ESTIMATE ON LOGARITHMIC COEFFICIENTS OF NEW SUBCLASSES OF ANALYTIC FUNCTION LINKED WITH TAN HYPERBOLIC FUNCTIONS

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Abstract. Function theory research has long struggled with the challenge of deriving sharp estimates for the coefficients of analytic and univalent functions. Researchers have advanced the field by developing and applying a variety of approaches to get these bounds. In the current paper, we apply the technique of subordination. The logarithmic coefficients play an important role for different estimates in the theory of univalent functions. Due to the significance of the recent studies about the logarithmic coefficients, the problem of obtaining the sharp bounds for the modulus of these coefficients has received attention. If S denotes the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ analytic and univalent in the open unit disk $\mathbb{D} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$, then the logarithmic coefficients γ_n of the function $f \in \mathcal{S}$ are defined by $\log\left(\frac{f(z)}{z}\right) = 2\sum_{n=1}^{\infty} \gamma_n z^n, \quad (z \in \mathbb{D})$. In this paper, due to the significant importance of the study of logarithmic coefficients involving tan hyperbolic function, we find the upper bounds for some expressions associated with the logarithmic coefficients of functions that belong to some classes defined by subordination. Consequently, by employing the hyperbolic tangent function tanhz, a subfamily $\mathbb{TS}^*(\beta)$ of starlike functions is introduced and investigated and coefficient functionals can be scrutinized. This study will tackle several coefficient problems by applying the methodology to the aforementioned family of functions. We explore the bounds of certain initial coefficients, including the Fekete-Szego inequality and other results concerning logarithmic coefficients for functions within this class.

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1. Introduction

The logarithmic functions find application in various branches of mathematics and other scientific disciplines. In order to provide a comprehensive grasp of the principal outcomes detailed in this paper,

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we elucidate the fundamental terminology employed throughout our key findings, accompanied by preliminary definitions and relevant results.

We denote by $\mathcal{A}$ the class of analytic, holomorphic normalized functions $f : \mathbb{D} \longrightarrow \mathbb{C}$ defined in an open unit disk $\mathbb{D} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$, which satisfy the following normalization conditions

$$f(0) = 0 = f'(0) - 1$$

Thus, for each $f \in \mathcal{A}$ we have

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \mathbb{D}). \tag{1.1}$$

Moreover, we denote by $\mathbb{S}$ the subclass of $\mathcal{A}$ of functions which are univalent in $\mathbb{D}$. For two functions $g_1, g_2 \in \mathcal{A}$, we say that the function g_1 is subordinate to the function g_2 (written as $g_1 \prec g_2$) if there exists an analytic function ω with the property

$$|\omega(z)| \leq 1$$

and

$$\omega(0) = 0$$

such that

$$g_1(z) = g_2(\omega(z)) \qquad (z \in \mathbb{D}) \tag{1.2}$$

In particular, if g_2 is univalent in $\mathbb{S}$, then we have the following equivalence

$$g_1(z) \prec g_2(z) \iff g_1(0) = g_2(0) \text{ and } g_1(|z| < 1) \subset g_2(|z| < 1). \tag{1.3}$$

In 1992, Ma and Minda [1] introduced the $\mathcal{S}^*(\phi)$ as follows

$$\mathcal{S}^*(\phi) = \left\{ f \in \mathcal{A} : \left(\frac{zf'(z)}{f(z)} \right) \prec \phi(z), \quad (z \in \mathbb{D}) \right\}. \tag{1.4}$$

The function $\phi(z)$ is expected to be analytic within a region $\mathbb{D}$, where its real part is positive. In simple terms, $\mathcal{S}^*(\phi)$ is imagined to have a symmetric shape like a star, but it's confined within a certain area $\phi(0) = 1$ and $\phi'(0) > 0$.

Additionally, they explored several beneficial geometric characteristics like expansion, deformation, and coverage outcomes. This was achieved by implementing

$$\phi(z) = (1 + z)(1 - z)^{-1}$$

Specifically, we observe that the class of function $\mathbb{S}^*(\phi)$ resembles the well-established class of starlike functions. Depending on the particular function $\phi(z)$ chosen, we encounter the following function classes which are distinct.

1. We adopt the class of functions
$$\phi(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^2,$$
then we secure the class
$$\mathcal{S}^*_{Card} = \mathcal{S}^*\left(1 + \frac{4}{3}z + \frac{2}{3}z^2\right),$$
which is the class of starlike functions whose image under open unit is cardioid shaped given by $(9x^2 + 9y^2 - 18x + 5)^2 - 16(9x^2 + 9y^2 - 6x + 1) = 0$. and this was introduced by Sharma et al.(see [2]).

2. We put the class of functions
$$\phi(z) = 1 + z - \frac{1}{3}z^3,$$
then we get the class
$$\mathbb{S}^*_{nep} = \mathbb{S}^*\left(1 + z - \frac{1}{3}z^3\right),$$
the class of starlike functions exhibits a unique feature when visualized within the open unit disk $\mathbb{D}$. It forms an nephroid - shaped region. This distinct shape becomes apparent when examining their representation under the unit disk (see [3]).

3. If we opt the class of functions
$$\phi(z) = \sqrt{1 + z},$$
then we acquire the class
$$\mathbb{S}^*_{\mathcal{L}} = \mathcal{S}^*\left(\sqrt{1 + z}\right),$$
The function $\phi(z) = \sqrt{1 + z}$ transforms the domain $\mathbb{D}$ onto the image domain bounded by the right half of the Bernoulli lemniscate represented by $|w^2 - 1|$ (see [4]).

4. If we opt the class of functions
$$\phi(z) = e^z,$$
then we derive the class of function
$$\mathcal{S}^*_{Exp} = \mathcal{S}^*\left(e^z\right).$$

This is the class of starlike function associated with exponential function and this was introduced and studied by Mendiratta et al.(see [5]).

5. If we pick the class of functions

$$\phi(z) = (\sqrt{1+z}) + z,$$

then we procure the class

$$\mathcal{S}^*_{Cre} = \mathcal{S}^*\left((\sqrt{1+z}) + z\right),$$

which is the class of starlike functions associated with the crescent - shaped region as discussed in (see [6]).

6. If we let opt for

$$\phi(z) = 1 + \frac{4}{5}z + \frac{1}{5}z^4,$$

then we obtain the class

$$\mathcal{S}^*_{Three\ leaf} = \mathcal{S}^*\left(1 + \frac{4}{5}z + \frac{1}{5}z^4\right),$$

now, we have identified a group of starlike functions linked to the defined geometric area called three leaf shaped domain and studied in (see [7]).

7. If we assign the class of functions

$$\phi(z) = 1 + \frac{5}{6}z + \frac{1}{6}z^5,$$

then we attain the class

$$\mathcal{S}^*_{Four\ leaf} = \mathcal{S}^*\left(1 + \frac{5}{6}z + \frac{1}{6}z^5\right),$$

this is possess the category of starlike functions linked with outlined four - shaped region which was introduced and studied in (see [8]).

