N}(T^*) \cap \overline{\mathcal{R}(T)} = \{0\}.$$

So $Ug(|T|)x = 0$. By using $P(1)$ and Lemma 2.2, we deduce that $g(|T|)x = U^*Ug(|T|)x = 0$. Hence, $x \in \mathcal{N}(g(|T|))$. Finally, each $x_n \in \mathcal{N}(\Delta_{f,g}(T))^{\perp}$ and $\Delta_{f,g}(T)x_n \to 0$, which is a contradiction with the fact that $\mathcal{R}(\Delta_{f,g}(T))$ is closed. □

3. On the polar decomposition of the (f,g)-Aluthge transform

We show below that under some conditions the (f,g)-Aluthge transform possesses the polar decomposition. The proof of Theorem 3.1, in the particular case $f(x) = g(x) = x^{\frac{1}{2}}$, $(x \geqslant 0)$ can be found in [4].

Theorem 3.1. *Let $T = U|T|$ be the polar decomposition of $T \in \mathcal{B}(\mathcal{H})$ and $f, g \in \mathcal{C}_+$ such that $f(x)g(x) = x$, for all $x \geq 0$ and let $f(|T|)g(|T^*|) = V|f(|T|)g(|T^*|)|$ be the polar decomposition too. Then*

$$\Delta_{f,g}(T) = VU|\Delta_{f,g}(T)|$$

is also the polar decomposition of $\Delta_{f,g}(T)$.

Proof. (i) Firstly, we show that $\Delta_{f,g}(T) = VU|\Delta_{f,g}(T)|$. By Lemma 2.1 (ii), we easily obtain

$$(Ug(|T|)Sg(|T|)U^*)^n = U(g(|T|)Sg(|T|))^n U^*,$$

for any positive operator $S \in \mathcal{B}(\mathcal{H})$ and all $n \in \mathbb{N}^*$. Which implies

$$P(Ug(|T|)Sg(|T|)U^*) = UP(g(|T|)Sg(|T|))U^*,$$

for any polynomial $P(t)$ with no constant term. Since $K(t) = t^{\alpha}$, $(\alpha > 0)$ is a continuous function in $[0, \infty[$, so there exist a sequence of polynomial $(P_n)_{n\in\mathbb{N}^*}$ such that $P_n(0) = 0$, for each $n \in \mathbb{N}^*$, and $(P_n(t))_{n\in\mathbb{N}^*}$ converges uniformly to $K(t)$ on the interval $[0, \||T|\|]$. Hence,

$$\begin{aligned} K(Ug(|T|)Sg(|T|)U^*) &= \lim_{n\to+\infty} P_n(Ug(|T|)Sg(|T|)U^*) \\ &= \lim_{n\to+\infty} UP_n(g(|T|)Sg(|T|))U^* \\ &= UK(g(|T|)Sg(|T|))U^*. \end{aligned}$$

So,

$$(Ug(|T|)Sg(|T|)U^*)^{\alpha} = U(g(|T|)Sg(|T|))^{\alpha}U^*, \tag{3.1}$$

for any positive operator $S \in \mathcal{B}(\mathcal{H})$ and all $\alpha > 0$. It follows that

$$\begin{aligned}
VU|\Delta_{f,g}(T)| &= VUU^*U(g(|T|)U^*f(|T|)f(|T|)Ug(|T|))^{\frac{1}{2}} \\
&= VU(g(|T|)U^*f(|T|)f(|T|)Ug(|T|))^{\frac{1}{2}}U^*U \\
&= V(Ug(|T|)U^*f(|T|)f(|T|)Ug(|T|)U^*)^{\frac{1}{2}}U \quad \text{by } (3.1) \\
&= V(g(|T^*|)f(|T|)f(|T|)g(|T^*|))^{\frac{1}{2}}U \quad \text{by Lemma 2.1 } (v) \\
&= V|f(|T|)g(|T^*|)|U \\
&= f(|T|)g(|T^*|)U \\
&= f(|T|)Ug(|T|) \quad \text{by Lemma 2.1}(i) \\
&= \Delta_{f,g}(T).
\end{aligned}$$

(ii) Secondly, we will show that $\mathcal{N}(\Delta_{f,g}(T)) = \mathcal{N}(VU)$. For $x \in \mathcal{B}(\mathcal{H})$, we have

$$\begin{aligned}
VUx = 0 &\Leftrightarrow f(|T|)g(|T^*|)Ux = 0 \quad \text{since } \mathcal{N}(V) = \mathcal{N}(f(|T|)g(|T^*|)) \\
&\Leftrightarrow f(|T|)Ug(|T|)x = 0 \quad \text{by Lemma 2.1 } (i) \\
&\Leftrightarrow \Delta_{f,g}(T)x = 0.
\end{aligned}$$

Therefore, $\mathcal{N}(VU) = \mathcal{N}(\Delta_{f,g}(T))$.

