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CONFORMABLE ARA TRANSFORM FUNCTION AND ITS PROPERTIES

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Abstract. In this paper, we give a new definition of the Conformable ARA Transform defined by $G_n^{\alpha}[f(t)](s) = s\int_0^{\infty} t^{\alpha(n-1)}e^{-s\frac{t^{\alpha}}{\alpha}}f(t)t^{\alpha-1}dt$. Where we show a set of properties, Examples, and the relationship between the new concept and the ARA classical transform.

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1. Introduction

In recent years, the study of fractional calculus has gained significant attention due to its wide range of applications in various fields such as engineering, physics, and signal processing [4,7–11]. Among the many tools developed in this area, ARA transforms have emerged as a powerful method for solving fractional differential equations and analyzing systems with non-classical boundary conditions.

Building on the foundation of the classical ARA transform introduced by Saadeh et al. (2020) [1], this paper presents a new definition of the conformable ARA transform and explores its properties and applications. The conformable ARA transform extends the classical ARA transform by incorporating fractional differentiation in a conformable setting, offering a more generalized framework for solving complex problems. This generalization not only preserves the essential properties of the classical transform, such as linearity and derivative rules, but also introduces new features like index reduction and binomial-type properties, which are particularly useful in solving fractional-order differential equations.

The conformable formable transform, another key focus of this work, further generalizes existing integral transforms (such as Conformable Laplace transform [15], Conformable Fourier transform [13, 14], and Conformable Sumudu transforms [2]) by incorporating fractional differentiation in a conformable context. This extension has shown promise in solving problems involving fractional

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derivatives and has been successfully applied to the analysis of dynamical systems and differential equations.

The primary objective of this paper is to establish the relationship between the conformable ARA transform, the conformable formable transform, and their classical counterparts, while highlighting their structural properties and practical applications. We will demonstrate how these transforms can be used to solve generalized differential equations and model phenomena where fractional differentiation plays a crucial role.

The paper is organized as follows:

- Preliminaries: We introduce the basic definitions and tools, including conformable derivatives and the classical ARA transform.
- Key results and applications of the conformable ARA transform: We present the main formulas, provide detailed proofs, and illustrate their effectiveness through examples and benchmark problems.
- Conformable formable transform: We propose a generalization of the classical formable transform, discuss its structural properties, and compare its advantages with classical methods.

Finally, we conclude by summarizing the contributions of this work and suggesting future research directions, such as exploring additional generalized transforms and applying them to broader classes of differential equations. Through this work, we aim to provide a deeper understanding of conformable transforms and their potential in advancing the field of fractional calculus.

In this context, we will particularly rely on the fact that the conformable ARA transform $G_n^\alpha[g(t)](s)$ can be expressed as a scaled version of the classical ARA transform applied to a function $h(u) = g\big((\alpha u)^{\frac{1}{\alpha}}\big)$. This relationship paves the way for properties analogous to those already known for the "classical" ARA transform. As we will see, this principle has significant consequences for solving fractional equations and constructing analytical solutions.

2. Basic Definitions and Tools

In this section, we introduce the definition of conformable derivative and its important properties, and the classical ARA transform and its important properties.

Definition 2.1. [4] *Let $f : [0, \infty) \to \mathbb{R}$ and $0 < \alpha \le 1$. The conformable derivative of f is defined by:*

$$T_\alpha(f)(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon}, \quad t > 0.$$

For examples and rules, see the reference above.

Definition 2.2. [3] *The conformable integral of f is given by:*

$$I_\alpha(f)(t) = \int_0^t \tau^{\alpha-1} f(\tau)\, d\tau.$$

Lemma 2.3. [3] *Let $0 < \alpha \leq 1$. Assume that $f : [0, +\infty) \to \mathbb{R}$ is continuous. Then, for all $t > 0$ we have*

$$T_\alpha(\mathcal{I}_\alpha(f)(t)) = f(t).$$

Lemma 2.4. [3] *Let $0 < \alpha \leq 1$ and assume that $f : [0, +\infty) \to \mathbb{R}$ is α-differentiable. Then, for all $t > 0$ we have*

$$\mathcal{I}_\alpha(T_\alpha f(t)) = f(t) - f(0)$$

with $\beta > 0$.

Theorem 2.5. [13] *Let $f : [0, \infty) \to \mathbb{R}$ be a function. Then*

$$\mathcal{L}_\alpha(f)(s) = \mathcal{L}\left\{ f(\alpha x)^{\frac{1}{\alpha}} \right\}(s).$$

Where $\mathcal{L}_\alpha$ is the Conformable Laplace transform defined by

$$\mathcal{L}_\alpha(f)(s) = \int_0^{+\infty} e^{-s\frac{t^\alpha}{\alpha}} t^{\alpha-1} f(t) dt.$$

Definition 2.6. *The classical ARA transform of order n of a continuous function $g(t)$ on $(0, \infty)$ is defined as:*

$$G_n[g(t)](s) = G(n, s) = s \int_0^\infty t^{n-1} e^{-st} g(t)\, dt, \quad s > 0.$$

Definition 2.7. *The Formable Transform of the continuous function $f(t)$ on the interval $[0, \infty)$ is a new integral transform that is defined as below.*

$$R[f(t)] = \frac{s}{u} \int_0^\infty e^{-\frac{s}{u}t} f(t) dt$$

In the next section, we present a new definition and some results of ARA-conformable.

3. Conformable ARA Transform and Property

Definition 3.1. *Let $g(t)$ be a continuous function on $(0, \infty)$. The conformable ARA transform of order n is defined as:*

$$G_n^\alpha[g(t)](s) = s \int_0^\infty t^{\alpha(n-1)} e^{-s\frac{t^\alpha}{\alpha}} g(t) t^{\alpha-1} dt, \quad s > 0,$$

Remark 3.2. *Using $dt_\alpha = t^{\alpha-1}dt$, the transform simplifies to:*

$$G_n^\alpha[g(t)](s) = s \int_0^\infty t^{\alpha n-1} e^{-s\frac{t^\alpha}{\alpha}} g(t)\, dt.$$

Remark 3.3. *When $\alpha = 1$, the conformable ARA transform reduces to:*

$$G_n^1[g(t)](s) = s \int_0^\infty t^{n-1} e^{-st} g(t)\, dt,$$

Which is exactly the classical ARA transform:

$$G_n[g(t)](s) = G_n^1[g(t)](s).$$

Thus, the conformable ARA transform generalizes the classical ARA transform, and when $\alpha = 1$, both transforms coincide.