8. If we assign the class of function

$$\phi(z) = 1 + \sinh^{-1}(z),$$

then the class of function leads to the class

$$\mathcal{S}^*_{Petal} = \mathcal{S}^*\left(1 + \sinh^{-1}(z)\right),$$

which is the class of starlike functions associated with the petal - shaped region as discussed in (see [9]).

9. Moreover, if we take
$$\phi(z) = \cosh(z),$$
then we derive the class
$$\mathcal{S}^*_{cosh} = \mathcal{S}^*\left(\cosh(z)\right),$$
whose image is bounded by the cosine of the function which were contributed by A. Alotaibi, M. Arif, M. A. Alghamdi, and S. Hussain (see [10]) .

10. If we let
$$\phi(z) = 1 + \sin(z),$$
then we obtain the class
$$\mathbb{S}^*_{sin} = \mathbb{S}^*\left(1 + \sin(z)\right),$$
the class of starlike functions maps to an eight-shaped figure within the open unit disk $\mathbb{D}$. This distinctive shape emerges when considering their image under the unit disk (see [11]).

11. If we put the class of functions
$$\delta(z) = \cos(z),$$
then we obtain the class
$$\mathcal{S}^*_{cos} = \mathcal{S}^*\left(\cos(z)\right).$$
This is the class of starlike functions associated with the Cosine function as discussed in (see [12]).

12. If we assign the class of functions
$$\phi(z) = \frac{1 + (1 - 2\varphi)z}{1 - z}$$
with $0 \le \varphi < 1$, we acquire the class
$$\mathcal{S}^* = \mathcal{S}^*\left(\frac{1 + (1 - 2\varphi)z}{1 - z}\right)$$
of starlike functions of order φ (see [13]).

The two most important and extensively studied families of univalent functions are the class $\mathcal{S}^*(\phi(z))$ which represents star-like functions with respect to symmetric points of order $\phi(z)$, $(0 \le \phi < 1)$ analytically defined by
$$\mathbb{S}^*(\phi) = \left\{ f \in \mathcal{A} : \text{Re}\left(\frac{2zf'(z)}{f(z) - f(-z)}\right) \prec \phi(z), \quad (z \in \mathbb{D}) \right\}. \tag{1.5}$$

The class $\mathcal{K}(\phi) \subset \mathbb{S}^*$ of convex functions with respect to symmetric points of order $\phi(z)$, $(0 \leq \phi < 1)$ is defined by

$$\mathcal{K}(\phi) = \left\{ f \in \mathcal{A} : \text{Re}\left(\frac{2z\{f'(z)\}'}{zf'(z) + zf'(-z)} \right) < \phi(z), \quad (z \in \mathbb{D}) \right\} \tag{1.6}$$

where ϕ is a holomorphic functions with $\phi'(0) > 0$ and has a positive real part in $\mathbb{D}$. Also, the function ϕ maps $\mathbb{D}$ onto a star-shaped region with respect to $\phi(0) = 1$ and is symmetric about the real axis. In the 1960s Zalcman posed a conjecture that if $f \in \mathbb{S}$ then

$$|a_n^2 - a_{2n-1}| \leq (n-1)^2 for\ n = 2, 3, \cdots. \tag{1.7}$$

The equality holds only for the Koebe function $K(z) = \dfrac{z}{(1-z)^2}$ or its rotations. For functions in $\mathbb{S}$, Krushkal proved the Zalcman conjecture for $n = 3$ [14] and recently $n = 4, 5, 6, \cdots$. This remarkable conjecture was investigated by many researchers and is still an open problem for functions belonging to class $\mathbb{S}$ when $n > 6$. The Zalcman conjecture was proved for certain special subclasses of $\mathbb{S}$, such as starlike, typically real, and close-to-convex functions(see [15], [16]). Recently, Abu Muhanna et al. [17] solved the Zalcman conjecture for the class $\mathcal{F}$ consisting of the function $f \in \mathcal{A}$ satisfying the analytic condition

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > -\frac{1}{2}, \quad z \in \mathbb{D}$$

Functions in the class $\mathcal{F}$ are known to be convex in some direction (and hence close-to-convex) in $\mathbb{D}$. Ma [18] proved the Zalcman conjecture for close-to-convex functions. For $f \in \mathbb{S}$, Ma [19] proposed a generalized Zalcman conjecture

$$|a_n a_m - a_{n+m-1}| \leq (n-1)(m-1) \ \ for\ \ m, n = 2, 3, \cdots \tag{1.8}$$

which is still an open problem, and proved it for classes $\mathbb{S}^*$ and $\mathbb{S}_{\mathbb{R}}$, where $\mathbb{S}_{\mathbb{R}}$ denotes the type of all functions in $\mathcal{A}$ which are typically real. Bansal and Sokol [20] studied the Zalcman conjecture for some subclasses of analytic functions. Ravichandran and Verma [21] proved this conjecture for the classes of starlike and convex functions of a certain order and the class of functions with bounded turning. Motivated by the results mentioned above, which are associated with the Zalcman conjecture and the Hankel determinants, the logarithmic coefficient γ_n of $f \in \mathcal{S}^*$ are defined by

$$\mathcal{F}_{af}(z) = \log\left(\frac{f(z)}{z}\right) = 2\sum_{n=1}^{\infty} \gamma_n z^n, \quad (z \in \mathbb{D}) \quad . \tag{1.9}$$

The logarithmic coefficients γ_n play a central role in the theory of univalent functions [40,43–45]. A very few exact upper bounds for γ_n seem to have been established. The significance of this problem in the context of Bieberbach conjecture was pointed by Milin [22] in his conjecture. Milin [22] conjectured that for $f \in \mathcal{S}^*$ and $n \geq 2$

$$\sum_{m=1}^{n}\sum_{k=1}^{m}\left(k|\gamma_k|^2 - \frac{1}{k}\right) \leq 0 \quad , \tag{1.10}$$

which led De Branges, by proving this conjecture, to the proof of Bieberbach conjecture [23]. For the Koebe function $k(z) = \dfrac{z}{(1-z)^2}$, the logarithmic coefficients are $\gamma_n = \dfrac{1}{n}$. Since the Koebe function k plays the role of extremal function for most of the extremal problems in the class $\mathcal{S}^*$, it is expected that $\gamma_n \leq \dfrac{1}{n}$ holds for functions in $\mathcal{S}^*$. But this is not true in general, even in order of magnitude. Indeed, there exists a bounded function f in the class $\mathcal{S}^*$ with logarithmic coefficients $\gamma_n \neq \bigcirc(n^{-0.83})$. By differentiating (1.9) and the equating coefficients we obtain

$$\gamma_1 = \frac{1}{2}a_2 \, , \tag{1.11}$$

$$\gamma_2 = \frac{1}{2}\left(a_3 - \frac{1}{2}a_2^2\right) \, , \tag{1.12}$$

$$\gamma_3 = \frac{1}{2}\left(a_4 - a_2a_3 + \frac{1}{3}a_2^3\right) \, , \tag{1.13}$$

$$\gamma_4 = \frac{1}{2}\left(a_5 - a_2a_4 + a_2^2a_3 - \frac{1}{2}a_3^2 - \frac{1}{4}a_2^4\right) \, . \tag{1.14}$$

