Asia Pac. J. Math. 2025 **12**:6

A NOVEL APPROACH TO LYAPUNOV UNIFORM STABILITY OF CAPUTO FRACTIONAL DYNAMIC EQUATIONS ON TIME SCALE USING A NEW GENERALIZED DERIVATIVE

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Received Oct. 30, 2024

Abstract. This work introduces a novel approach to achieving uniform stability by applying a new and more generalized derivative, the Caputo fractional delta Dini derivative of order $\alpha \in (0,1)$. By developing comparison results and uniform stability criteria for Caputo fractional dynamic equations, a unified framework for stability analysis on time scales is created which bridges continuous and discrete time domains. The established uniform stability results are demonstrated through an illustrative example, showcasing their relevance, effectiveness, and applicability over traditional integer-order methods.

2020 Mathematics Subject Classification. 34A08; 34A34; 34D20; 34N05.

Key words and phrases. uniform stability; Caputo derivative; Lyapunov function; fractional dynamic equation; time scale.

1. Introduction

The study of dynamic systems and their stability is a cornerstone of research in control theory, signal processing, and engineering, as these systems often operate under varying and complex conditions. Ensuring predictable behavior and maintaining stability, particularly uniform stability, where a system's response remains bounded and consistent despite fluctuations in initial conditions or external influences, is critical for their reliable operation [18]. However, as dynamic systems become increasingly intricate, traditional methods of analysis often fall short of capturing their full complexity. This has led to growing interest in fractional calculus, a generalization of traditional calculus, which has proven to be a powerful framework for modeling and analyzing complex systems. Fractional differential equations,

DOI: 10.28924/APJM/12-6

in particular, offer enhanced flexibility by incorporating memory effects and hereditary properties, characteristics frequently observed in real-world systems. Despite these advantages, extending stability criteria such as uniform stability to fractional dynamic equations on arbitrary time scales remains a significantly unexplored.

Time scale theory, introduced by Hilger in [9], provides a critical bridge between continuous and discrete analysis, enabling a unified study of dynamic systems across different temporal domains. This unification is particularly important for fractional dynamic equations, as it allows for the modeling of systems that exhibit both smooth and abrupt changes, such as those encountered in hybrid or multi-scale phenomena. By leveraging the time scale framework, researchers can analyze fractional systems in a more comprehensive manner, addressing gaps in traditional stability analyses and advancing the theoretical understanding of uniform stability in these contexts [12].

This work introduces a novel approach to the Lyapunov uniform stability of Caputo fractional dynamic equations on time scales. The proposed method is based on a new generalized derivative, which extends the Caputo fractional derivative to time scales. This generalized derivative, referred to as the Caputo fractional delta derivative and the Caputo fractional delta Dini derivative of order $\alpha \in (0,1)$, provides a unified framework for analyzing stability across different time domains.

In [1,13–17,19], the Dini fractional derivative defined as follows:

$$\begin{aligned} {}^{C}_{t_0}D^{\alpha}_{+}V(t,\chi(t)) \quad = \quad & \limsup_{\kappa\to 0+} \frac{1}{\kappa^{\alpha}}\Bigg\{V(t,\chi(t)) - V(t_0,\chi(t_0)) \\ & - \sum_{r=1}^{[\frac{t-t_0}{\kappa}]} (-1)^{r+1}\, {}^{\alpha}C_r[V(t-r\kappa,\chi(t)-\kappa^{\alpha}f(t,\chi(t))) - V(t_0,\chi(t_0))]\Bigg\}, \end{aligned} \qquad (1)$$

where $V:\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}_+$ is continuous, $f \in C[\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n]$, κ is a positive number, and ${}^{\alpha}C_r = \frac{\alpha(\alpha-1)\cdots(\alpha-r+1)}{r!}$, has been used to investigate several forms of stability for Caputo fractional differential equations (FrDE) with continuous domain. This derivative maintains the concept of fractional derivatives, depending on both the present point t, the initial point t_0, and the initial state $V(t_0,\chi_0)$. However, a comprehensive analysis that includes different time domains is still lacking. Previous works, such as those by [1,18,19], focused primarily on continuous time, often neglecting the intricacies of discrete domains. On the other hand, studies like [6] have addressed stability in discrete domains.

The examination of fractional dynamic systems on time scales is a relatively recent and evolving area of research, offering significant advantages in fields like modeling, mechanics, and population dynamics. Current literature on fractional dynamic systems on time scales primarily focuses on the existence and uniqueness of solutions for fractional dynamic equations on time scale (FrDET), with Caputo-type derivatives gaining prominence [7,8,26,29]. Notably, [24] explored the existence, stability, and controllability of fractional dynamic systems on time scales with applications to population dynamics using Hyers-Ulam type stability, and [21] explored Lyapunov stability analysis of Caputo fractional dynamic systems on time scales, but with focus on only stability and asymptotic stability.

By establishing comparison results and uniform stability criteria for Caputo fractional dynamic equations, this approach not only extends the classical Lyapunov stability analysis to fractional-order systems but also extends recent literature, [11,20,21]. This work provides a unified framework for stability analysis on time scales, bridging continuous and discrete time domains. The significance of this research lies in its potential to ensure predictable and consistent outcomes in dynamic systems, enabling reliable applications and offering new insights and methodologies for researchers and practitioners in the field.

For the purpose of this work, we consider the Caputo fractional dynamic system of order α with $0 < \alpha < 1$

$$\begin{aligned} {}^{C\mathbb{T}}D^{\alpha}x^{\Delta} &= f(t,x),\ t \in \mathbb{T}, \\ x(t_0) &= x_0,\ t_0 \geq 0, \end{aligned} \tag{2}$$

where $f \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n]$, $f(t,0) \equiv 0$ and ${}^{C\mathbb{T}}D^{\alpha}x^{\Delta}$ is the Caputo fractional delta derivative of $x \in \mathbb{R}^n$ of order α with respect to $t \in \mathbb{T}$. Let $x(t) = x(t,t_0,x_0) \in C^{\alpha}_{rd}[\mathbb{T}, \mathbb{R}^n]$ (the fractional derivative of order α of $x(t)$ exist and it is rd-continuous) be a solution of (2) and assume the solution exists and is unique (results on existence and uniqueness of (2) are contained in [3,24,31]), this work aims to investigate the uniform stability and uniform asymptotic stability of the system (2).