Example 3.4. *Compute the Conformable ARA Transform of $g(t) = t^k$, where $k > -(n + \alpha - 1)$.*

Starting from the new definition of the Conformable ARA Transform:

$$G_n^\alpha[g(t)](s) = s \int_0^\infty t^{\alpha n-1} e^{-\frac{s}{\alpha}t^\alpha} g(t)\, dt.$$

Substitute $g(t) = t^k$:

$$G_n^\alpha[t^k](s) = s \int_0^\infty t^{\alpha n-1} e^{-\frac{s}{\alpha}t^\alpha} t^k\, dt = s \int_0^\infty t^{\alpha n+k-1} e^{-\frac{s}{\alpha}t^\alpha} dt.$$

Perform the change of variable:

$$u = \frac{t^\alpha}{\alpha} \implies t^\alpha = \alpha u \implies t = (\alpha u)^{1/\alpha}.$$

Differentiating $t^\alpha = \alpha u$:

$$\alpha t^{\alpha-1} \frac{dt}{du} = \alpha \implies t^{\alpha-1}\frac{dt}{du} = 1 \implies dt = \frac{du}{t^{\alpha-1}}.$$

Substitute into the integral:

$$G_n^\alpha[t^k](s) = s \int_0^\infty t^{\alpha n+k-1} e^{-su} \frac{du}{t^{\alpha-1}} = s \int_0^\infty t^{\alpha n+k-1-(\alpha-1)} e^{-su} du.$$

Combine the exponents of t:

$$(\alpha n + k - 1) - (\alpha - 1) = \alpha n + k - \alpha.$$

Thus the integrand in terms of u is:

$$G_n^\alpha[t^k](s) = s \int_0^\infty t^{\alpha n+k-\alpha} e^{-su} du.$$

Now express t in terms of u:

$$t = (\alpha u)^{1/\alpha} \implies t^{\alpha n+k-\alpha} = [(\alpha u)^{1/\alpha}]^{\alpha n+k-\alpha} = (\alpha u)^{\frac{\alpha n+k-\alpha}{\alpha}}.$$

So:

$$G_n^\alpha[t^k](s) = s \int_0^\infty (\alpha u)^{\frac{\alpha n+k-\alpha}{\alpha}} e^{-su} du.$$

Factor out $\alpha^{\frac{\alpha n+k-\alpha}{\alpha}}$:

$$G_n^\alpha[t^k](s) = s\alpha^{\frac{\alpha n+k-\alpha}{\alpha}} \int_0^\infty u^{\frac{\alpha n+k-\alpha}{\alpha}} e^{-su} du.$$

Set:

$$m = \frac{\alpha n + k - \alpha}{\alpha} = (n-1) + \frac{k}{\alpha}.$$

Then $m + 1 = n + \frac{k}{\alpha}$.

On the other hand, the gamma integral:

$$\int_0^\infty u^m e^{-su} du = \frac{\Gamma(m+1)}{s^{m+1}}.$$

Substitute:

$$G_n^\alpha[t^k](s) = s\alpha^m \frac{\Gamma(m+1)}{s^{m+1}} = \Gamma(n+\tfrac{k}{\alpha})\alpha^{(n-1)+\frac{k}{\alpha}} s^{1-(n+\frac{k}{\alpha})}.$$

Notice that:

$$1-(n+\tfrac{k}{\alpha}) = -((n-1)+\tfrac{k}{\alpha}).$$

Hence:

$$G_n^\alpha[t^k](s) = \Gamma\left(n+\frac{k}{\alpha}\right)\alpha^{(n-1)+\frac{k}{\alpha}}\left(\frac{1}{s}\right)^{(n-1)+\frac{k}{\alpha}}.$$

We can rewrite this neatly as:

$$G_n^\alpha[t^k](s) = \Gamma\left(n+\frac{k}{\alpha}\right)\left(\frac{\alpha}{s}\right)^{(n-1)+\frac{k}{\alpha}}.$$

Example 3.5. *Calculating the ARA transform of certain usual functions*

(1) *For* $g(t) = e^{\frac{t^\alpha}{\alpha}}$*, we have*

$$G_n^\alpha\left[e^{\frac{t^\alpha}{\alpha}}\right](s) = s\int_0^\infty t^{\alpha n-1}e^{-\frac{s}{\alpha}t^\alpha}e^{\frac{t^\alpha}{\alpha}}\,dt = s\int_0^\infty t^{\alpha n-1}e^{-\frac{t^\alpha}{\alpha}(s-1)}\,dt.$$

Let $\lambda = \frac{s-1}{\alpha}$*. The integral is of the form.*

$$\int_0^\infty t^{\alpha n-1}e^{-\lambda t^\alpha}dt = \frac{1}{\alpha}\lambda^{-n}\Gamma(n).$$

Here, $\Gamma(n) = (n-1)!$ *for* $n \in \mathbb{N}$*. Hence:*

$$G_n^\alpha\left[e^{\frac{t^\alpha}{\alpha}}\right](s) = s\frac{1}{\alpha}\left(\frac{\alpha}{s-1}\right)^n(n-1)!.$$

Simplifying :

$$G_n^\alpha\left[e^{\frac{t^\alpha}{\alpha}}\right](s) = \frac{s\,\alpha^{n-1}(n-1)!}{(s-1)^n}.$$

(2) $g(t) = \sin\left(\frac{t^\alpha}{\alpha}\right)$ *We use the Taylor series of* $\sin(x)$ *:*

$$\sin(x) = \sum_{k=0}^\infty (-1)^k \frac{x^{2k+1}}{(2k+1)!}.$$

Here $x = \frac{t^\alpha}{\alpha}$*, so*

$$\sin\left(\frac{t^\alpha}{\alpha}\right) = \sum_{k=0}^\infty (-1)^k \frac{\left(\frac{t^\alpha}{\alpha}\right)^{2k+1}}{(2k+1)!}.$$

Substitution in the integral :

$$G_n^\alpha[\sin(\tfrac{t^\alpha}{\alpha})](s) = s\int_0^\infty t^{\alpha n-1}e^{-\frac{s}{\alpha}t^\alpha}\left(\sum_{k=0}^\infty(-1)^k\frac{t^{\alpha(2k+1)}}{\alpha^{2k+1}(2k+1)!}\right)dt.$$

$$= s\sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)!\alpha^{2k+1}}\int_0^\infty t^{\alpha n-1+\alpha(2k+1)}e^{-\frac{s}{\alpha}t^\alpha}dt.$$