If $f \in \mathcal{S}^*$, it is easy to see that $|\gamma_1| \leq 1$, because $|a_2| \leq 2$. Using the Fekete-Szego inequality for functions in $\mathcal{S}^*$ in (1.16), we obtain sharp estimate

$$|\gamma_2| \leq \frac{1}{2}(1 + 2e^{-2}) = 0.635\cdots .$$

For $n \geq 3$, the problem seems much harder, and no significant bound for $|\gamma_n|$ when $f \in \mathcal{S}^*$ appear to be known. In 2017, Ali and Allu [24] obtained the initial logarithmic coefficients bounds for the initial three logarithmic coefficients for a subclass of $f \in \mathcal{S}^*$. The problem of computing the bound of the logarithmic coefficients is also considered in [6, 16, 20] for several subclasses of close to convex functions. In 2021, Zaprawa [25] obtained the sharp bounds of the initial logarithmic coefficients γ_n for functions in the classes $\mathcal{S}_s^*$ and $\mathcal{K}_s$.

We recall the definition of the Hankel determinant with k as a parameter and $n \in \mathbb{N} := \{1, 2, 3, ...\}$

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+q-1} & a_{n+k} & \cdots & a_{n+2q-2} \end{vmatrix} \quad (n, q \in \mathbb{N} = 1, 2, 3, ...) \tag{1.15}$$

for example,

$$H_{2,1}(f) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix}, \quad H_{2,2}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}, \quad H_{3,1}(f) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}. \tag{1.16}$$

The evaluation of the upper bound of $H_{q,n}(f)$ across different subfamilies of $\mathbb{A}$ is an intriguing area of research within Geometric Function Theory in Complex Analysis. Noonan and Thomas [26], as well as Noor [27], examined the growth rate of $H_{q,n}(f)$ as $n \longrightarrow \infty$ for fixed q and n, focusing on various subfamilies of the class of univalent function $\mathcal{S}^*$. The Hankel determinant $H_{2,1}(f) = a_3 - a_2^2$ and $H_{2,2}(f) = a_2a_4 - a_3^2$ are known as the Fekete-Szego functional and second Hankel determinant respectively. The functional $H_{2,1}(f)$ is further generalized as $a_3 - \mu a_2^2$ for some real or complex parameter μ. Various researchers have obtained upper bounds of $H_{2,1}(f)$ for different subfamilies of class $\mathcal{S}^*$ (refer to [28–30]). Recently, Srivastava et al. [28] derived bounds for the second Hankel determinant for q-starlike and q-convex functions. Additionally, several studies have focused on obtaining bounds for initial coefficients, exploring the Fekete-Szegö inequality, and estimating Hankel determinants of different orders for various subclasses of univalent and bi-univalent functions [29–32].

Building upon the previous concepts, we suggest investigating the Hankel determinant, where its elements comprise of logarithmic coefficients of $\mathcal{S}^*$. This exploration could unveil fascinating insights into the interplay between logarithmic coefficients and Hankel determinants given by

$$H_{q,n}(f) = \begin{vmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+k-1} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+k} \\ \vdots & \vdots & \cdots & \vdots \\ \gamma_{n+k-1} & \gamma_{n+k} & \cdots & \gamma_{n+2k-2} \end{vmatrix} \quad (n, k \in \mathbb{N} = 1, 2, 3, ...) \tag{1.17}$$

In the present paper, we are attempting to find sharp upper bounds for the logarithmic coefficient inequalities of tan hyperbolic function specified in the abstract for the functions belonging to a certain subclass of analytic function defined as follows.

Definition 1.1. *A mapping $f \in \mathcal{A}$ is said to be in the class $\mathbb{TS}^*(\beta)$ $(0 \leq \beta \leq 1)$ if*

$$\operatorname{Re}\left[\frac{2\left\{zf'(z) + \beta z^2 f''(z)\right\}}{(1-\beta)\left\{f(z) - f(-z)\right\} + \beta\left\{zf'(z) + zf'(-z)\right\}}\right] \geq 0, \qquad z \in \mathbb{D}. \tag{1.18}$$

For $\beta = 0$ and $\beta = 1$, we get $\mathbb{TS}^*(0) = \mathbb{S}_s^*$, consisting of starlike functions with respect to symmetric points, interpreted and studied by Sakaguchi [32], $\mathbb{TS}^*(1) = \mathbb{K}_s^*$, consisting of convex functions with respect to symmetric points, analyzed by Das and Singh [33] for which analytic conditions are given in (1.5) and (1.6).

Definition 1.2. [40] *Consider the hyperbolic function*

$$\phi(z) = 1 + tanhz, \quad (\phi(0) = 1) \quad .$$

We define the following family of functions

$$\mathbb{TS}^*(\phi) := \left\{ f : f \in \mathbb{A} \text{ and } \mathrm{Re}\left(\frac{2zf'(z)}{f(z) - f(-z)}\right) \prec \phi(z), \quad (z \in \mathbb{D}) \right\}. \tag{1.19}$$

In other words, a function f is in the class $\mathbb{TS}^*$ if and only if there exists a holomorphic function q, fulfilling

$$q(z) \prec q_0(z) := 1 + tanhz \ ,$$

such that

$$f(z) = zexp\left(\int_0^z \frac{q(t) - 1}{t} dt\right) \tag{1.20}$$

by taking

$$q(z) = q_0(z) = 1 + tanhz$$

In (1.20), we get the function that plays the role of the extremal function in many problems of the class $\mathbb{TS}^*$, given by

$$f_0(z) = zexp\left(\int_0^z \frac{tanht}{t} dt\right) = z + z^2 + \frac{1}{2}z^3 + \frac{1}{18}z^4 + \cdots \tag{1.21}$$

Let $p \in \mathcal{P}$ represent the class of all functions p that are holomorphic in $\mathbb{D}$ with $\mathcal{R}p(z) > 0$ and has series representation given in the form of

$$p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k \ , \tag{1.22}$$

where every such function is called Caratheodory function. [35].
To prove the main results, we need the following lemmas.