To do this, we shall use the Caputo fractional dynamic system of the form

$$ {}^{C\mathbb{T}}D^{\alpha}u^{\Delta} = g(t,u),\ u(t_0) = u_0 \geq 0, \tag{3}$$

where $u \in \mathbb{R}_+$, $g : \mathbb{T} \times \mathbb{R}_+ \to \mathbb{R}_+$ and $g(t,0) \equiv 0$. System (3) is called the comparison system. For this work, we will assume that the function $g \in [\mathbb{T} \times \mathbb{R}_+, \mathbb{R}_+]$, is such that for any initial data $(t_0, u_0) \in \mathbb{T} \times \mathbb{R}_+$, the system (3) with $u(t_0) = u_0$ has a unique solution $u(t) = u(t; t_0, u_0) \in C^{\alpha}_{rd}[\mathbb{T}, \mathbb{R}_+]$ see [3].

This work is organized as follows: Section 2 delves into essential terminologies, remarks, and fundamental lemmas that form the basis for the subsequent developments. It also introduces definitions and significant remarks. In Section 3, the main result (Uniform Stability) is presented, Section 4 provides a practical example to illustrate the relevance and application of our approach. Lastly, Section 5 offers a conclusion, summarizing the key findings and the implications of this study.

2. Preliminaries, Definitions, and Notations

In this section, we lay the groundwork by introducing key notations and definitions that will be instrumental in developing the main results.

Definition 2.1. [5] *For $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined as*

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ *is defined as*

$$\rho(t) = sup\{s \in \mathbb{T} : s < t\}.$$

(i) *if* $\sigma(t) > t$, t *is right scattered,*

(ii) *if* $\rho(t) < t$, t *is left scattered,*

(iii) *if* $t < max\mathbb{T}$ *and* $\sigma(t) = t$, *then* t *is called right dense,*

(iv) *if* $t > min\mathbb{T}$ *and* $\rho(t) = t$, *then* t *is called left dense.*

Definition 2.2. [5] *The graininess function* $\mu : \mathbb{T} \to [0, \infty)$ *for* $t \in \mathbb{T}$ *is defined as*

$$\mu(t) = \sigma(t) - t.$$

Definition 2.3 (Delta Derivative)**.** [2] *Let* $h : \mathbb{T} \to \mathbb{R}$ *and* $t \in \mathbb{T}^k$. *We define the delta derivative* h^Δ *also known as the Hilger derivative as*

$$h^\Delta(t) = \lim_{s \to t} \frac{h(\sigma(t)) - h(s)}{\sigma(t) - s}, \quad s \neq \sigma(t).$$

provided the limit exists.

The function $h^\Delta : \mathbb{T} \to \mathbb{R}$ *is called the (Delta) derivative of* h *on* $\mathbb{T}^k$.

If t is right dense, the delta derivative of $h : \mathbb{T} \to \mathbb{R}$, becomes

$$h^\Delta(t) = \lim_{s \to t} \frac{h(t) - h(s)}{t - s},$$

and if t is right scattered, the Delta derivative becomes

$$h^\Delta(t) = \frac{h^\sigma(t) - h(t)}{\mu(t)}.$$

For a function $h : \mathbb{T} \to \mathbb{R}$, h^σ denotes $h(\sigma(t))$.

Definition 2.4. [10] *A function* $h : \mathbb{T} \to \mathbb{R}$ *is right dense continuous if it is continuous at all right dense points of* $\mathbb{T}$ *and its left sided limits exist and is finite at left dense points of* $\mathbb{T}$. *The set of all right dense continuous functions are denoted by*

$$C_{rd} = C_{rd}(\mathbb{T}).$$

Remark 2.1. [10] *All right dense continuous functions are delta integrable.*

Definition 2.5. [10] *A function* $\phi : [0, r] \to [0, \infty)$ *is of class* $\mathcal{K}$ *if it is continuous, and strictly increasing on* $[0, r]$ *with* $\phi(0) = 0$.

Definition 2.6. [10] *A continuous function* $\mathcal{V} : \mathbb{R}^n \to \mathbb{R}$ *with* $\mathcal{V}(0) = 0$ *is called positive definite(negative definite) on the domain* D *if there exists a function* $\phi \in \mathcal{K}$ *such that* $\phi(|x|) \leq \mathcal{V}(x)$ $(\phi(|x|) \leq -\mathcal{V}(x))$ *for* $x \in D$.

Definition 2.7. [10] *A continuous function* $\mathcal{V}:\mathbb{R}^n\to\mathbb{R}$ *with* $\mathcal{V}(0)=0$ *is called positive semidefinite (negative semi-definite) on* D *if* $\mathcal{V}(x)\geq 0$ $(\mathcal{V}(x)\leq 0)$ *for all* $x\in D$ *and it can also vanish for some* $x\neq 0$.

Definition 2.8. [2] *Let* $a,b\in\mathbb{T}$ *and* $h\in C_{rd}$, *then the integration on a time scale* $\mathbb{T}$ *is defined as follows:*

(i) *If* $\mathbb{T}=\mathbb{R}$, *then*

$$\int_a^b h(t)\Delta t=\int_a^b h(t)dt,$$

where $\int_a^b h(t)dt$ *is the usual Riemann integral from calculus.*

(ii) *If* $[a,b]$ *consists of only isolated points, then*

$$\int_a^b h(t)\Delta t=\begin{cases}\sum_{t\in[a,b)}\mu(t)h(t) & \text{if}\quad a<b\\ 0 & \text{if}\quad a=b\\ -\sum_{t\in[b,a)}\mu(t)h(t) & \text{if}\quad a>b.\end{cases}$$