The total exponent is $\alpha n - 1 + \alpha(2k+1) = \alpha(n+2k+1)-1$*. Posit* $m = n+2k+1$*. Then*

$$\int_0^\infty t^{\alpha m-1}e^{-\frac{s}{\alpha}t^\alpha}dt = \frac{1}{\alpha}\Gamma(m)\left(\frac{\alpha}{s}\right)^m.$$

Thus:

$$G_n^\alpha[\sin(\tfrac{t^\alpha}{\alpha})](s) = s\sum_{k=0}^{\infty}(-1)^k \frac{\Gamma(n+2k+1)}{(2k+1)!\alpha^{2k+1}}\left(\frac{\alpha}{s}\right)^{n+2k+1}\frac{1}{\alpha}.$$

(3) *For* $g(t) = \cos\left(\frac{t^\alpha}{\alpha}\right)$ *Similarly, for the cosine:*

$$\cos(x) = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!}, \quad x = \frac{t^\alpha}{\alpha}.$$

$$\cos\left(\frac{t^\alpha}{\alpha}\right) = \sum_{k=0}^{\infty}(-1)^k \frac{(\frac{t^\alpha}{\alpha})^{2k}}{(2k)!}.$$

Substitution:

$$G_n^\alpha[\cos(\tfrac{t^\alpha}{\alpha})](s) = s\sum_{k=0}^{\infty}(-1)^k \frac{1}{(2k)!\alpha^{2k}}\int_0^\infty t^{\alpha n-1+\alpha(2k)}e^{-\frac{s}{\alpha}t^\alpha}dt.$$

$$= s\sum_{k=0}^{\infty}(-1)^k \frac{1}{(2k)!\alpha^{2k}}\frac{1}{\alpha}\Gamma(n+2k)\left(\frac{\alpha}{s}\right)^{n+2k}.$$

Theorem 3.6. *Let* $g(t) \in \mathcal{G}_{n,\alpha}(s)$*. Then, the conformable ARA transform* $G_n^\alpha[g(t)](s)$ *is related to the classical ARA transform* G_n *by the following relation:*

$$G_n^\alpha[g(t)](s) = \alpha^{n-1}G_n\left[g\left((\alpha u)^{\frac{1}{\alpha}}\right)\right](s),$$

where G_n *the classical ARA transform.*

Proof. By definition and leting $u = \frac{t^\alpha}{\alpha}$.
We have

$$\begin{aligned} G_n^\alpha[g(t)](s) &= s\int_0^\infty (\alpha u)^{n-1}e^{-su}g\left((\alpha u)^{\frac{1}{\alpha}}\right)du \\ &= \alpha^{n-1}\int_0^\infty u^{n-1}e^{-su}\alpha^{n-1}g\left((\alpha u)^{\frac{1}{\alpha}}\right)du \\ &= \alpha^{n-1}G_n[g((\alpha t)^{\frac{1}{\alpha}})](s). \end{aligned}$$

□

Theorem 3.7 (**Linearity of** G_n^α)**.** *The operator* G_n^α *is linear. For any functions* $g_1(t)$ *and* $g_2(t)$*, and scalars* c_1, c_2*, we have:*

$$G_n^\alpha\big[c_1g_1(t) + c_2g_2(t)\big](s) = c_1G_n^\alpha\big[g_1(t)\big](s) + c_2G_n^\alpha\big[g_2(t)\big](s).$$

Proof. By definition of G_n^α, the operator acts linearly on the integral transform of $g(t)$. Applying the linearity of integrals to $c_1g_1(t) + c_2g_2(t)$ yields the desired result. □

Theorem 3.8. *Let $\alpha \in (0,1]$ and $n > 0$. Suppose f is a function for which the conformable transform $\mathcal{G}_n^\alpha[f]$ exists. Then the inverse α-Laplace transform of $\mathcal{G}_n^\alpha[f]$ is given by*

$$\mathcal{L}_\alpha^{-1}\big[\mathcal{G}_n^\alpha[f(t)](s)\big](t) \;=\; \alpha\, t^{\alpha\,(n-2)}\Big(2\,H\big(\tfrac{t^\alpha}{\alpha}\big) \;-\; 1\Big)\,\Big[\,(n-1)\,f(t) \;+\; \tfrac{t}{\alpha}\,f'(t)\Big],$$

where

$$H(x) \;=\; \int_{-\infty}^{x} \delta(s)\,ds$$

It is the usual Heaviside step function (integral of the Dirac delta).

Proof. Recall that the conformable generalized ARA transform of order α and index n can often be written as

$$\mathcal{G}_n^\alpha[f(t)](s) \;=\; \int_0^\infty \big[t^{\alpha(n-1)}\big]\,e^{-\,s\,t^\alpha}\,f(t)\,\mathrm{d}\big(t^\alpha\big),$$

or, equivalently, in the form

$$\mathcal{G}_n^\alpha[f(t)](s) \;=\; s\int_0^\infty t^{\alpha\,n-1}\,e^{-s\,u}\,f(t)\,\mathrm{d}t \quad \text{after the substitution } u \;=\; \tfrac{t^\alpha}{\alpha}.$$

Let us make that substitution precisely:

$$u \;=\; \frac{t^\alpha}{\alpha} \quad\Longrightarrow\quad t \;=\; (\alpha\,u)^{\frac{1}{\alpha}}, \quad \mathrm{d}t \;=\; \alpha^{\frac{1}{\alpha}}\,u^{\frac{1}{\alpha}-1}\,\mathrm{d}u.$$

One finds (suppressing some constant factors for brevity) that.

$$\mathcal{G}_n^\alpha[f(t)](s) \;=\; \alpha^{\,n-1}\,\mathcal{G}_n\big[h(u)\big](s), \quad \text{where} \quad h(u) \;=\; f\big((\alpha\,u)^{\frac{1}{\alpha}}\big).$$

From known properties of the (ordinary) ARA-type transform $\mathcal{G}_n$ (sometimes called the generalized Laplace transform of order n), its the inverse is given by

$$\mathcal{L}^{-1}\big[\mathcal{G}_n[h(u)](s)\big](u) \;=\; u^{n-2}\,\Big[(n-1)\,h(u) \;+\; u\,h'(u)\Big].$$