Lemma 1.3. [36] *Let $p \in \mathcal{P}$, be given by (1.22), then*

$$|c_k| \leq 2 \ for \ k \geq 1 \ , \tag{1.23}$$

$$|c_{n+k} - \mu c_n c_k| < 2 \ for \ 0 \leq \mu \leq 1 \ , \tag{1.24}$$

$$|c_n c_k - c_n c_l| \leq 4 \ for \ m + k = k + l, \tag{1.25}$$

$$\left|c_2 - \frac{1}{2}c_1^2\right| \leq 2 - \frac{1}{2}|c_1|^2 \ . \tag{1.26}$$

Lemma 1.4. [34,35,38] *Let $p \in \mathcal{P}$, be given by (1.22), for complex number μ, we have*

$$|c_{n+k} - \mu c_n c_k| \leq 2max\{1, |2\mu - 1|\} = \begin{cases} 2, & \text{for } 0 \leq \mu \leq 1 \\ 2|2\mu - 1|, & \text{otherwise} \ . \end{cases} \tag{1.27}$$

Also, if $B \in [0,1]$ and $B(2B-1) \le D \le B$, we get

$$\left|c_3 - 2Bc_1c_2 + Dc_1^3\right| \le 2 \quad . \tag{1.28}$$

Lemma 1.5. [13] *Let α, β, r and a satisfy the inequalities $0 < \alpha < 1$, $0 < a < 1$ and*

$$8a(1-a)((\alpha\beta - 2r)^2 + (\alpha(a+\alpha) - \beta)^2) + \alpha(1-\alpha)(\beta - 2a\alpha)^2 \le 4a\alpha^2(1-\alpha)^2(1-a). \tag{1.29}$$

If $p \in \mathcal{P}$, be given by (1.22), then

$$\left|rc_1^4 + ac_2^2 + 2\alpha c_1c_3 - \frac{3}{2}\beta c_1^2c_2 - c_4\right| \le 2 \quad . \tag{1.30}$$

Lemma 1.6. [41], [37] *Let $p \in \mathcal{P}$, be given by (1.22), then for some complex valued x with $|x| \le 1$, some complex valued τ with $|\tau| \le 1$ and some complex valued ρ with $|\rho| \le 1$. We have*

$$2c_2 = c_1^2 + x\left(4 - c_1^2\right), \tag{1.31}$$

$$4c_3 = c_1^3 + 2\left(4-c_1^2\right)c_1x - c_1\left(4-c_1^2\right)x^2 + 2\left(4-c_1^2\right)\left(1-|x|^2\right)\tau, \tag{1.32}$$

$$\begin{aligned} 8c_4 &= c_1^4 + \left(4-c_1^2\right)x\left[c_1^2\left(x^2-3x+3\right)+4x\right] \\ &- 4\left(4-c_1^2\right)\left(1-|x|^2\right)\left[c\left(x-1\right)\tau + \bar{x}\tau^2 - \left(1-|\tau|^2\rho\right)\right] \quad . \end{aligned} \tag{1.33}$$

In this section, we start with finding the bounds of the first few initial logarithmic coefficients for the class of functions $\mathbb{TS}^*(\beta)$ of star - like functions linked with caratheodory functions.

2. Coefficient Estimates for Logarithmic Function $\mathbb{TS}^*(\beta)$

Theorem 2.1. *If $f \in \mathbb{TS}^*(\beta)$, $(0 \le \beta \le 1)$ then we have the sharp bounds*

$$|\gamma_1| \le \frac{(1-\beta)}{4}, \tag{2.1}$$

$$|\gamma_2| \le \frac{1-\beta}{2(2+7\beta)}, \tag{2.2}$$

$$|\gamma_3| \le \frac{1-\beta}{2(4+8\beta)}, \tag{2.3}$$

$$|\gamma_4| \le \frac{1-\beta}{2(4+21\beta)} \quad . \tag{2.4}$$

Proof. Let $f \in \mathbb{TS}^*(\beta)$. Then the equation (1.18) can be written in the form of a hyperbolic function ω as follows

$$\left[\frac{2\left\{zf'(z) + \beta z^2 f''(z)\right\}}{(1-\beta)\left\{f(z) - f(-z)\right\} + \beta\left\{zf'(z) + zf'(-z)\right\}}\right] = 1 + tanh\omega(z) \quad . \tag{2.5}$$

Expressing $p(z)$ in terms of Schwarz function ω, we have

$$p(z) = \frac{1+\omega(z)}{1-\omega(z)} := 1 + c_1z + c_2z^2 + c_3z^3 + c_4z^4 + \cdots \tag{2.6}$$

or equivalently,

$$\omega(z) = \frac{p(z)-1}{p(z)+1} = \frac{c_1z + c_2z^2 + c_3z^3 + c_4z^4 + \cdots}{2 + c_1z + c_2z^2 + c_3z^3 + c_4z^4 + \cdots}$$

where

$$\omega(z)=\frac{1}{2}c_1z+\left(\frac{1}{2}c_2-\frac{1}{4}c_1^2\right)z^2+\left(\frac{1}{8}c_1^3-\frac{1}{2}c_1c_2+\frac{1}{2}c_3\right)z^3+\left(\frac{1}{2}c_4-\frac{1}{2}c_1c_3-\frac{1}{4}c_2^2-\frac{1}{16}c_1^4+\frac{3}{8}c_1^2c_2\right)z^4+\cdots \quad (2.7)$$

After some calculation and by using the series expansion given by (2.7), we get

$$1+tanh(\omega(z))=1+\frac{1}{2}c_1z+\left(-\frac{1}{4}c_1^2+\frac{1}{2}c_2\right)z^2+\left(\frac{1}{12}c_1^3-\frac{1}{2}c_1c_2+\frac{1}{2}c_3\right)z^3+\left(\frac{1}{2}c_4+\frac{1}{4}c_1^2c_2-\frac{1}{2}c_1c_3-\frac{1}{4}c_2^2\right)z^4+\cdots \quad (2.8)$$

and

$$\begin{aligned}&\left\{\frac{2\left\{zf'(z)+\beta z^2f''(z)\right\}}{(1-\beta)\left\{f(z)-f(-z)\right\}+\beta\left\{zf'(z)+zf'(-z)\right\}}\right\}\\ &=\left\{\frac{1+2a_2(1+\beta)z+3a_3(1+2\beta)z^2+4a_4(1+3\beta)z^3+5a_5(1+4\beta)z^4+\cdots}{(1-\beta)+a_3(1-\beta)z^2+a_5(1-\beta)z^4+2a_2\beta z+4a_4\beta z^3+6a_6\beta z^5+\cdots}\right\}. \quad (2.9)\end{aligned}$$

Comparing (2.8) and (2.9), we obtain

$$a_2 = \frac{c_1(1-\beta)}{4}, \quad (2.10)$$

$$a_3 = \frac{c_2(1-\beta)}{2(2+7\beta)}+\frac{c_1^2(1-\beta)^2}{4(2+\beta)}, \quad (2.11)$$

$$a_4 = \frac{1}{4+8\beta}\left\{c_1^3\left(\frac{24\beta^3-38\beta^2+13\beta+1}{24(2+7\beta)}\right)-c_1c_2\left(\frac{7\beta^3-20\beta^2+10\beta+3}{4(2+7\beta)}\right)+c_3\left(\frac{1-\beta}{2}\right)\right\}, \quad (2.12)$$

$$\begin{aligned}a_5 &= c_1^4\left(\frac{-40\beta^4-68\beta^3+88\beta^2+8\beta+12}{48(2+7\beta)(4+8\beta)4+21\beta}\right)+c_1^2c_2\left(\frac{92\beta^4-84\beta^3-64\beta^2+48\beta+8}{8(2+7\beta)(4+8\beta)4+21\beta}\right)\\ &+ c_1c_3\left(\frac{8\beta^3-16\beta^2+8}{4(4+8\beta)4+21\beta}\right)-c_2^2\left(\frac{-8\beta^2+7\beta+1}{4(2+7\beta)4+21\beta}\right)+c_4\left(\frac{1-\beta}{2(4+21\beta)}\right) \quad (2.13)\end{aligned}$$