(iii) Finally, we shall prove that VU is a partial isometry. By (ii), we get that

$$\mathcal{N}(VU)^{\perp} = \mathcal{N}(|\Delta_{f,g}(T)|)^{\perp} = \overline{\mathcal{R}(|\Delta_{f,g}(T)|)}.$$

So, for every $x \in \overline{\mathcal{R}(|\Delta_{f,g}(T)|)}$, there exists a sequence $(y_n)_n \subset \mathcal{H}$ such that $x = \lim\limits_{n\to+\infty} |\Delta_{f,g}(T)|y_n$. Hence, we have

$$\begin{aligned}
\|VUx\| &= \|VU \lim_{n\to\infty} |\Delta_{f,g}(T)|y_n\| \\
&= \|\lim_{n\to\infty} VU|\Delta_{f,g}(T)|y_n\| \\
&= \|\lim_{n\to\infty} \Delta_{f,g}(T)y_n\| \quad \text{by } (i) \\
&= \lim_{n\to\infty} \|\Delta_{f,g}(T)y_n\| \\
&= \lim_{n\to\infty} \||\Delta_{f,g}(T)|y_n\| \\
&= \|\lim_{n\to\infty} |\Delta_{f,g}(T)|y_n\| \\
&= \|x\|,
\end{aligned}$$

that is VU is a partial isometry.

□

The following is a new characterization of binormal operators which is an extension of Theorem 3.1 in [4].

Theorem 3.2. *Let $T = U|T|$ be the polar decomposition of $T \in \mathcal{B}(\mathcal{H})$ and let $f, g \in \mathcal{C}_+$ be two increasing functions such that $f(x)g(x) = x$, for all $x \geq 0$. Then*

$$T \text{ is binormal} \iff \Delta_{f,g}(T) = U|\Delta_{f,g}(T)|.$$

Proof.

($\Longrightarrow$). This implication is true without the increasing condition of f and g. Suppose that T is binormal. This means that $|T||T^*| = |T^*||T|$. Since $f, g \in \mathcal{C}_+$, by the continuous functional calculus we have $f(|T|)g(|T^*|) = g(|T^*|)f(|T|)$. It follows that $f(|T|)g(|T^*|) \geqslant 0$ and so $f(|T|)g(|T^*|) = |f(|T|)g(|T^*|)|$. From this equality and Lemma 2.1, we get

$$\begin{aligned}
\Delta_{f,g}(T) &= f(|T|)Ug(|T|) \\
&= f(|T|)g(|T^*|)U \\
&= |f(|T|)g(|T^*|)|U \\
&= UU^*(g(|T^*|)f(|T|)f(|T|)g(|T^*|))^{\frac{1}{2}}U \\
&= U(U^*g(|T^*|)f(|T|)f(|T|)g(|T^*|)U)^{\frac{1}{2}} \\
&= U((g(|T^*|)U)^*f(|T|)f(|T|)g(|T^*|)U)^{\frac{1}{2}} \\
&= U((Ug(|T|))^*f(|T|)f(|T|)Ug(|T|))^{\frac{1}{2}} \\
&= U(g(|T|)U^*f(|T|)f(|T|)Ug(|T|))^{\frac{1}{2}} \\
&= U((\Delta_{f,g}(T))^*\Delta_{f,g}(T))^{\frac{1}{2}} \\
&= U|\Delta_{f,g}(T)|.
\end{aligned}$$

($\Longleftarrow$). Assume that $\Delta_{f,g}(T) = U|\Delta_{f,g}(T)|$. Then we have

$$f(|T|)g(|T^*|) = f(|T|)Ug(|T|)U^* = \Delta_{f,g}(T)U^* = U|\Delta_{f,g}(T)|U^*,$$

and

$$g(|T^*|)f(|T|) = Ug(|T|)U^*f(|T|) = U(\Delta_{f,g}(T))^* = U(U|\Delta_{f,g}(T)|)^* = U|\Delta_{f,g}(T)|U^*.$$

Hence, $f(|T|)g(|T^*|) = g(|T^*|)f(|T|)$. Since f and g are increasing, they have inverses. So by the continuous functional calculus, we get

$$f^{-1}f(|T|)g^{-1}g(|T^*|) = g^{-1}g(|T^*|)f^{-1}f(|T|).$$

Therefore, $|T||T^*| = |T^*||T|$ and so T is binormal.