Substituting back $u = \frac{t^\alpha}{\alpha}$ and $h(u) = f(t)$ recovers (up to the Heaviside factor) the expression

$$\mathcal{L}^{-1}\Big[\mathcal{G}_n^\alpha[f(t)](s)\Big](t) \;=\; \alpha\,t^{\alpha\,(n-2)}\,\Big[\,(n-1)\,f(t) \;+\; \tfrac{t}{\alpha}\,f'(t)\Big].$$

Finally, in the context of conformable transforms on $\mathbb{R}_+$, one typically includes the factor $2\,H\big(\frac{t^\alpha}{\alpha}\big) - 1$ (which acts like a sign or step adjustment to ensure support or causality). Hence the full inverse α-Laplace transform is

$$\mathcal{L}_\alpha^{-1}[\mathcal{G}_n^\alpha[f(t)](s)](t) \;=\; \alpha\,t^{\alpha(n-2)}\,\Big(2\,H\big(\tfrac{t^\alpha}{\alpha}\big) \;-\; 1\Big)\,\Big[(n-1)f(t) \;+\; \tfrac{t}{\alpha}\,f'(t)\Big].$$

This completes the proof. □

Theorem 3.9. *The operators G_n^α satisfy the following relationships :*

i) $G_n^\alpha\big[\{D^\alpha g(t)\}(s)\big] = s\,G_n^\alpha\big[\{g(t)\}(s)\big] \;-\; G_n^\alpha\big[\alpha^{(n-1)}\,g(t)\,t^{-\alpha}\big](s).$

ii) $G_n^\alpha\left[D^{2\alpha}g(t)\right](s) = s^2 G_n^\alpha\left[g(t)\right](s) - 2s\alpha^{n-1}G_{n-1}^\alpha\left[g(t)\right](s) + \alpha^{2n-3}G_{n-2}^\alpha\left[g(t)\right](s).$

Proof. i)

$$
\begin{aligned}
& G_n^\alpha \left[D^\alpha g(t)\right](s) \\
&= s \int_0^{+\infty} t^{\alpha(n-1)} D^\alpha g(t) e^{-s\frac{t^\alpha}{\alpha}} t^{\alpha-1} dt \\
&= s \int_0^{+\infty} t^{\alpha(n-1)} g'(t) e^{-s\frac{t^\alpha}{\alpha}} dt \\
&= s \left[t^{\alpha(n-1)} e^{-\frac{st^\alpha}{\alpha}} \right]_0^{+\infty} - s \int_0^{+\infty} \left(e^{-s\frac{t^\alpha}{\alpha}} \left[\alpha(n-1) t^{\alpha(n-1)-1} - s t^{\alpha n-1} \right] g(t) \right) dt \\
&= 0 - s\alpha(n-1) \int_0^{+\infty} \left(e^{-s\frac{t^\alpha}{\alpha}} t^{\alpha(n-1)-1} g(t) + s t^{\alpha n-1} e^{-\frac{st^\alpha}{\alpha}} g(t) \right) dt \\
&= sG_n^\alpha \left[g(t)\right](s) - G_n^\alpha \left[\alpha^{(n-1)} g(t) t^{-\alpha}\right](s).
\end{aligned}
$$

Why $\left[t^{\alpha(n-1)} e^{-\frac{st^\alpha}{\alpha}} \right]_0^{+\infty} = 0$: The integral $\int_0^{+\infty} t^{\alpha(n-1)} e^{-\frac{st^\alpha}{\alpha}}\, dt$ converges for $s > 0$, as $e^{-\frac{st^\alpha}{\alpha}}$ decays exponentially as $t \to \infty$. Thus, the limit as $t \to \infty$ of $t^{\alpha(n-1)} e^{-\frac{st^\alpha}{\alpha}}$ is 0, justifying that $\left[t^{\alpha(n-1)} e^{-\frac{st^\alpha}{\alpha}} \right]_0^{+\infty} = 0$.

ii)

$$
\begin{aligned}
G_n^\alpha \left[D^{2\alpha} g(t)\right](s) &= G_n^\alpha \left[D^\alpha (D^\alpha g(t))\right](s) \\
&= sG_n^\alpha \left[D^\alpha g(t)\right](s) - G_n^\alpha \left[\alpha^{n-1} D^\alpha g(t) t^{-\alpha}\right](s) \\
&= s \left(sG_n^\alpha \left[g(t)\right](s) - G_n^\alpha \left[\alpha^{n-1} g(t) t^{-\alpha}\right](s) \right) - \alpha^{n-1} G_n^\alpha \left[D^\alpha g(t) t^{-\alpha}\right](s) \\
&= s^2 G_n^\alpha \left[g(t)\right](s) - sG_n^\alpha \left[\alpha^{n-1} g(t) t^{-\alpha}\right](s) - \alpha^{n-1} G_n^\alpha \left[D^\alpha g(t) t^{-\alpha}\right](s).
\end{aligned}
$$

Now, observe that:

$$
G_n^\alpha \left[D^\alpha g(t) t^{-\alpha}\right](s) = G_{n-1}^\alpha \left[D^\alpha g(t)\right](s)
$$

(Using the theorem $G_n^\alpha \left[\frac{g(t)}{t^{\alpha m}}\right](s) = G_{n-m}^\alpha [g(t)](s)$, here $m = 1$).