By making use of (2.10) − (2.13) in (1.11) − (1.14), we get

$$\gamma_1 = \frac{c_1(1-\beta)}{8}, \quad (2.14)$$

$$\gamma_2 = \frac{c_2(1-\beta)}{4(2+7\beta)}-\frac{c_1^2(1-\beta)^2(6-7\beta)}{64(2+7\beta)}, \quad (2.15)$$

$$\begin{aligned}\gamma_3 &= \frac{1-\beta}{4(4+8\beta)}\left\{c_3-\left[\frac{22\beta^3-52\beta^2+20\beta+10}{4(2+7\beta)}\right]c_1c_2\right\}\\ &+ \frac{1-\beta}{4(4+8\beta)}\left\{\left[\frac{-56\beta^5+28\beta^4+220\beta^3-234\beta^2-15\beta+57}{48(2+7\beta)}\right]c_1^3\right\}, \quad (2.16)\end{aligned}$$

$$\begin{aligned}\gamma_4 &= -\frac{1-\beta}{4(4+21\beta)}\left[\frac{24696\beta^8-11172\beta^7-229824\beta^6+371704\beta^5-3544\beta^4}{1536(2+7\beta)^2(4+8\beta)(1-\beta)}\right]c_1^4\\ &- \frac{1-\beta}{4(4+21\beta)}\left[\frac{-157156\beta^3-12928\beta^2+16240\beta+1984}{1536(2+7\beta)^2(4+8\beta)(1-\beta)}\right]c_1^4\\ &+ \frac{1-\beta}{4(4+21\beta)}\left[\frac{-91\beta^3+28\beta^2+55\beta+8}{4(2+7\beta)^2(1-\beta)}\right]c_2^2\\ &- \frac{1-\beta}{4(4+21\beta)}\left[\frac{37\beta^3-70\beta^2+13\beta+20}{8(4+8\beta)(1-\beta))}\right]2c_1c_3\\ &+ \frac{1-\beta}{4(4+21\beta)}\left[\frac{-3234\beta^6+11242\beta^5-5644\beta^4-6130\beta^3+1898\beta^2+1660\beta+208}{24(2+7\beta)^2(4+8\beta)(1-\beta))}\right]\frac{3}{2}c_1c_2^2\end{aligned}$$

$$+ \left[\frac{(1-\beta)}{4(4+21\beta)}\right] c_4. \tag{2.17}$$

For γ_1, using Lemma (1.3) in (2.14) we get

$$|\gamma_1| \leq \frac{1-\beta}{4} \tag{2.18}$$

For γ_2, using Lemma (1.4) in (2.15) we get

$$|\gamma_2| \leq \frac{1-\beta}{2(2+7\beta)} \tag{2.19}$$

For γ_3, using Lemma (1.4) in (2.16) we get

$$\gamma_3 = \frac{1-\beta}{4(4+8\beta)} \left\{ c_3 - 2\left[\frac{11\beta^3 - 26\beta^2 + 10\beta + 5}{4(2+7\beta)}\right] c_1 c_2 + \left[\frac{-56\beta^5 + 28\beta^4 + 220\beta^3 - 234\beta^2 - 15\beta + 57}{48(2+7\beta)}\right] c_1^3 \right\}. \tag{2.20}$$

Comparing the bracket portion of (2.20) with Lemma (1.4), we obtain

$$B = \frac{11\beta^3 - 26\beta^2 + 10\beta + 5}{4(2+7\beta)} \tag{2.21}$$

$$D = \frac{-56\beta^5 + 28\beta^4 + 220\beta^3 - 234\beta^2 - 15\beta + 57}{48(2+7\beta)} \tag{2.22}$$

Clearly $0 \leq B \leq 1$ and $B \geq D$. Moreover,

$$B\,(2B-1) = \frac{121\beta^6 - 572\beta^5 + 896\beta^4 - 410\beta^3 - 160\beta^2 + 100\beta + 25}{8(2+7\beta)^2} \leq D \quad . \tag{2.23}$$

Given that all conditions out lined in Lemma (1.4) are met, applying (2.16) yields the following conclusion

$$|\gamma_3| \leq \frac{1-\beta}{2(4+8\beta)} \quad . \tag{2.24}$$

To obtain the bound of γ_4, we use lemma(1.5) in (2.17) so that

$$\gamma_4 = -\frac{1-\beta}{4(4+21\beta)} \left[r c_1^4 + a c_2^2 + 2\alpha c_1 c_3 - \frac{3}{2}\eta c_1^2 c_2 - c_4 \right] \tag{2.25}$$

$$r = \left[\frac{24696\beta^8 - 11172\beta^7 - 229824\beta^6 + 371704\beta^5 - 3544\beta^4 - 157156\beta^3 - 12928\beta^2 + 16240\beta + 1984}{1536(2+7\beta)^2(4+8\beta)(1-\beta)}\right],$$

$$a = \left[\frac{-91\beta^3 + 28\beta^2 + 55\beta + 8}{4(2+7\beta)^2(1-\beta)}\right],$$

$$\alpha = \left[\frac{37\beta^3 - 70\beta^2 + 13\beta + 20}{8(4+8\beta)(1-\beta)}\right],$$

$$\eta = \left[\frac{-3234\beta^6 + 11242\beta^5 - 5644\beta^4 - 6130\beta^3 + 1898\beta^2 + 1660\beta + 208}{24(2+7\beta)^2(4+8\beta)(1-\beta)}\right] \quad .$$

Consequently, all the conditions outlined in Lemma (1.4) and (1.5) are fulfilled, and utilizing equation (1.30), we get

$$|\gamma_4| \leq \frac{1-\beta}{2(4+21\beta)} \tag{2.26}$$

which completes the proof of theorem (2.1). □

Theorem 2.2. *If $f \in \mathbb{TS}^*(\beta)$, $(0 \leq \beta \leq 1)$ then*

$$\left|\gamma_2 - \lambda\gamma_1^2\right| \leq \frac{1-\beta}{2(2+7\beta)} \max\left\{1, \left|\frac{7\beta^2 - 13\beta + |\lambda|(2+7\beta)(1-\beta) - 2}{8}\right|\right\} \tag{2.27}$$

Proof. Making the use of (2.14) and (2.15). We get

$$\left|\gamma_2 - \lambda\gamma_1^2\right| = \left|\left[\frac{c_2(1-\beta)}{4(2+7\beta)} - \frac{c_1^2(1-\beta)^2(6-7\beta)}{64(2+7\beta)}\right] - \lambda\left[\frac{c_1^2(1-\beta)^2}{64}\right]\right| \tag{2.28}$$

$$= \frac{1-\beta}{4(2+7\beta)}\left|c_2 - \left[\frac{(1-\beta)(6-7\beta)}{16} + \frac{\lambda(1-\beta)(2+7\beta)}{16}\right] c_1^2\right| \tag{2.29}$$

$$= \frac{1-\beta}{4(2+7\beta)}\left|c_2 - \left[\frac{(1-\beta)(6+2\lambda-7\beta)+7\lambda\beta}{16}\right] c_1^2\right| \tag{2.30}$$

$$= \frac{1-\beta}{4(2+7\beta)}\left|c_2 - \mu c_1^2\right| \tag{2.31}$$

$$\tag{2.32}$$

where,

$$\mu = \frac{(1-\beta)(6+2\lambda-7\beta)+7\lambda\beta}{16} \tag{2.33}$$

Using Lemma (1.4), we get

$$\left|\gamma_2 - \lambda\gamma_1^2\right| \leq \frac{1-\beta}{2(2+7\beta)} \max\left\{1, \left|\frac{7\beta^2 - 13\beta + |\lambda|(2+7\beta)(1-\beta) - 2}{8}\right|\right\} \tag{2.34}$$

which completes the proof of theorem (2.2). □

Putting $\lambda = 1$ in theorem (2.2), we get the following result.