(iii) *If there exists a point* $\sigma(t)>t$, *then*

$$\int_t^{\sigma(t)} h(s)\Delta s=\mu(t)f(t).$$

Definition 2.9. [22] *Assume* $V\in C[\mathbb{T}\times\mathbb{R}^n,\mathbb{R}_+]$, $h\in C_{rd}[\mathbb{T}\times\mathbb{R}^n,\mathbb{R}^n]$ *and* $\mu(t)$ *is the graininess function then the dini derivative of* $V(t,x)$ *is defined as:*

$$D_-V^{\Delta}(t,x)=\liminf_{\mu(t)\to 0}\frac{V(t,x)-V(t-\mu(t),x-\mu(t)h(t,x))}{\mu(t)},\tag{4}$$

$$D^+V^{\Delta}(t,x)=\limsup_{\mu(t)\to 0}\frac{V(t+\mu(t),x+\mu(t)h(t,x))-V(t,x)}{\mu(t)}.\tag{5}$$

If V is differentiable, then $D_-V^{\Delta}(t,x)=D^+V^{\Delta}(t,x)=V^{\Delta}(t,x)$.

Definition 2.10. (Fractional Integral on Time Scales) [4]. *Let* $\alpha\in(0,1)$, $[a,b]$ *be an interval on* $\mathbb{T}$ *and* h *an integrable function on* $[a,b]$. *Then the fractional integral of order* α *of* h *is defined by*

$${}_a^{\mathbb{T}}I_t^{\alpha}h^{\Delta}(t)=\int_a^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\Delta s.$$

Definition 2.11. (Caputo Derivative on Time Scale) [3] *Let* $\mathbb{T}$ *be a time scale,* $t\in\mathbb{T},\ 0<\alpha<1$, *and* $h:\mathbb{T}\to\mathbb{R}$. *The Caputo fractional derivative of order* α *of* h *is defined by*

$${}_a^{\mathbb{T}}D_t^{\alpha}h^{\Delta}(t)=\frac{1}{\Gamma(1-\alpha)}\int_a^t(t-s)^{-\alpha}h^{\Delta^n}(s)\Delta s.$$

Lemma 2.1. [23] *Let* $\mathbb{T}$ *be a time scale with minimal element* $t_0\geq 0$. *Assume that for any* $t\in\mathbb{T}$, *there is a statement* $\mathbf{S}(t)$ *such that the following conditions are verified:*

(i) $\mathbf{S}(t_0)$ *is true;*

(ii) *If* t *is right scattered and* $\mathbf{S}(t)$ *is true, then* $\mathbf{S}(\sigma(t))$ *is also true;*

(iii) *For each right-dense t, there exists a neighborhood $\mathcal{U}$ such that whenever $\mathbf{S}(t)$ is true, $\mathbf{S}(t^*)$ is also true for all $t^* \in \mathcal{U}$, $t^* \geq t$;*

(iv) *For left dense t, $\mathbf{S}(t^*)$ is true for all $t^* \in [t_0, t)$ implies $\mathbf{S}(t)$ is true.*

Then the statement $\mathbf{S}(t)$ is true for all $t \in \mathbb{T}$.

Remark 2.2. *When $\mathbb{T} = \mathbb{N}$, then Lemma 2.1 reduces to the well-known principle of mathematical induction. That is,*

(1) *$\mathbf{S}(t_0)$ is true is equivalent to the statement is true for $n = 1$;*

(2) *$\mathbf{S}(t)$ is true then $\mathbf{S}(\sigma(t))$ is true is equivalent to if the statement is true for $n = k$, then the statement is true for $n = k + 1$.*

Now, we give the following definitions and remarks.

Definition 2.12. *Let $\mathbb{T}$ be a time scale. A point $t_0 \in \mathbb{T}$ is said to be a minimal element of $\mathbb{T}$ if, for any $t \in \mathbb{T}$, $t > t_0$ whenever $t \neq t_0$.*

Remark 2.3. *The concept of minimal element is essential in studying dynamic equations because it establishes a starting point, a reference time from which the dynamics of the system evolve. In the study of difference equations (a discrete-time setting), t_0 represents the initial time step. Similarly, in differential equations (a continuous-time setting), t_0 represents the initial time instant.*

Definition 2.13. *Let $h \in C^{\alpha}_{rd}[\mathbb{T}, \mathbb{R}^n]$, the Grunwald-Letnikov fractional delta derivative is given by*

$$^{GL\mathbb{T}}D^{\alpha}_{0} h^{\Delta}(t) = \lim_{\mu \to 0+} \frac{1}{\mu^{\alpha}} \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\,\alpha}C_r[h(\sigma(t) - r\mu)], \quad t \geq t_0, \tag{6}$$

and the Grunwald-Letnikov fractional delta dini derivative is given by

$$^{GL\mathbb{T}}D^{\alpha}_{0+} h^{\Delta}(t) = \limsup_{\mu \to 0+} \frac{1}{\mu^{\alpha}} \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\,\alpha}C_r[h(\sigma(t) - r\mu)], \quad t \geq t_0. \tag{7}$$

where $0 < \alpha < 1$, $^{\alpha}C_r = \frac{q(q-1)\ldots(q-r+1)}{r!}$, and $[\frac{(t-t_0)}{\mu}]$ denotes the integer part of the fraction $\frac{(t-t_0)}{\mu}$.