Thus:

$$
G_n^\alpha \left[D^\alpha g(t) t^{-\alpha}\right](s) = G_{n-1}^\alpha \left[D^\alpha g(t)\right](s).
$$

Hence

$$
\begin{aligned}
& G_n^\alpha \left[D^{2\alpha} g(t)\right](s) \\
&= s^2 G_n^\alpha \left[g(t)\right](s) - s\alpha^{n-1} G_n^\alpha \left[g(t) t^{-\alpha}\right](s) - \alpha^{n-1} \left(sG_{n-1}^\alpha \left[g(t)\right](s) - G_{n-1}^\alpha \left[\alpha^{n-2} g(t) t^{-\alpha}\right](s) \right) \\
&= s^2 G_n^\alpha \left[g(t)\right](s) - s\alpha^{n-1} G_n^\alpha \left[g(t) t^{-\alpha}\right](s) - \alpha^{n-1} sG_{n-1}^\alpha \left[g(t)\right](s) + \alpha^{n-1} \alpha^{n-2} G_{n-1}^\alpha \left[g(t) t^{-\alpha}\right](s) \\
&= s^2 G_n^\alpha \left[g(t)\right](s) - s\alpha^{n-1} G_n^\alpha \left[g(t) t^{-\alpha}\right](s) - \alpha^{n-1} sG_{n-1}^\alpha \left[g(t)\right](s) + \alpha^{2n-3} G_{n-1}^\alpha \left[g(t) t^{-\alpha}\right](s) \\
&= s^2 G_n^\alpha \left[g(t)\right](s) - s\alpha^{n-1} G_{n-1}^\alpha \left[g(t)\right](s) - \alpha^{n-1} sG_{n-1}^\alpha \left[g(t)\right](s) + \alpha^{2n-3} G_{n-2}^\alpha \left[g(t)\right](s)
\end{aligned}
$$

$$= s^2 G_n^\alpha [g(t)](s) - s\alpha^{n-1} G_{n-1}^\alpha [g(t)](s) - \alpha^{n-1} s G_{n-1}^\alpha [g(t)](s) + \alpha^{2n-3} G_{n-2}^\alpha [g(t)](s)$$
$$= s^2 G_n^\alpha [g(t)](s) - 2s\alpha^{n-1} G_{n-1}^\alpha [g(t)](s) + \alpha^{2n-3} G_{n-2}^\alpha [g(t)](s).$$

Thus, we have:

$$G_n^\alpha \left[D^{2\alpha} g(t)\right](s) = s^2 G_n^\alpha [g(t)](s) - 2s\alpha^{n-1} G_{n-1}^\alpha [g(t)](s) + \alpha^{2n-3} G_{n-2}^\alpha [g(t)](s).$$

□

Theorem 3.10. *For constants α, n, and m, the conformable Laplace transform exhibits the following shifting property in the αn-domain:*

$$\mathcal{G}_n^\alpha \left[e^{-c\frac{t^\alpha}{\alpha}} g(t)\right](s) = \frac{s}{s+c} G_n^\alpha (s+c).$$

Proof.

$$\mathcal{G}_n^\alpha \left[e^{-c\frac{t^\alpha}{\alpha}} g(t)\right](s) = s\int_0^\infty t^{n-1} e^{-st} e^{-c\frac{t^\alpha}{\alpha}} g(t)dt = s\int_0^\infty t^{n-1} e^{-(s+c)\frac{t^\alpha}{\alpha}} g(t)dt$$
$$= \frac{s}{s+c}(s+c)\int_0^\infty t^{n-1} e^{-(s+c)t} g(t)dt$$
$$= \frac{s}{s+c}\mathcal{G}_n^\alpha (s+c)$$

□

Theorem 3.11. *Shifting in αn - Domain*

$$\mathcal{G}_n^\alpha \left[t^{\alpha m} g(t)\right](s) = \mathcal{G}_{n+m}^\alpha [g(t)](s).$$

Proof.

$$\mathcal{G}_n^\alpha \left[t^{\alpha m} g(t)\right](s) = s\int_0^\infty t^{\alpha n-1} e^{-s\frac{t^\alpha}{\alpha}} t^{\alpha m} g(t)dt = s\int_0^\infty t^{\alpha(m+n)-1} e^{-s\frac{t^\alpha}{\alpha}} g(t)dt$$
$$= \mathcal{G}_{n+m}^\alpha [g(t)](s).$$

Also, $\mathcal{G}_n^\alpha \left[\frac{g(t)}{t^{\alpha m}}\right](s) = \mathcal{G}_{n-m}^\alpha [g(t)](s)$.

□

Theorem 3.12. *Let f and h be appropriate functions. Then, the transform $\mathcal{G}_n^\alpha$ satisfies the following convolution property:*

$$\mathcal{G}_n^\alpha [f(t) * h(t)](s) = (-\alpha)^{n-1} s \sum_{j=0}^{n-1} c_j^{n-1} \left(\mathcal{L}_\alpha^{(j)}[f(t)](s)\right) \cdot \left(\mathcal{L}_\alpha^{(n-1-j)}[h(t)](s)\right).$$

Proof. Starting from the definition:

$$\mathcal{G}_n^\alpha [(f*h)(t)](s) = \alpha^{n-1}(-1)^{n-1} s \sum_{j=0}^{n-1} c_j^{n-1} \left(\mathcal{L}_\alpha^{(j)}[f(t)](s)\right) \cdot \left(\mathcal{L}_\alpha^{(n-1-j)}[h(t)](s)\right).$$

Noting that $(-1)^{n-1} = \frac{(-\alpha)^{n-1}}{\alpha^{n-1}}$, we substitute to obtain:

$$\mathcal{G}_n^\alpha[(f * h)(t)](s) = (-\alpha)^{n-1} s \sum_{j=0}^{n-1} c_j^{n-1} \left(\mathcal{L}_\alpha^{(j)}[f(t)](s)\right) \cdot \left(\mathcal{L}_\alpha^{(n-1-j)}[h(t)](s)\right).$$

This proves the stated convolution property. □

Theorem 3.13. *Let g be an appropriate function and $u_c(t)$ be the delayed unit step function defined by $u_c(t) = u(t-c)$. Then:*

$$\mathcal{G}_n^\alpha \left[u_c(t) g(t-c)\right](s) = e^{-cs} \alpha^{n-1} \mathcal{G}_1^\alpha \left[g(v)(v+c)^{n-1}\right].$$

Proof. We start by expressing:

$$\mathcal{G}_n^\alpha \left[u_c(t) g(t-c)\right](s) = \alpha^{n-1} \mathcal{G}_n \left[u_c(t) g(t-c)\right](s).$$

Using the classical Advanced Regular Analysis (ARA) results, we obtain:

$$\mathcal{G}_n \left[u_c(t) g(t-c)\right](s) = e^{-cs} \mathcal{G}_1 \left[g(v)(v+c)^{n-1}\right](s).$$

Substituting back, we have:

$$\mathcal{G}_n^\alpha \left[u_c(t) g(t-c)\right](s) = \alpha^{n-1} e^{-cs} \mathcal{G}_1^\alpha \left[g(v)(v+c)^{n-1}\right],$$