Corollary 2.3. *Let f given by (1.1), be in the class $\mathbb{T}\mathcal{S}^*(\beta)$; $(0 \leq \beta \leq 1)$. Then for any $\lambda \in \mathbb{C}$, we have*

$$\left|\gamma_2 - \lambda\gamma_1^2\right| \leq \frac{1}{2}\max\{1, |2\mu - 1|\}. \tag{2.35}$$

Theorem 2.4. *If $f \in \mathbb{TS}^*(\beta)$, $(0 \leq \beta \leq 1)$ then*

$$|\gamma_1\gamma_2 - \gamma_3| \leq \frac{1-\beta}{2(4+8\beta)}. \tag{2.36}$$

Proof. Making the use of (2.14), (2.15) and (2.16). We get,

$$|\gamma_1\gamma_2 - \gamma_3| = \frac{1-\beta}{4(4+8\beta)}\left|c_3 - 2\left[\frac{11\beta^3 - 30\beta^2 + 12\beta + 7}{8(2+7\beta)}\right]c_1c_2 + \left[\frac{-56\beta^5 - 28\beta^4 + 352\beta^3 - 306\beta^2 - 43\beta + 81}{96(2+7\beta)}\right]c_1^3\right|. \tag{2.37}$$

An application Lemma (1.4), we get

$$0 \leq B = \frac{11\beta^3 - 30\beta^2 + 12\beta + 7}{8(2+7\beta)} \leq 1,$$

$$B = \frac{11\beta^3 - 30\beta^2 + 12\beta + 7}{8(2+7\beta)} \geq D = \frac{-56\beta^5 - 28\beta^4 + 352\beta^3 - 306\beta^2 - 43\beta + 81}{96(2+7\beta)} \tag{2.38}$$

and

$$B(2B-1)=\frac{121\beta^6-660\beta^5+856\beta^4+186\beta^3-372\beta^2-124\beta-7}{32(2+7\beta)^2}$$
$$<D=\frac{-56\beta^5-28\beta^4+352\beta^3-306\beta^2-43\beta+81}{96(2+7\beta)}. \tag{2.39}$$

An application Lemma (1.4), we get

$$|\gamma_1\gamma_2-\gamma_3|\le\frac{1-\beta}{2(4+8\beta)}.$$

which completes the proof of theorem (2.4). □

Theorem 2.5. *If $f\in\mathbb{TS}^*(\beta)$, $(0\le\beta\le 1)$ then*

$$\left|\gamma_4-\gamma_2^2\right|\le\frac{(1-\beta)}{2(4+21\beta)}. \tag{2.40}$$

Proof. Making the use of (2.15) and (2.17). We get,

$$\begin{aligned}
\left|\gamma_4-\gamma_2^2\right| &= \frac{(1-\beta)}{4(4+21\beta)}\left\{\frac{197568\beta^8-88200\beta^7-1844724\beta^6}{12288(2+7\beta)^2(4+8\beta)(1-\beta)}\right\}c_1^4\\
&+ \frac{(1-\beta)}{4(4+21\beta)}\left\{\frac{2986256\beta^5-40616\beta^4-1252952\beta^3}{12288(2+7\beta)^2(4+8\beta)(1-\beta)}\right\}c_1^4\\
&+ \frac{(1-\beta)}{4(4+21\beta)}\left\{\frac{-101684\beta^2+128048\beta+16304}{12288(2+7\beta)^2(4+8\beta)(1-\beta)}\right\}c_1^4\\
&+ \frac{(1-\beta)}{4(4+21\beta)}\left\{\frac{-364\beta^3+113\beta^2+218\beta+33}{16(2+7\beta)^2(1-\beta)}\right\}c_2^2\\
&+ \frac{(1-\beta)}{4(4+21\beta)}\left\{\frac{-37\beta^3-70\beta^2+13\beta+20}{8(4+8\beta)(1-\beta)}\right\}2c_1c_3\\
&- \frac{(1-\beta)}{4(4+21\beta)}\left\{\frac{-25872\beta^6+89880\beta^5-44964\beta^4-49224\beta^3+15228\beta^2+13332\beta+1640}{192(2+7\beta)^2(4+8\beta)(1-\beta)}\right\}\frac{3}{2}c_1^2c_2\\
&- \frac{(1-\beta)}{4(4+21\beta)}c_4.
\end{aligned} \tag{2.41}$$

After simplifying we get

$$\begin{aligned}
\left|\gamma_4-\gamma_2^2\right| &\le \frac{(1-\beta)}{4(4+21\beta)}\left|\left\{\frac{197568\beta^8-88200\beta^7-1844724\beta^6}{12288(2+7\beta)^2(4+8\beta)(1-\beta)}\right\}c_1^4\right|\\
&+ \frac{(1-\beta)}{4(4+21\beta)}\left|\left\{\frac{2986256\beta^5-40616\beta^4-1252952\beta^3}{12288(2+7\beta)^2(4+8\beta)(1-\beta)}\right\}c_1^4\right|\\
&+ \frac{(1-\beta)}{4(4+21\beta)}\left|\left\{\frac{-101684\beta^2+128048\beta+16304}{12288(2+7\beta)^2(4+8\beta)(1-\beta)}\right\}c_1^4\right|\\
&+ \frac{(1-\beta)}{4(4+21\beta)}\left|\left\{\frac{-364\beta^3+113\beta^2+218\beta+33}{16(2+7\beta)^2(1-\beta)}\right\}c_2^2\right|\\
&+ \frac{(1-\beta)}{4(4+21\beta)}\left|\left\{\frac{-37\beta^3-70\beta^2+13\beta+20}{8(4+8\beta)(1-\beta)}\right\}2c_1c_3\right|\\
&- \frac{(1-\beta)}{4(4+21\beta)}\left|\left\{\frac{-25872\beta^6+89880\beta^5-44964\beta^4-49224\beta^3+15228\beta^2+13332\beta+1640}{192(2+7\beta)^2(4+8\beta)(1-\beta)}\right\}\frac{3}{2}c_1^2c_2\right|\\
&- \frac{(1-\beta)}{4(4+21\beta)}\left|c_4\right|.
\end{aligned} \tag{2.42}$$