Observe that if the domain is $\mathbb{R}$, then (7) becomes

$$^{GL\mathbb{T}}D^{\alpha}_{0+} h^{\Delta}(t) = \limsup_{d \to 0+} \frac{1}{d^{\alpha}} \sum_{r=0}^{[\frac{(t-t_0)}{d}]} (-1)^{r\,\alpha}C_r[h(t - rd)], \quad t \geq t_0.$$

Remark 2.4. *It is necessary to note that the relationship between the Caputo fractional delta derivative and the Grunwald-Letnikov fractional delta derivative is given by*

$$^{C\mathbb{T}}D^{\alpha}_{0} h^{\Delta}(t) = ^{GL\mathbb{T}} D^{\alpha}_{0}[h(t) - h(t_0)]^{\Delta}, \tag{8}$$

substituting (6) into (8) we have that the Caputo fractional delta derivative becomes

$$^{C\mathbb{T}}D_0^{\alpha}h^{\Delta}(t) = \lim_{\mu\to 0+}\frac{1}{\mu^{\alpha}}\sum_{r=0}^{[\frac{(t-t_0)}{\mu}]}(-1)^{r\,\alpha}C_r[h(\sigma(t)-r\mu)-h(t_0)] \quad t\geq t_0$$

$$^{C\mathbb{T}}D_0^{\alpha}h^{\Delta}(t) = \lim_{\mu\to 0+}\frac{1}{\mu^{\alpha}}\Big\{h(\sigma(t))-h(t_0)+\sum_{r=1}^{[\frac{(t-t_0)}{\mu}]}(-1)^{r\,\alpha}C_r[h(\sigma(t)-r\mu)-h(t_0)]\Big\}, \tag{9}$$

and the Caputo fractional delta Dini derivative becomes

$$^{C\mathbb{T}}D_{0+}^{\alpha}h^{\Delta}(t) = \limsup_{\mu\to 0+}\frac{1}{\mu^{\alpha}}\sum_{r=0}^{[\frac{(t-t_0)}{\mu}]}(-1)^{r\,\alpha}C_r[h(\sigma(t)-r\mu)-h(t_0)], \quad t\geq t_0. \tag{10}$$

Which is equivalent to

$$^{C\mathbb{T}}D_{0+}^{\alpha}h^{\Delta}(t) = \limsup_{\mu\to 0+}\frac{1}{\mu^{\alpha}}\Big\{h(\sigma(t))-h(t_0)+\sum_{r=1}^{[\frac{(t-t_0)}{\mu}]}(-1)^{r\,\alpha}C_r[h(\sigma(t)-r\mu)-h(t_0)]\Big\}, \quad t\geq t_0. \tag{11}$$

For notation simplicity, we shall represent the Caputo fractional delta derivative of order α as $^{C\mathbb{T}}D^{\alpha}$ and the Caputo fractional delta dini derivative of order α as $^{C\mathbb{T}}D_+^{\alpha}$.

Now, we introduce the derivative of the Lyapunov function using the Caputo fractional delta Dini derivative of $h(t)$ given in (10).

Definition 2.14. *We define the Caputo fractional delta Dini derivative of the Lyapunov function $V(t,x)\in C[\mathbb{T}\times\mathbb{R}^n,\mathbb{R}_+]$ (which is locally Lipschitzian with respect to its second argument and $V(t,0)\equiv 0$) along the trajectories of solutions of the system (2) as:*

$$^{C\mathbb{T}}D_+^{\alpha}V^{\Delta}(t,x) = \limsup_{\mu\to 0^+}\frac{1}{\mu^{\alpha}}\Bigg[\sum_{r=0}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^{\alpha}C_r)[V(\sigma(t)-r\mu,x(\sigma(t))-\mu^{\alpha}f(t,x(t))-V(t_0,x_0)]\Bigg],$$

and can be expanded as

$$^{C\mathbb{T}}D_+^{\alpha}V^{\Delta}(t,x) = \limsup_{\mu\to 0^+}\frac{1}{\mu^{\alpha}}\Big\{V(\sigma(t),x(\sigma(t))-V(t_0,x_0) \tag{12}$$
$$-\sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^{r+1}({}^{\alpha}C_r)[V(\sigma(t)-r\mu,x(\sigma(t))-\mu^{\alpha}f(t,x(t))-V(t_0,x_0)]\Big\},$$

where $t\in\mathbb{T}, x, x_0\in\mathbb{R}^n$, $\mu=\sigma(t)-t$ and $x(\sigma(t))-\mu^{\alpha}f(t,x)\in\mathbb{R}^n$.

If $\mathbb{T}$ is discrete and $V(t,x(t))$ is continuous at t, the Caputo fractional delta Dini derivative of the Lyapunov function in discrete times, is given by:

$$^{C\mathbb{T}}D_+^{\alpha}V^{\Delta}(t,x) = \frac{1}{\mu^{\alpha}}\Bigg[\sum_{r=0}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^{\alpha}C_r)(V(\sigma(t),x(\sigma(t)))-V(t_0,x_0))\Bigg], \tag{13}$$

and if $\mathbb{T}$ is continuous, that is $\mathbb{T} = \mathbb{R}$, and $V(t, x(t))$ is continuous at t, we have that

$$
{}^{C\mathbb{T}}D_{+}^{\alpha}V^{\Delta}(t,x) = \limsup_{d\to 0^{+}} \frac{1}{d^{\alpha}}\Big\{V(t,x(t)) - V(t_0,x_0) - \sum_{r=1}^{[\frac{t-t_0}{d}]}(-1)^{r+1}({}^{\alpha}C_r)[V(t-rd,x(t)) - d^{\alpha}f(t,x(t)) - V(t_0,x_0)]\Big\}. \tag{14}
$$

Notice that (14) is the same in [1] where $d > 0$.

Given that $\lim_{N\to\infty}\sum_{r=0}^{N}(-1)^{r\alpha}C_r = 0$ where $\alpha \in (0,1)$, and $\lim_{\mu\to 0^{+}}[\frac{(t-t_0)}{\mu}] = \infty$ then it is easy to see that

$$
\lim_{\mu\to 0^{+}} \sum_{r=1}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha}C_r = -1. \tag{15}
$$

Also from (10) and since the Caputo and Riemann-Liouville formulations coincide when $h(t_0) = 0$, ([1]) then we have that

$$
\limsup_{\mu\to 0^{+}} \frac{1}{\mu^{\alpha}} \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha}C_r = {}^{RL\mathbb{T}}D^{\alpha}(1) = \frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}, \quad t \ge t_0. \tag{16}
$$

Definition 2.15. *The trivial solution of (2) is said to be Uniformly stable if for every $\epsilon > 0$ and $t_0 \in \mathbb{T}$, there exist $\delta = \delta(\epsilon) > 0$(independent of t) such that for any $x_0 \in \mathbb{R}^n$, the inequality $\|x_0\| < \delta$ implies $\|x(t;t_0,x_0)\| < \epsilon$ for $t \ge t_0$.*