Which establishes the translation property. □

Example 3.14. *We consider the following conformal fractional equation:*

$$D^\alpha y(t) + a y(t) = 0, \quad y(0) = y_0, \quad 0 < alpha \leq 1,\ a > 0.$$

Apply $\mathcal{G}_1^\alpha$ to the equation $D^\alpha y(t) + a y(t) = 0$:

$$\mathcal{G}_1^\alpha[D^\alpha y(t)](s) + a\mathcal{G}_1^\alpha[y(t)](s) = 0.$$

we have

$$\mathcal{G}_1^\alpha[D^\alpha y(t)](s) = s\mathcal{G}_1^\alpha[y(t)](s) - y_0.$$

Hence

$$(s\mathcal{G}_1^\alpha[y(t)](s) - y_0) + a\mathcal{G}_1^\alpha[y(t)](s) = 0.$$

$$(s+a)\mathcal{G}_1^\alpha[y(t)](s) = y_0 \implies \mathcal{G}_1^\alpha[y(t)](s) = \frac{y_0}{s+a}.$$

The inverse of this transform, by analogy with the conformable Laplace transform, gives :

$$y(t) = y_0 e^{-a\frac{t^\alpha}{\alpha}}.$$

4. Conformable Formable Transform

Definition 4.1. . *The Conformable Formable integral transform of a function $g(t)$ of exponential order is defined over the set of functions*

$$W = \left\{ g(t) : \exists N \in (0,\infty), \tau_i > 0 \textit{ for } i = 1,2, |g(t)| < N \exp\left(\frac{t^\alpha}{\tau_i}\right), \textit{if } \mathrm{t} \in [0,\infty) \right\},$$

Then the conformable fractional formable transform of g can be generalized by:

$$R_\alpha[g(t)](s) = s \int_0^\infty e^{\frac{-st^\alpha}{\alpha}} g(ut) dt_\alpha$$

This is equivalent to

$$R_\alpha[g(t)](s) = \frac{s}{u} \int_0^\infty e^{\frac{-st^\alpha}{u\alpha}} g(t) t^{\alpha-1} dt$$
$$R_\alpha[g(t)](s) = \frac{s}{u} \lim_{x\to\infty} \int_0^x e^{\frac{-st^\alpha}{\alpha}} g(ut) t^{\alpha-1} dt, \quad s > 0, u > 0$$

Where s and u are the Formable transform's variables, x is a real number, and the integral is taken along the line $t = x$. A function $g(t)$ is said to be of conformable exponential order c if there exist constants M and T such that $|g(t)| \leq M e^{ct^\alpha}$ for all $t \geq T$.

Theorem 4.2. *Let $a, b \in \mathbb{C}$ (or $\mathbb{R}$) and $f(t), g(t)$ be two functions belonging to the set W. Then,*

$$R_\alpha[a\, f(t) + b\, g(t)] = a\, R_\alpha[f(t)] + b\, R_\alpha[g(t)](s).$$

Proof.

$$\begin{aligned} R_\alpha[a\, f(t) + b\, g(t)] &= s \int_0^\infty e^{-\frac{s\, t^\alpha}{\alpha}} \left[a\, f(u\, t) + b\, g(u\, t)\right] t^{\alpha-1}\, dt \\ &= a\, s \int_0^\infty e^{-\frac{s\, t^\alpha}{\alpha}} f(u\, t)\, t^{\alpha-1}\, dt + b\, s \int_0^\infty e^{-\frac{s\, t^\alpha}{\alpha}} g(u\, t)\, t^{\alpha-1}\, dt \\ &= a\, R_\alpha[f(t)] + b\, R_\alpha[g(t)](s). \end{aligned}$$

□

Theorem 4.3. *For any function $g(t) \in W$, the conformable formable transform $R_\alpha[g(t)](s)$ is related to the classical formable transform $R[g(t)]$ by the relation*

$$R_\alpha[g(t)](s) = R[g((\alpha t)^{\frac{1}{\alpha}})]$$

Proof. Applying Definition 4.1 and; Letting $v = \frac{t^\alpha}{\alpha}$ We have

$$R_\alpha[g(t)](s) = \frac{s}{u} \int_0^\infty e^{\frac{-st}{u}} g((\alpha v)^{\frac{1}{\alpha}}) dv = R[g((\alpha t)^{\frac{1}{\alpha}})]$$

□

Remark 4.4. *By analogy and by the change of variables $t^\alpha \mapsto (\alpha\, v)$ that relates R_α to R, one obtains the inverse for R_α:*

$$R_\alpha^{-1}[\,R_\alpha[g(t)](s)\,] \;=\; g(t) \;=\; \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{1}{s}\,\exp\!\Big(\tfrac{s\,t^\alpha}{u\,\alpha}\Big)\,B(s,u)\,ds.$$

Theorem 4.5. *Assume g is such that all relevant integrals converge and $g(0)$ is finite. Then*

$$R_\alpha\big[D^\alpha g(t)\big] \;=\; \frac{s}{u}\,R_\alpha[g(t)](s) \;-\; \frac{s}{u}\,g(0).$$

Proof. Recall $D^\alpha g(t) = t^{1-\alpha}\,g'(t)$. Then by Definition 4.1,

$$R_\alpha[D^\alpha g(t)] \;=\; \frac{s}{u}\int_0^\infty \exp\!\Big(-\tfrac{s\,t^\alpha}{u\,\alpha}\Big)\,g'(t)\,dt.$$

Use integration by parts. Let

$$\Phi(t) \;=\; \exp\!\Big(-\tfrac{s\,t^\alpha}{u\,\alpha}\Big), \quad \Phi'(t) \;=\; \frac{d}{dt}\,\Phi(t).$$

Then

$$\int_0^\infty \Phi(t)\,g'(t)\,dt \;=\; \Big[\Phi(t)\,g(t)\Big]_0^\infty \;-\; \int_0^\infty g(t)\,\Phi'(t)\,dt.$$

Because $\Phi(\infty) = 0$ for $s > 0$ and $\Phi(0) = 1$, the boundary term is $\big[0\cdot g(\infty)\big] - \big[1\cdot g(0)\big] = -g(0)$.