Comparing the rightside of (2.14) with

$$\left|rc_1^4+ac_2^2+2\alpha c_1c_3-\frac{3}{2}\eta c_1^2c_2-c_4\right| \tag{2.43}$$

where,

$$\begin{aligned} r &= \left\{\frac{197568\beta^8-88200\beta^7-1844724\beta^6+2986256\beta^5-40616\beta^4-1252952\beta^3-101684\beta^2+128048\beta+16304}{12288(2+7\beta)^2(4+8\beta)(1-\beta))}\right\}\\ a &= \left\{\frac{-364\beta^3+113\beta^2+218\beta+33}{16(2+7\beta)^2(1-\beta)}\right\}\\ \alpha &= \left\{\frac{-37\beta^3-70\beta^2+13\beta+20}{8(4+8\beta)(1-\beta)}\right\}\\ \eta &= \left\{\frac{-25872\beta^6+89880\beta^5-44964\beta^4-49224\beta^3+15228\beta^2+13332\beta+1640}{192(2+7\beta)^2(4+8\beta)(1-\beta)}\right\} \end{aligned} \tag{2.44}$$

are such that

$$8a(1-a)\left((\alpha\eta-2r)^2+(\alpha(a+\alpha)-\beta)^2\right)+\alpha(1-\alpha)(\beta-2a\alpha)^2\le 4a\alpha^2(1-\alpha)^2(1-a),\ 0<\alpha<1,\ \ 0<a<1.$$

By equations (2.15) and (2.17) and the lemma (1.6), we get

$$\left|\gamma_4-\gamma_2^2\right|\le\frac{1-\beta}{2(4+21\beta)}$$

which completes the proof of theorem (2.5). □

Theorem 2.6. *If* $f\in\mathbb{TS}^*(\beta)$, $(0\le\beta\le 1)$ *then*

$$\left|\gamma_1\gamma_3-\gamma_2^2\right|\le\frac{(1-\beta)^2}{(4(2+7\beta)^2)} \tag{2.45}$$

Proof. Making the use of (2.14), (2.15) and (2.16), we get

$$\begin{aligned} \gamma_1\gamma_3-\gamma_2^2 &= \left\{\frac{12544\beta^7-15379\beta^6-47544\beta^5+87410\beta^4-17468\beta^3-31947\beta^2+8844\beta+3540}{12288(2+7\beta)^2}\right\}c_1^4\\ &+ \left\{\frac{(1-\beta)^2}{32(4+8\beta)}\right\}c_1c_3\\ &+ \left\{\frac{616\beta^5-1889\beta^4+1397\beta^3+335\beta^2-385\beta-74}{128(2+7\beta)^2}\right\}c_1^2c_2\\ &- \left\{\frac{(1-\beta)^2}{16(2+7\beta)}\right\}c_2^2. \end{aligned} \tag{2.46}$$

Using (1.23) and (1.28) we express c_2 and c_3 in terms of c_1 and without loss of generality we can write $c_1=c$ with $0\le c\le 2$. We get

$$\begin{aligned} \left|\gamma_1\gamma_3-\gamma_2^2\right| &\le \left|\left\{\frac{-100352\beta^8+72856\beta^7+205324\beta^6+98000\beta^5}{12288(2+7\beta)^2(4+8\beta)}\right\}(-c^4)\right|\\ &+ \left|\left\{\frac{-377608\beta^4-77752\beta^3+136236\beta^2+42096\beta+1200}{12288(2+7\beta)^2(4+8\beta)}\right\}(-c^4)\right|\\ &- \left|\left\{\frac{-4928\beta^6+12648\beta^5-3368\beta^4-8532\beta^3+1560\beta^2+2276\beta+344}{256(2+7\beta)^2(4+8\beta)}\right\}c^2(4-c^2)x\right| \end{aligned}$$

$$
\begin{aligned}
& - \left|\left\{\frac{(1-\beta)^2}{128(4+8\beta)}\right\} c^2(4-c^2)x^2\right| \\
& + \left|\left\{\frac{(1-\beta)^2}{64(4+8\beta)}\right\} c(4-c^2)(1-|x|^2)\tau\right| \\
& - \left|\left\{\frac{(1-\beta)^2}{64(2+7\beta)^2}\right\} x^2(4-c^2)^2\right|. \qquad (2.47)
\end{aligned}
$$

Since $|\tau| \leq 1$ and replcing $|x| = b$ where $b \leq 1$ (2.47) becomes

$$
\begin{aligned}
\left|\gamma_1\gamma_3 - \gamma_2^2\right| \leq & \left\{\frac{-100352\beta^8 + 72856\beta^7 + 205324\beta^6 + 98000\beta^5 - 377608\beta^4 - 77752\beta^3}{12288(2+7\beta)^2(4+8\beta)}\right\} c^4 \\
& + \left\{\frac{136236\beta^2 + 42096\beta + 1200}{12288(2+7\beta)^2(4+8\beta)}\right\} c^4 \\
& + \left\{\frac{-4928\beta^6 + 12648\beta^5 - 3368\beta^4 - 8532\beta^3 + 1560\beta^2 + 2276\beta + 344}{256(2+7\beta)^2(4+8\beta)}\right\} c^2(4-c^2)b \\
& + \left\{\frac{(1-\beta)^2}{128(4+8\beta)}\right\} c^2(4-c^2)b^2 \\
& + \left\{\frac{(1-\beta)^2}{64(4+8\beta)}\right\} c(4-c^2)(1-b^2) \\
& + \left\{\frac{(1-\beta)^2}{64(2+7\beta)^2}\right\} b^2(4-c^2)^2 \\
= & \ \phi(c,b). \qquad (2.48)
\end{aligned}
$$

It is a simple exercise to show that $\dfrac{\partial\phi}{\partial b} \geq 0$ on $[0,1]$, so that $\phi(c,b) \leq \phi(c,1)$.
Putting b=1 in (2.48) we get

$$
\begin{aligned}
\left|\gamma_1\gamma_3 - \gamma_2^2\right| \leq & \left\{\frac{-100352\beta^8 + 72856\beta^7 + 205324\beta^6 + 98000\beta^5 - 377608\beta^4 - 77752\beta^3}{12288(2+7\beta)^2(4+8\beta)}\right\} c^4 \\
& + \left\{\frac{136236\beta^2 + 42096\beta + 1200}{12288(2+7\beta)^2(4+8\beta)}\right\} c^4 \\
& + \left\{\frac{-4928\beta^6 + 12648\beta^5 - 3368\beta^4 - 8532\beta^3 + 1560\beta^2 + 2276\beta + 344}{256(2+7\beta)^2(4+8\beta)}\right\} c^2(4-c^2) \\
& + \left\{\frac{(1-\beta)^2}{128(4+8\beta)}\right\} c^2(4-c^2) \\
& + \left\{\frac{(1-\beta)^2}{64(2+7\beta)^2}\right\} (4-c^2)^2 \\
= & \ \phi(c,1) \qquad (2.49)
\end{aligned}
$$

since $\dfrac{\partial\phi(c,1)}{\partial c} < 0$, $\phi(c,1)$ is a decreasing function, and obtains its maximum value at c=0. Therefore we get

$$
\left|\gamma_1\gamma_3 - \gamma_2^2\right| \leq \frac{(1-\beta)^2}{(4(2+7\beta)^2)}. \qquad (2.50)
$$

which completes the proof of theorem (2.6). □

Theorem 2.7. *If* $f \in \mathbb{TS}^*(\beta)$, $(0 \leq \beta \leq 1)$ *then*

$$\left|\gamma_2^2 - \gamma_3\right| \leq \frac{(1-\beta)^2}{2(2+7\beta)^2}. \tag{2.51}$$