Lemma 2.2. [*see* [21]] *Assume that*

(*i*) $g \in C_{rd}[\mathbb{T}\times\mathbb{R}_+,\mathbb{R}_+]$ *and $g(t,u)\mu$ is non-decreasing in u.*

(*ii*) $V \in C[\mathbb{T}\times\mathbb{R}^n,\mathbb{R}_+]$ *be locally Lipschitzian in the second variable such that*

$$
{}^{C\mathbb{T}}D_{+}^{\alpha}V^{\Delta}(t,x) \le g(t,V(t,x)), (t,x) \in \mathbb{T}\times\mathbb{R}^n. \tag{17}
$$

(*iii*) $z(t) = z(t;t_0,u_0)$ *is the maximal solution of (3) existing on $\mathbb{T}$.*

Then

$$
V(t,x(t)) \le z(t), \quad t \ge t_0, \tag{18}
$$

provided that

$$
V(t_0,x_0) \le u_0, \tag{19}
$$

where $x(t) = x(t;t_0,x_0)$ is any solution of (2), $t \in \mathbb{T}, \ \ t \ge t_0$.

3. Main Result

In this section, we will obtain sufficient conditions for the uniform stability of system (12).

Theorem 3.1 (Uniform Stability)**.** *Assume the following*

(1) *$g \in C_{rd}[\mathbb{T} \times \mathbb{R}_+, \mathbb{R}_+]$ and $g(t,u)$ is non-decreasing in u with $g(t,u) \equiv 0$.*

(2) *$V(t,x(t)) \in C[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}_+]$ be such that*

(i) V is locally Lipschitzian in x with $V(t,0) \equiv 0$.

(ii) $b(\|x\|) \le V(t,x) \le a(\|x\|)$ where $a, b \in \mathcal{K}$.

(iii) For any points $t, t_0 \ge 0$ and $x, x_0 \in \mathbb{R}^n$, the inequality

$$^{C\mathbb{T}}D_+^{\alpha} V^{\Delta}(t,x(t)) \le g(t, V(t,x(t))),$$

holds.

(3) *The trivial solution of FrDE (3) is uniformly stable.*

Then the trivial solution of the FrDE (2) is uniformly stable.

Proof. Let $\epsilon \in (0,\rho)$ and $t_0 \in \mathbb{T}$ be given. Assume that the trivial solution $u = 0$ of (3) is uniformly stable. Then given $b(\epsilon) > 0$ and $t_0 \in \mathbb{T}$, there exist a $\delta = \delta(\epsilon) > 0$ such that

$$u_0 < \delta \quad \Longrightarrow \quad u(t;t_0,u_0) < b(\epsilon), \quad \text{for } t \ge t_0, \tag{20}$$

where $u(t;t_0,u_0)$ is any solution of (3). Let $z(t) = z(t;t_0,u_0)$ be the maximal solution of (3), then consequent of (20),

$$u_0 < \delta \quad \Longrightarrow \quad z(t) < b(\epsilon), \quad \text{for } t \ge t_0. \tag{21}$$

$V(t,0) = 0$ and $V \in C_{rd}$ this implies that V is continuous at the origin, then given $\delta > 0$, we can find a $\delta_1 = \delta_1(\delta) > 0$, such that, for $x_0 \in \mathbb{R}^N$, we have that,

$$\|x_0\| < \delta_1 \quad \Longrightarrow \quad V(t_0,x_0) < \delta. \tag{22}$$

Let $x(t) = x(t;t_0,x_0)$ be any solution of (2), with $\|x_0\| < \delta_1$.
Claim:

$$\|x(t)\| < \epsilon, \ t \ge t_0. \tag{23}$$

Assuming (23) is not true, then there would exist a time $t_1 > t_0$, such that $\|x(t_1)\| = \epsilon$ and $\|x(t)\| < \epsilon$ for all $t \in [t_0, t_1)$.
Let $V(t_0,x_0) \le u_0$, then it follows from Lemma 2.2 that

$$V(t,x(t)) \le z(t). \tag{24}$$

Combining (21), (24) and from condition (ii) of the Theorem, we have that, at $t = t_1$,

$$b(\|x(t_1)\|) \le V(t_1,x(t_1)) \le z(t_1) < b(\epsilon).$$

Then by our assumption that $\|x(t_1)\| = \epsilon$, we have that,

$$b(\epsilon) \leq V(t_1, x(t_1)) \leq z(t_1) < b(\epsilon),$$

which is a contradiction. This contradiction proves that (23) is true. i.e for arbitrary $\epsilon \in (0, \rho), t_0 \in \mathbb{R}_+$, there exist $\delta_1(\epsilon)$ (independent of t_0) such that $\|x_0\| < \delta_1$ implies $\|x(t)\| < \epsilon$, for all $t \geq t_0$. Hence , we conclude that the trivial solution $x = 0$ of (2) is uniformly stable. □

4. Application

Consider the system of dynamic equations

$$\begin{aligned} {}^{C\mathbb{T}}D^{\alpha}\chi_1^{\Delta}(t) &= \chi_1 - \chi_1 \exp(\chi_1) - 2\chi_2 - \chi_1 \exp(\chi_2) + \chi_1\chi_2^2 \\ {}^{C\mathbb{T}}D^{\alpha}\chi_2^{\Delta}(t) &= 2\chi_1 + \chi_2 - \chi_2 \exp(\chi_1) - \chi_2 \exp(\chi_2) - \chi_1^2\chi_2, \end{aligned} \tag{25}$$

for $t \geq t_0$, with initial conditions

$$\chi_1(t_0) = \chi_{10} \quad \text{and} \quad \chi_2(t_0) = \chi_{20},$$

where $\chi = (\chi_1, \chi_2)$ and $f = (f_1, f_2)$

Consider $V(t, \chi_1, \chi_2) = \chi_1^2 + \chi_2^2$, for $t \in \mathbb{T}$, $(\chi_1, \chi_2) \in \mathbb{R}^2$ and choose $\alpha = 1$, so that (25) becomes an integer (first) order system. Then we compute the delta Dini derivative of $V(t, \chi_1, \chi_2) = \chi_1^2 + \chi_2^2$ along the solution path of (25) as follows:

From (5) we have that

$$\begin{aligned} D^+V^{\Delta}(t,\chi) &= \limsup_{\mu(t)\to 0} \frac{V(t+\mu(t), \chi + \mu(t)f(t,\chi)) - V(t,\chi)}{\mu(t)} \\ &= \limsup_{\mu(t)\to 0} \frac{(\chi_1 + \mu(t)f_1(t,\chi_1,\chi_2))^2 + (\chi_2 + \mu(t)f_2(t,\chi_1,\chi_2))^2 - [\chi_1^2 + \chi_2^2]}{\mu(t)} \\ &= \limsup_{\mu(t)\to 0} \frac{\chi_1^2 + 2\chi_1\mu(t)f_1 + \mu^2(t)f_1^2 + \chi_2^2 + 2\chi_2\mu(t)f_2 + \mu^2(t)f_2^2 - [\chi_1^2 + \chi_2^2]}{\mu(t)} \\ &= \limsup_{\mu(t)\to 0} \frac{2\chi_1\mu(t)f_1 + \mu^2(t)f_1^2 + 2\chi_2\mu(t)f_2 + \mu^2(t)f_2^2}{\mu(t)} \\ &\leq 2\chi_1 f_1 + 2\chi_2 f_2 \\ &= 2\chi_1(\chi_1 - \chi_1\exp(\chi_1) - 2\chi_2 - \chi_1\exp(\chi_2) + \chi_1\chi_2^2) \\ &\quad +2\chi_2(2\chi_1 + \chi_2 - \chi_2\exp(\chi_1) - \chi_2\exp(\chi_2) - \chi_1^2\chi_2) \\ &= 2\left[\chi_1^2 - \chi_1^2\exp(\chi_1) - \chi_1^2\exp(\chi_2) + \chi_2^2 - \chi_2^2\exp(\chi_1) - \chi_2^2\exp(\chi_2)\right] \\ &= 2\left[\chi_1^2 + \chi_2^2 - (\chi_1^2 + \chi_2^2)(\exp(\chi_1) + \exp(\chi_2))\right] \\ &\leq 2\left[\chi_1^2 + \chi_2^2\right] \\ D^+V^{\Delta}(t,x) &\leq 2V^2(t,\chi_1,\chi_2). \end{aligned}$$

Now consider the consider the comparison equation

$$D^+u^\Delta = 2u^2 > 0,\ u(0) = u_0. \tag{26}$$

Even though conditions (i)-(iii) of [22] are satisfied that is $V \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}_+]$, $D^+V^\Delta(t,\chi) \leq g(t, V(t,\chi))$ and $\sqrt{\chi_1^2+\chi_2^2} \leq \chi_1^2+\chi_2^2 \leq 2(\chi_1^2+\chi_2^2)$, for $b(\|\chi\|) = r$ and $a(\|\chi\|) = 2r^2$, it is obvious to see that the solution of the comparison system (26) is not uniformly stable, so we can not deduce the uniform stability properties of the system (25) by applying the basic definition of the Dini-derivative of a Lyapunov function of dynamic equation on time scale to the Lyapunov function $V(t,\chi_1,\chi_2) = \chi_1^2+\chi_2^2$.

Let us consider (25) with $\alpha \in (0,1)$ and apply the new definition (12).

For $V(t,\chi_1,\chi_2) = \chi_1^2+\chi_2^2$, for $t \in \mathbb{T}$ and $(\chi_1,\chi_2) \in \mathbb{R}^2$. Then condition 1 of Theorem (3.1) is satisfied, for $b(\|\chi\|) \leq V(t,\chi) \leq a(\|\chi\|)$, with $b(r) = r$, $a(r) = 2r^2$, $a, b \in \mathcal{K}$, so that the associated norm $\|\chi\| = \sqrt{\chi_1^2+\chi_2^2}$.

Since

$$V(t,\chi_1,\chi_2) = \chi_1^2+\chi_2^2,$$

then $\sqrt{\chi_1^2+\chi_2^2} \leq \chi_1^2+\chi_2^2 \leq 2(\chi_1^2+\chi_2^2)$. From (12), we compute the Caputo fractional Dini derivative for $V(t,\chi_1,\chi_2) = \chi_1^2+\chi_2^2$ as follows

$$\begin{aligned}
{}^{C\mathbb{T}}D_+^\alpha V^\Delta(t,\chi) &= \limsup_{\mu\to 0^+} \frac{1}{\mu^\alpha}\Big\{ V(\sigma(t),\chi(\sigma(t)) - V(t_0,\chi_0) \\
&\quad - \sum_{r=1}^{[\frac{t-t_0}{\mu}]} (-1)^{r+1}({}^\alpha C_r)[V(\sigma(t)-r\mu, \chi(\sigma(t)) - \mu^\alpha f(t,\chi(t))) - V(t_0,\chi_0)]\Big\} \\
&= \limsup_{\mu\to 0^+} \frac{1}{\mu^\alpha}\Big\{ \left[(\chi_1(\sigma(t)))^2 + (\chi_2^2(\sigma(t)))^2\right] - \left[(\chi_{10})^2+(\chi_{20})^2\right] \\
&\quad + \sum_{r=1}^{[\frac{t-t_0}{\mu}]} (-1)^r({}^\alpha C_r)[(\chi_1(\sigma(t)) - \mu^\alpha f_1(t,\chi_1,\chi_2))^2 \\
&\quad + (\chi_2(\sigma(t)) - \mu^\alpha f_2(t,\chi_1,\chi_2))^2((\chi_{10})^2+(\chi_{20})^2)]\Big\} \\
&= \limsup_{\mu\to 0^+} \frac{1}{\mu^\alpha}\Big\{ \left[(\chi_1(\sigma(t)))^2 + (\chi_2^2(\sigma(t)))^2\right] - \left[(\chi_{10})^2+(\chi_{20})^2\right] \\
&\quad + \sum_{r=1}^{[\frac{t-t_0}{\mu}]} (-1)^r({}^\alpha C_r)[(\chi_1(\sigma(t)))^2 - 2\chi_1(\sigma(t))\mu^\alpha f_1(t,\chi_1,\chi_2) + \mu^{2\alpha}(f_1(t,\chi_1,\chi_2))^2 \\
&\quad + (\chi_2(\sigma(t)))^2 - 2\chi_2(\sigma(t))\mu^\alpha f_2(t,\chi_1,\chi_2) + \mu^{2\alpha}(f_2(t,\chi_1,\chi_2))^2 - ((\chi_{10})^2+(\chi_{20})^2)]\Big\} \\
&= \limsup_{\mu\to 0^+} \frac{1}{\mu^\alpha}\Big\{ - \sum_{r=0}^{[\frac{t-t_0}{\mu}]} (-1)^r({}^\alpha C_r)\left[(\chi_{10})^2+(\chi_{20})^2\right]
\end{aligned}$$