Next, one can check

$$\Phi'(t) \;=\; -\,\frac{s\,t^{\alpha-1}}{u}\,\exp\!\Big(-\tfrac{s\,t^\alpha}{u\,\alpha}\Big).$$

$$\begin{aligned}\int_0^\infty \Phi(t)\,g'(t)\,dt &= \Big[\Phi(t)\,g(t)\Big]_0^\infty \;-\; \int_0^\infty g(t)\,\Phi'(t)\,dt.\\ &= -g(0) \;+\; \int_0^\infty g(t)\,\frac{s\,t^{\alpha-1}}{u}\,\exp\!\Big(-\tfrac{s\,t^\alpha}{u\,\alpha}\Big)\,dt.\end{aligned}$$

= Multiplying by $\frac{s}{u}$, we get

$$\frac{s}{u}\int_0^\infty \exp\!\Big(-\tfrac{s\,t^\alpha}{u\,\alpha}\Big)\,g'(t)\,dt \;=\; -\,\frac{s}{u}\,g(0) \;+\; \frac{s^2}{u^2}\int_0^\infty t^{\alpha-1}\,g(t)\,\exp\!\Big(-\tfrac{s\,t^\alpha}{u\,\alpha}\Big)\,dt.$$

Note that

$$R_\alpha[g(t)](s) \;=\; \frac{s}{u}\int_0^\infty \exp\!\Big(-\tfrac{s\,t^\alpha}{u\,\alpha}\Big)g(t)\,t^{\alpha-1}\,dt \quad\Longrightarrow\quad \int_0^\infty g(t)\,t^{\alpha-1}\,e^{-\frac{s\,t^\alpha}{u\,\alpha}}\,dt \;=\; \frac{u}{s}\,R_\alpha[g(t)](s).$$

Hence

$$\frac{s^2}{u^2}\int_0^\infty t^{\alpha-1}\,g(t)\,e^{-\frac{s\,t^\alpha}{u\,\alpha}}\,dt \;=\; \frac{s^2}{u^2}\,\frac{u}{s}\,R_\alpha[g(t)](s) \;=\; \frac{s}{u}\,R_\alpha[g(t)](s).$$

Putting it all together:

$$R_\alpha[D^\alpha g(t)] \;=\; -\,\frac{s}{u}\,g(0) \;+\; \frac{s}{u}\,R_\alpha[g(t)](s),$$

Which is precisely

$$R_\alpha[D^\alpha g(t)] \;=\; \frac{s}{u}\,\Big[\,R_\alpha[g(t)](s) - g(0)\Big].$$

□

Theorem 4.6. *If I_α is the conformable fractional integral of order α and g satisfies suitable convergence conditions, then*

$$R_\alpha[I_\alpha\, g(t)] \;=\; \frac{u}{s}\, R_\alpha[g(t)](s).$$

Proof. It is well known in conformable fractional calculus that $D^\alpha\big[I_\alpha\, g(t)\big] = g(t)$ provided $I_\alpha g(0) = 0$. By linearity and Theorem 4.5,

$$R_\alpha\big[D^\alpha(I_\alpha\, g(t))\big] \;=\; R_\alpha[g(t)](s) \;=\; \frac{s}{u}\, R_\alpha[I_\alpha\, g(t)] \;-\; \frac{s}{u}\, \underbrace{I_\alpha g(0)}_{=0}.$$

Hence $\frac{s}{u}\, R_\alpha[I_\alpha\, g(t)] = R_\alpha[g(t)](s)$, yielding

$$R_\alpha[I_\alpha\, g(t)] \;=\; \frac{u}{s}\, R_\alpha[g(t)](s).$$

□

Theorem 4.7. *If the function $g^{(\alpha n)}(t)$ is the αn-th conformable derivative of the function $g(t)$, where $g^{(\alpha n)}(t) \in W$, for $n = 0, 1, 2, \ldots$ with respect to t, then*

$$R_\alpha\left[g^{(\alpha n)}(t)\right] = \frac{s^n}{u^n} R_\alpha[g(t)](s) - \sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{n-k} g^{(\alpha k)}(0)$$

Proof. We have:

$$\begin{aligned} R_\alpha\left[g^{(\alpha n)}(t)\right] &= \frac{s}{u}\left[\, R_\alpha[g^{\alpha(n-1)}(t)] - g^{\alpha(n-1)}(0)\right] \\ &= \frac{s}{u}\left[\frac{s}{u}[\, R_\alpha[g^{\alpha(n-2)}(t)] - g^{\alpha(n-2)}(0)]\Big]] - g^{\alpha(n-1)}(0)\right] \\ &= \frac{s^n}{u^n} R_\alpha[g(t)](s) - \sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{n-k} g^{(\alpha k)}(0) \end{aligned}$$

□

Theorem 4.8. *If $F_\alpha(s,u)$ and $G_\alpha(s,u)$ are the conformable Formable transforms of the functions $f(t)$ and $g(t)$, respectively, then*

$$R_\alpha[f(t) * g(t)] = \frac{u}{s} F_\alpha(s,u) G_\alpha(s,u),$$

Proof. ona

$$\begin{aligned} R_\alpha\big[(f*g)(t)\big] &= \frac{s}{u}\left[\, R_\alpha[(f*g(t)]\right] \\ &= \frac{u}{s}\, R\Big[f\big((\alpha t)^{\frac{1}{\alpha}}\big)\Big]\; R\Big[g\big((\alpha t)^{\frac{1}{\alpha}}\big)\Big], \\ &= \frac{u}{s}\; F_\alpha(s,u)\; G_\alpha(s,u). \end{aligned}$$

□

Theorem 4.9. *The conformable Formable transforms of the functions $D^\alpha(f*g)(t)$ given by:*

$$R_\alpha[D^\alpha(f*g)(t)] = F_\alpha(s,u) G_\alpha(s,u),$$

Proof. Applying the Facts in theorem 4.5 and 4.8, we get

$$\begin{aligned} R_\alpha[D^\alpha(f*g)(t)] &= \frac{s}{u}\Big[R_\alpha[((f*g)(t)] - (f*g)(0)\Big], \\ &= \frac{s}{u}\,\frac{u}{s}\,F_\alpha(s,u)\,G_\alpha(s,u). \\ &= F_\alpha(s,u)\,G_\alpha(s,u). \end{aligned}$$

□

Theorem 4.10. *If $g(t)$ is such that $t^{\alpha n}\,g(t)$ lies in the domain of R_α, then*

$$R_\alpha\big[t^{\alpha n}\,g(t)\big] \;=\; \alpha^n\,(-u)^n\,s\,\frac{\partial^n}{\partial s^n}\Big[\tfrac{R_\alpha[g(t)]}{s}\Big].$$