Proof. Making the use of (2.15) and (2.16). We get,

$$\begin{aligned}
\gamma_2^2 - \gamma_3 &= \left\{\frac{49\beta^6 - 280\beta^5 + 666\beta^4 - 844\beta^3 + 601\beta^2 - 228\beta + 36}{4096(2+7\beta)^2}\right\} c_1^4 \\
&- \left\{\frac{56\beta^6 - 84\beta^5 - 192\beta^4 + 454\beta^3 - 219\beta^2 - 72\beta + 57}{192(2+7\beta)(4+8\beta)}\right\} c_1^3 \\
&+ \left\{\frac{(1-\beta)^2}{64(2+7\beta)^2}\right\} c_2^2 \\
&- \left\{\frac{7\beta^4 - 27\beta^3 + 39\beta^2 - 25\beta + 6}{128(2+7\beta)^2}\right\} c_1^2 c_2 \\
&+ \left\{\frac{-22\beta^4 + 74\beta^3 - 72\beta^2 + 10\beta + 10}{16(2+7\beta)(4+8\beta)}\right\} c_1 c_2 \\
&- \left\{\frac{(1-\beta)}{4(4+8\beta)}\right\} c_3
\end{aligned} \tag{2.52}$$

using (1.23) and (1.28) we express c_2 and c_3 in terms of c_1 and without loss of generality we can write $c_1 = c$ with $0 \leq c \leq 2$, we get

$$\begin{aligned}
\left|\gamma_2^2 - \gamma_3\right| &\leq \left|\left\{\frac{49\beta^6 - 280\beta^5 + 666\beta^4 - 844\beta^3 + 601\beta^2 - 228\beta + 36}{4096(2+7\beta)^2}\right\} (c^4)\right| \\
&- \left|\left\{\frac{56\beta^6 - 84\beta^5 - 192\beta^4 + 454\beta^3 - 219\beta^2 - 72\beta + 57}{192(2+7\beta)(4+8\beta)}\right\} c^3\right| \\
&+ \left|\left\{\frac{(1-\beta)^2}{64(2+7\beta)^2}\right\} x^2(4-c^2)^2\right| \\
&+ \left|\left\{\frac{-7\beta^4 + 27\beta^3 + 25\beta^2 - 103\beta + 58}{256(2+7\beta)^2}\right\} c^2(4-c^2)x\right| \\
&+ \left|\left\{\frac{-22\beta^4 + 74\beta^3 - 44\beta^2 - 10\beta + 2}{32(2+7\beta)(4+8\beta)}\right\} c(4-c^2)x\right| \\
&- \left|\left\{\frac{(1-\beta)}{8(4+8\beta)}\right\} (4-c^2)(1-|x|^2)\tau\right| \\
&+ \left|\left\{\frac{(1-\beta)}{16(4+8\beta)}\right\} cx^2(4-c^2)\right|.
\end{aligned} \tag{2.53}$$

Since $|\tau| \leq 1$ and replcing $|x| = b$ where $b \leq 1$ (2.53) becomes

$$\begin{aligned}
\left|\gamma_2^2 - \gamma_3\right| &\leq \left\{\frac{49\beta^6 - 280\beta^5 + 554\beta^4 - 412\beta^3 + 41\beta^2 + 44\beta + 4}{4096(2+7\beta)^2}\right\} c^4 \\
&+ \left\{\frac{-56\beta^6 + 84\beta^5 - 72\beta^4 + 434\beta^3 + 561\beta^2 + 132\beta + 39}{192(2+7\beta)(4+8\beta)}\right\} c^3 \\
&+ \left\{\frac{(1-\beta)^2}{64(2+7\beta)^2}\right\} b^2(4-c^2)^2
\end{aligned}$$

$$
\begin{aligned}
&+ \left\{\frac{-7\beta^4+27\beta^3+25\beta^2-103\beta+58}{256(2+7\beta)^2}\right\} c^2(4-c^2)b \\
&+ \left\{\frac{-22\beta^4+74\beta^3-44\beta^2-10\beta+2}{32(2+7\beta)(4+8\beta)}\right\} c(4-c^2)b \\
&+ \left\{\frac{(1-\beta)}{16(4+8\beta)}\right\} cb^2(4-c^2) \\
&+ \left\{\frac{(1-\beta)}{8(4+8\beta)}\right\} (4-c^2)(1-b^2) \\
&= \phi(c,b). \qquad (2.54)
\end{aligned}
$$

It is a simple exercise to show that $\dfrac{\partial\phi(c,b)}{\partial b} \geq 0$ on [0,1], so that $\phi(c,b) \leq \phi(c,1)$. Putting b=1 in (2.54) we get

$$
\begin{aligned}
\left|\gamma_2^2-\gamma_3\right| &\leq \left\{\frac{49\beta^6-280\beta^5+554\beta^4-412\beta^3+41\beta^2+44\beta+4}{4096(2+7\beta)^2}\right\} c^4 \\
&+ \left\{\frac{-56\beta^6+84\beta^5-72\beta^4+434\beta^3-561\beta^2+132\beta+39}{192(2+7\beta)(4+8\beta)}\right\} c^3 \\
&+ \left\{\frac{(1-\beta)^2}{32(2+7\beta)^2}\right\} (4-c^2)^2 \\
&+ \left\{\frac{-7\beta^4+27\beta^3+25\beta^2-103\beta+58}{256(2+7\beta)^2}\right\} c^2(4-c^2) \\
&+ \left\{\frac{22\beta^4-74\beta^3+44\beta^2+10\beta-2}{32(2+7\beta)(4+8\beta)}\right\} c(4-c^2) \\
&+ \left\{\frac{(1-\beta)}{8(4+8\beta)}\right\} c(4-c^2) \\
&= \phi(c,1). \qquad (2.55)
\end{aligned}
$$

Since $\dfrac{\partial\phi(c,1)}{\partial c} < 0$, so $\phi(c,1)$ is a decreasing function, and obtains its maximum value at c=0. Therefore we get

$$
\left|\gamma_2^2-\gamma_3\right| \leq \frac{(1-\beta)^2}{2(2+7\beta)^2} \qquad (2.56)
$$

which completes the proof of theorem (2.7). □

Conclusion

In the current paper, we obtained the upper bounds of Hankel determinant and Zalcman conjecture associated with the logarithmic coefficients $\gamma_n (n \in \mathbb{N})$ involving tan hyperbolic functions. All the estimations were proven to be sharp.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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