$$+\sum_{r=0}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^{\alpha}C_r)\left[(\chi_1(\sigma(t)))^2+(\chi_2(\sigma(t)))^2\right]$$

$$-\sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^{\alpha}C_r)\left[2\chi_1(\sigma(t))\mu^{\alpha}f_1(t,\chi_1,\chi_2)+2\chi_2(\sigma(t))\mu^{\alpha}f_2(t,\chi_1,\chi_2)\right]$$

$$+\sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^{\alpha}C_r)\left[\mu^{2\alpha}(f_1(t,\chi_1,\chi_2))^2+\mu^{2\alpha}(f_2(t,\chi_1,\chi_2))^2\right]\Bigg\}$$

$$= -\limsup_{\mu\to 0^+}\frac{1}{\mu^{\alpha}}\left\{\sum_{r=0}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^{\alpha}C_r)\left[(\chi_{10})^2+(\chi_{20})^2\right]\right\}$$

$$+\limsup_{\mu\to 0^+}\frac{1}{\mu^{\alpha}}\left\{\sum_{r=0}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^{\alpha}C_r)\left[(\chi_1(\sigma(t)))^2+(\chi_2(\sigma(t)))^2\right]\right\}$$

$$-\limsup_{\mu\to 0^+}\frac{1}{\mu^{\alpha}}\left\{\sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^{\alpha}C_r)[2\chi_1(\sigma(t))\mu^{\alpha}f_1(t,\chi_1,\chi_2+2\chi_2(\sigma(t))\mu^{\alpha}f_2(t,\chi_1,\chi_2)\right\}.$$

Applying (15) and (16) we have

$$\begin{aligned}
&= -\frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}\left((\chi_{10})^2+(\chi_{20})^2\right)+\frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}((\chi_1(\sigma(t)))^2+(\chi_2(\sigma(t)))^2)\\
&-\left[2x_1(\sigma(t))f_1(t,\chi_1,\chi_2)+2\chi_2(\sigma(t))f_2(t,\chi_1,\chi_2)\right]\\
&\le \frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}((\chi_1(\sigma(t)))^2+(\chi_2(\sigma(t)))^2)\\
&-\left[2\chi_1(\sigma(t))f_1(t,\chi_1,\chi_2)+2\chi_2(\sigma(t))f_2(t,\chi_1,\chi_2)\right],
\end{aligned}$$

As $t\to\infty$, $\frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}((\chi_1(\sigma(t)))^2+(\chi_2(\sigma(t)))^2)\to 0$, then

$$\begin{aligned}
{}^{C\mathbb{T}}D_+^{\alpha}V^{\Delta}(t;\chi_1,\chi_2) &\le -\left[2\chi_1(\sigma(t))f_1(t,\chi_1,\chi_2)+2\chi_2(\sigma(t))f_2(t,\chi_1,\chi_2)\right]\\
&= -2\left[\chi_1(\sigma(t))f_1(t,\chi_1,\chi_2)+\chi_2(\sigma(t))f_2(t,\chi_1,\chi_2)\right],
\end{aligned}$$

applying $\chi(\sigma(t))\le\mu^{C\mathbb{T}}D^{\alpha}x(t)+x(t)$

$$\begin{aligned}
&= -2\Big[\mu(t)f_1^2(t,\chi_1,\chi_2)+\chi_1(t)f_1(t,\chi_1,\chi_2)+\mu(t)f_2^2(t,\chi_1,\chi_2)+\chi_2(t)f_2(t,\chi_1,\chi_2)\Big]\\
&= -2\Big[\mu(t)(\chi_1-\chi_1\exp(\chi_1)-2\chi_2-\chi_1\exp(\chi_2)+\chi_1\chi_2^2)^2\\
&\quad+\chi_1(\chi_1-\chi_1\exp(\chi_1)-2\chi_2-\chi_1\exp(\chi_2)+\chi_1\chi_2^2)\\
&\quad+\mu(t)(2\chi_1+\chi_2-\chi_2\exp(\chi_1)-\chi_2\exp(\chi_2)-\chi_1^2\chi_2)^2\\
&\quad+\chi_2(2\chi_1+\chi_2-\chi_2\exp(\chi_1)-\chi_2\exp(\chi_2)-\chi_1^2\chi_2)\Big]\\
&= -2\Big[\chi_1^2+\chi_2^2+\mu(t)(\chi_1-\chi_1\exp(\chi_1)-2\chi_2-\chi_1\exp(\chi_2)+\chi_1\chi_2^2)^2
\end{aligned}$$

$$+\mu(t)(2\chi_1+\chi_2-\chi_2\exp(\chi_1)-\chi_2\exp(\chi_2)-\chi_1^2\chi_2)^2\Big]$$
$$= -2\Big[\chi_1^2+\chi_2^2\Big]-2\mu(t)\Big[(\chi_1-\chi_1\exp(\chi_1)-2\chi_2-\chi_1\exp(\chi_2)+\chi_1\chi_2^2)^2$$
$$+(2\chi_1+\chi_2-\chi_2\exp(\chi_1)-\chi_2\exp(\chi_2)-\chi_1^2\chi_2)^2\Big]. \quad (27)$$