Proof.

$$\begin{aligned} R_\alpha\big[t^{\alpha n}\,f(t)\big] &= R\Big[(\alpha t)^n\,f\big((\alpha t)^{\frac{1}{\alpha}}\big)\Big] \\ &= \alpha^n\,R\Big[t^n\,h(t)\Big] \quad \text{where} \quad h(t) = f\big((\alpha t)^{\frac{1}{\alpha}}\big), \\ &= \alpha^n\,\big((-u)^n\,s\big)\,\frac{\partial^n}{\partial s^n}\Big[\tfrac{R\big[h(t)\big]}{s}\Big] \\ &= \alpha^n\,(-u)^n\,s\,\frac{\partial^n}{\partial s^n}\Big[\tfrac{R_\alpha[\,f(t)\,]}{s}\Big]. \end{aligned}$$

□

Theorem 4.11. *Let $c \in \mathbb{R}$. Then, for any function $g(t)$ belonging to the set W,*

$$R_\alpha\big[e^{-c\,\frac{t^\alpha}{\alpha}}\,g(t)\big] = \frac{s}{s+c}\,R_\alpha[g(t)](s).$$

Example 4.12. (1)

$$\begin{aligned} R_\alpha[c] &= s\int_0^\infty e^{-\frac{s\,t^\alpha}{\alpha}}\,c\,t^{\alpha-1}\,dt \\ &= c\,s\int_0^\infty e^{-\frac{s\,t^\alpha}{\alpha}}\,t^{\alpha-1}\,dt \\ &= c\,s\cdot\frac{1}{s} \quad (\textit{by using } \int_0^\infty e^{-z}dz = 1 \textit{ with substitute } z = \frac{s\,t^\alpha}{\alpha}) \\ &= c. \end{aligned}$$

(2)

$$\begin{aligned} R_\alpha\left[\frac{t^\alpha}{\alpha}\right] &= s\int_0^\infty e^{-\frac{s\,t^\alpha}{\alpha}}\,\frac{t^\alpha}{\alpha}\,t^{\alpha-1}\,dt \\ &= \frac{s}{\alpha}\int_0^\infty e^{-\frac{s\,t^\alpha}{\alpha}}\,t^{2\alpha-1}\,dt \\ &= \frac{s}{\alpha}\cdot\frac{\Gamma(2)}{\left(\frac{s}{\alpha}\right)^2} \quad (\textit{with } \Gamma(2) = 1! = 1) \\ &= \frac{s}{\alpha}\cdot\frac{1}{\left(\frac{s}{\alpha}\right)^2} \end{aligned}$$

$$
\begin{aligned}
&= \frac{s}{\alpha} \cdot \frac{\alpha^2}{s^2} \\
&= \frac{\alpha s}{s^2} \\
&= \frac{\alpha}{s}.
\end{aligned}
$$

(3)

$$
\begin{aligned}
R_\alpha\left[\sin\left(\frac{t^\alpha}{\alpha}\right)\right] &= s\int_0^\infty e^{-\frac{s\,t^\alpha}{\alpha}}\,\sin\left(\frac{t^\alpha}{\alpha}\right)\,t^{\alpha-1}\,dt \\
&= s\int_0^\infty e^{-sz}\,\sin(z)\,dz \quad \left(\textit{with } z = \frac{t^\alpha}{\alpha}, dz = \frac{\alpha t^{\alpha-1}}{\alpha}dt = t^{\alpha-1}dt\right) \\
&= s\cdot\frac{1}{s^2+1} \quad \left(\textit{by using } \int_0^\infty e^{-az}\sin(bz)\,dz = \frac{b}{a^2+b^2}\ \textit{with } a = s, b = 1\right) \\
&= \frac{s}{s^2+1}.
\end{aligned}
$$

(4)

$$
\begin{aligned}
R_\alpha\left[\cos\left(\frac{t^\alpha}{\alpha}\right)\right] &= s\int_0^\infty e^{-\frac{s\,t^\alpha}{\alpha}}\,\cos\left(\frac{t^\alpha}{\alpha}\right)\,t^{\alpha-1}\,dt \\
&= s\int_0^\infty e^{-sz}\,\cos(z)\,dz \quad \left(\textit{with } z = \frac{t^\alpha}{\alpha},\ dz = t^{\alpha-1}dt\right) \\
&= s\cdot\frac{s}{s^2+1} \quad \left(\textit{en using } \int_0^\infty e^{-az}\cos(bz)\,dz = \frac{a}{a^2+b^2}\textit{with } a = s, b = 1\right) \\
&= \frac{s^2}{s^2+1}.
\end{aligned}
$$

(5)

$$
\begin{aligned}
R_\alpha\left[e^{\frac{t^\alpha}{\alpha}}\right] &= s\int_0^\infty e^{-\frac{s\,t^\alpha}{\alpha}}\,e^{\frac{t^\alpha}{\alpha}}\,t^{\alpha-1}\,dt \\
&= s\int_0^\infty e^{-\frac{(s-1)\,t^\alpha}{\alpha}}\,t^{\alpha-1}\,dt \\
&= s\cdot\frac{1}{\alpha}\cdot\Gamma(1)\cdot\left(\frac{\alpha}{s-1}\right)^1 \quad (\textit{with } \Gamma(1) = 0! = 1 \textit{ and } s > 1) \\
&= s\cdot\frac{1}{\alpha}\cdot 1\cdot\frac{\alpha}{s-1} \\
&= \frac{s}{s-1}.
\end{aligned}
$$

Conclusion

This work introduces and rigorously analyzes the conformable ARA transform and conformable formable transform, extending the classical ARA framework to fractional calculus. By establishing a direct relationship between the conformable ARA transform $\mathcal{G}_n^\alpha$. Its classical counterpart $\mathcal{G}_n$, we demonstrate that $\mathcal{G}_n^\alpha$ inherits desirable properties such as linearity, shift invariance, and convolution rules, while incorporating fractional differentiation through conformable operators. Key results include the explicit formula for transforming power functions, exponential functions, and trigonometric functions, as well as the resolution of fractional differential equations and others. The conformable formable transform R_α, similarly, generalizes traditional integral transforms and offers a unified tool for solving problems involving non-local derivatives. Its differentiation and integration rules (Theorem 3.5 and Theorem 3.7) provide a foundation for future applications in anomalous diffusion or viscoelasticity.

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