□

The binormality of a bounded operator on Hilbert spaces does not imply the binormality of its (f,g)-Aluthge transform. As shown in [4, Example 3.4], for $f(t) = g(t) = t^{\frac{1}{2}}$, $(t \geq 0)$. Recently in [12], we showed that if T is a binormal operator such that the partial isometry factor U of its polar decomposition is unitary and satisfies $U^2|T| = |T|U^2$, then $\Delta_\lambda(T)$ is binormal, for any $\lambda \in]0,1[$. In our final result, we will show the binormality of $\Delta_{f,g}(T)$ under the same conditions.

Proposition 3.3. *let $f, g \in \mathcal{C}_+$ such that $f(x)g(x) = x \ (x \geq 0)$ and let $T = U|T|$ be the polar decomposition of a binormal operator $T \in \mathcal{B}(\mathcal{H})$. If in addition U is unitary and $U^2|T| = |T|U^2$, then $\Delta_{f,g}(T)$ is binormal.*

Proof. From the hypothesis $U^2|T| = |T|U^2$ and using the continuous functional calculus, we obtain $U^2 f(|T|) = f(|T|)U^2$, for $f \in \mathcal{C}_+$. This implies $Uf(|T^*|)U = f(|T|)U^2$, by Lemma 2.1 part (i). Multiplying this equality by U^* on the right side and since U is unitary, we get

$$Uf(|T^*|) = f(|T|)U, \tag{3.2}$$

and by taking the adjoint, we get also

$$f(|T^*|)U^* = U^* f(|T|). \tag{3.3}$$

Therefore, we have

$$\begin{aligned}
|\Delta_{f,g}(T)^*|^2|\Delta_{f,g}(T)|^2 &= f(|T|)Ug(|T|)g(|T|)U^*|T|U^*f(|T|)f(|T|)Ug(|T|) \\
&= f(|T|)g(|T^*|)g(|T^*|)UU^*|T|U^*f(|T|)f(|T|)Ug(|T|) \quad \text{by Lemma 2.1}(i) \\
&= f(|T|)[g(|T^*|)]^2|T|U^*f(|T|)f(|T|)Ug(|T|) \\
&= f(|T|)[g(|T^*|)]^2|T|(f(|T|)U)^*f(|T|)Ug(|T|) \\
&= f(|T|)[g(|T^*|)]^2|T|f(|T^*|)U^*f(|T|)Ug(|T|) \quad \text{by (3.2)} \\
&= f(|T|)[g(|T^*|)]^2|T|f(|T^*|)f(|T^*|)U^*Ug(|T|) \quad \text{by (3.3)} \\
&= f(|T|)[g(|T^*|)]^2|T|f(|T^*|)^2g(|T|).
\end{aligned} \tag{3.4}$$

On the other hand, since T is binormal, i.e. $|T||T^*| = |T^*||T|$, By using again the continuous functional calculus, we have

$$|T|f(|T^*|) = f(|T^*|)|T| \text{ and } |T^*|g(|T|) = g(|T|)|T^*|. \tag{3.5}$$

Then, by (3.4) and (3.5), we obtain

$$|\Delta_{f,g}(T)^*|^2|\Delta_{f,g}(T)|^2 = |T|^2|T^*|^2$$

With the same calculus, we have

$$\begin{aligned}
|\Delta_{f,g}(T)|^2|\Delta_{f,g}(T)^*|^2 &= g(|T|)U^*f(|T|)f(|T|)U|T|Ug(|T|)g(|T|)U^*f(|T|) \\
&= g(|T|)U^*f(|T|)f(|T|)U|T|g(|T^*|)Ug(|T|)U^*f(|T|) \\
&= g(|T|)U^*f(|T|)f(|T|)U|T|g(|T^*|)g(|T^*|)UU^*f(|T|) \ \text{ by Lemma 2.1}(i) \\
&= g(|T|)U^*f(|T|)f(|T|)U|T|[g(|T^*|)]^2f(|T|) \\
&= g(|T|)(f(|T|)U)^*f(|T|)U|T|[g(|T^*|)]^2f(|T|) \\
&= g(|T|)f(|T^*|)U^*f(|T|)U|T|[g(|T^*|)]^2f(|T|) \ \text{ by } (3.2) \\
&= g(|T|)f(|T^*|)f(|T^*|)U^*U|T|[g(|T^*|)]^2f(|T|) \ \text{ by } (3.3) \\
&= g(|T|)[f(|T^*|)]^2|T|[g(|T^*|)]^2f(|T|) \\
&= |T|^2|T^*|^2 \ \text{ by } (3.5).
\end{aligned}$$

And finally we deduce that $\Delta_{f,g}(T)$ is binormal. □

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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