If $\mathbb{T}=\mathbb{R}$ we have that $\mu=0$, so that (27) becomes;

$$=-2\Big[\chi_1^2+\chi_2^2\Big].$$

Therefore,

$$^{C\mathbb{T}}D_+^\alpha V^\Delta(t;\chi_1,\chi_2)\le -2V(t,\chi_1,\chi_2).$$

Consider the comparison system

$$^{C\mathbb{T}}D_+^\alpha u^\Delta = g(t,u)\le -2u \quad (28)$$
$$^{C\mathbb{T}}D_+^\alpha u^\Delta + 2u = 0.$$

Applying the Laplace transform method, we obtain

$$u(t)=u_0E_{\alpha,1}(-2t^\alpha),\quad \text{for } \alpha\in(0,1). \quad (29)$$

Now, let $u_0<\delta$, then from (29), we have $u(t)=2u_0E_{\alpha,1}<2E_{\alpha,1}<\epsilon$ whenever $u_0<\delta=\frac{\epsilon}{2E_{\alpha,1}}$.

Therefore given $\epsilon>0$, we can find a $\delta(\epsilon)>0$ (independent of t) such that $u(t)<\epsilon$ whenever $u_0<\delta$

If $\mathbb{T}=\mathbb{N}_0$ we have that $\mu=1$, so that (27) becomes;

$$= -2\Big[\chi_1^2+\chi_2^2\Big]-2\Big[(\chi_1-\chi_1\exp(\chi_1)-2\chi_2-\chi_1\exp(\chi_2)+\chi_1\chi_2^2)^2$$
$$+(2\chi_1+\chi_2-\chi_2\exp(\chi_1)-\chi_2\exp(\chi_2)-\chi_1^2\chi_2)^2\Big]$$
$$\le -2\Big[\chi_1^2+\chi_2^2\Big]$$
$$^{C\mathbb{T}}D_+^\alpha V^\Delta(t;x_1,x_2) \le -2\Big[\chi_1^2+\chi_2^2\Big].$$

We can also consider same comparison system as (28) leading to the same conclusion as (29) Since all the conditions of Theorem 3.1 are satisfied, and trivial solution of the comparison system (28) is stable, then we conclude that the trivial solution of system (25) is stable.

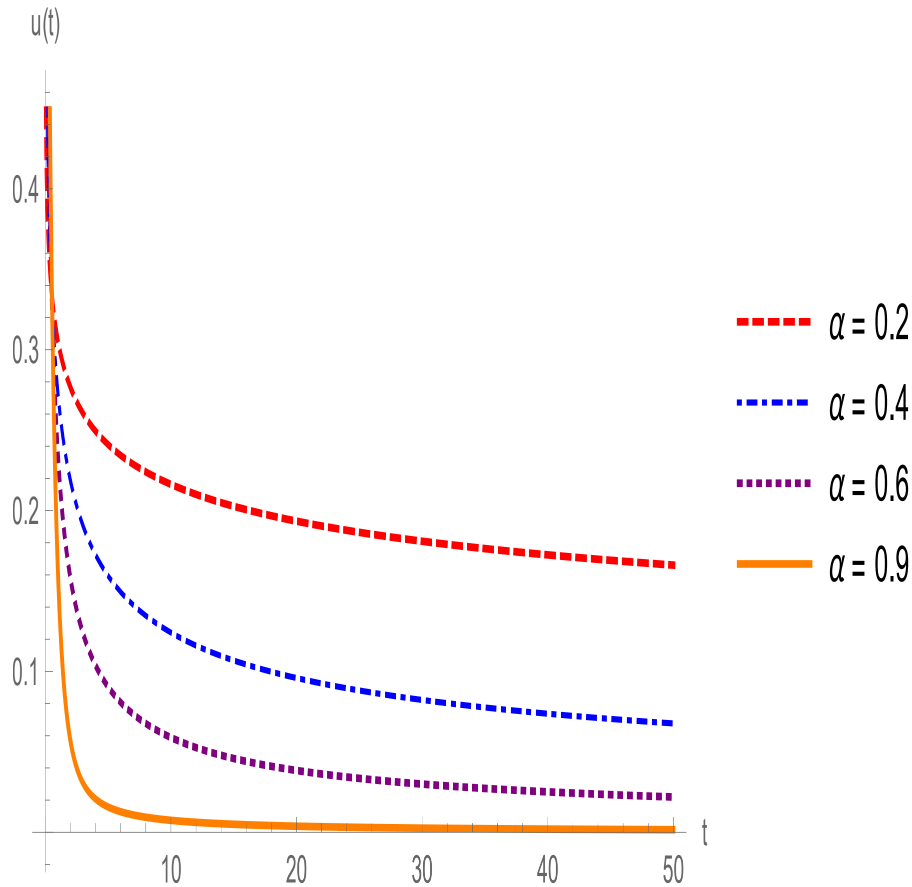

FIGURE 1. Graph of $u(t) = E_{\alpha,1}(-2t^{\alpha})$ against t for various values of α

Figure 1 above is the graphical representation of $u(t) = E_{\alpha,1}(-2t^{\alpha})$. The behaviour of the curves further buttresses the stability of the solution $u(t)$ of over time for different values of $\alpha \in (0,1)$.

5. Conclusion

In this paper, we have introduced a novel approach to establishing uniform stability criteria for Caputo fractional dynamic equations on arbitrary time domains. By developing comparison results and uniform stability criteria based on the Caputo fractional delta derivative and Caputo fractional delta Dini derivative, we have created a unified framework for uniform stability analysis on time scales. This framework bridges the gap between continuous and discrete time domains, providing a robust tool for predicting stable behavior in complex dynamic systems. The significance of this research lies in its potential to ensure reliable outcomes in various applications, including control theory, signal processing, and engineering. By providing a unified framework for stability analysis, we have paved the way for further research into the stability of fractional dynamic systems. We have also shown the practical applicability of our results as well as effectiveness using system (25).

Authors' contributions. All authors contributed equally to the manuscript.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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