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ASYMPTOTIC PROPERTIES OF SOLUTIONS TO AN INFECTIOUS DISEASE MODEL WITH VACCINATION, LATENCY, AND SWITCHING TRANSMISSION

LU CHRISTIAN S. ONG*, MARK ANTHONY C. TOLENTINO, TIMOTHY ROBIN Y. TENG

Department of Mathematics, Ateneo de Manila University, Philippines

*Corresponding author: lu.ong@student.ateneo.edu

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Abstract. During the COVID-19 pandemic, governments around the world resorted to the imposition of varying levels of lockdown measures to mitigate the spread of the disease. Mathematically, such measures can be incorporated in COVID-19 transmission models by using a switching transmission rate, where the switching times correspond to the dates when the imposed lockdown measures change in stringency. In this work, we employ such a switching transmission rate in a Susceptible-Vaccinated-Exposed-Infectious-Quarantined-Recovered (SVEIQR) compartmental model and carry out an analysis of this model and its solutions. We first discuss fundamental properties of solutions and determine the unique equilibrium that corresponds to the disease-free state. Using tools from the theory of switched systems, we then establish sufficient conditions for the global attractivity of this disease-free equilibrium. We also present several simulations to illustrate our results and to explore the rich dynamics of solutions in some scenarios. This paper extends previous studies in the literature that have focused on SIR, SEIR, and SEIQR models with switching transmission and provides a better understanding of the effects of lockdown measures on long-term disease transmission dynamics.

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1. Introduction

During the COVID-19 pandemic, governments around the world resorted to the imposition of varying levels of lockdown measures to mitigate the spread of the disease. In the Philippines, for example, a crucial component of the government's pandemic response plan is the imposition of Community Quarantines (CQs). These are policy restrictions that inhibit public mobility via lockdowns and restrict operations of various work sectors. CQs are classified into four levels according to their degree of stringency. Listed from most restrictive to least, the classifications are Enhanced Community Quarantine or ECQ; Modified Enhanced Community Quarantine or MECQ; General Community

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Quarantine or GCQ; and Modified General Community Quarantine or MGCQ. The guidelines for each CQ classification can be found in [1].

Throughout the pandemic, different CQ policies have been imposed based on the policymakers' assessment of the COVID-19 situation. To illustrate, we recount the changes in CQ policies for the National Capital Region (NCR) of the Philippines starting March 2020. Shortly after the announcement regarding the first local COVID-19 transmission in NCR, ECQ was imposed in the region to prevent further spread of the COVID-19 disease. This policy intervention constituted stay-at-home orders, school closures, skeletal workforce on agencies that provide essential services, among others. However, as this was not a sustainable intervention strategy, policymakers had to carefully consider how to safely exit from ECQ. This eventually led to the imposition of MECQ two months after ECQ was first announced. And then on June 1, 2020, the community quarantine restriction was further relaxed to GCQ, which increased human mobility and economic activity. However, this eventually led to a large wave of infections that threatened to overwhelm the health system in NCR. Thus, the government reverted the CQ level to MECQ on August 4, 2020, with health officials encouraging the general public to follow minimum health standards [2]. The CQ level was relaxed to GCQ after 2 weeks, when case numbers in NCR started to decline.

Mathematically, these multiple shifts in CQ policies (i.e., shifts in the stringency level of lockdown measures) can be incorporated in COVID-19 transmission models by using a switching transmission rate, where the switching times correspond to the dates when the imposed lockdown measures change in stringency. Particularly, compartmental models with this feature have been analyzed in the literature. In [3–5], SIR and SEIR models involving switching transmission rates have been analyzed. In [6], similar analysis has been carried out for a compartmental model that incorporates a quarantined state for individuals. In this work, we extend these previous studies by considering a Susceptible-Vaccinated-Exposed-Infectious-Quarantined-Recovered (SVEIQR) model with a switching transmission rate.

In the next section, we provide the details of the SVEIQR model to be studied in this paper. Fundamental properties of solutions to the SVEIQR model are then discussed in Section 3. Particularly, it will be determined that the model has a unique equilibrium that corresponds to the disease-free state. In Section 4, we use tools from the theory of switched systems (particularly [7]) to establish sufficient conditions for the global attractivity of this disease-free equilibrium. Finally, in Section 5, we present several simulations to illustrate our results and to explore the rich dynamics of solutions in some scenarios.

Pronouncements on community quarantine restrictions in NCR over the period of March 2020 to August 2020 are based on IATF Resolution Nos. 11, 12, 13, 20, 28, 35, 40, 41, 63 and 66. These documents are found in https://doh.gov.ph/diseases/covid-19/iatf-resolutions.

2. The SVEIQR Model

2.1. **Compartmental Structure.** The focus of this study is a nonautonomous nonlinear system of differential equations that correspond to an SVEIQR compartmental structure [8] for modelling the dynamics of an infectious disease in a population. We present a schematic of the compartmental model in Figure 1. The model's compartments and parameters are summarized in Table 1 and Table 2, respectively. Note that aside from the transmission rate $\beta(t)$, all other model parameters are assumed to be positive constants. Additional details on the model may be found in [8]. This compartmental structure has also been studied in [9] in the case where all model parameters are assumed to be constant.

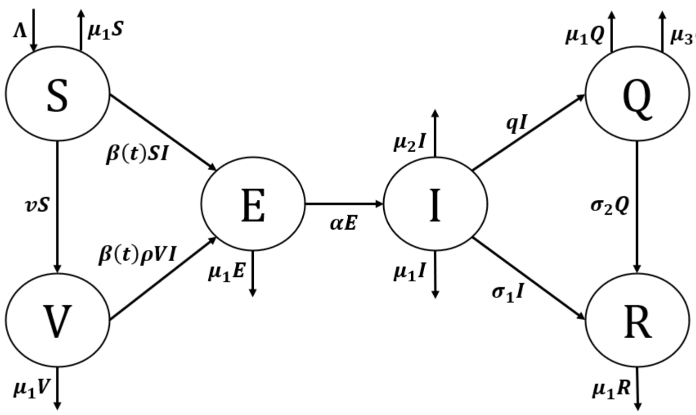


Figure 1. The SVEIQR model. The arrows depict inflows or outflows for each compartment. Shown on each arrow is the model parameter that represents the corresponding transfer rate. With the exception of Λ, all the transfer rates in the model are proportional rates.

Table 1. Compartments in the SVEIQR Model

Symbol	Compartment	Description
S	Susceptible	Individuals who are susceptible to the disease.
V	Vaccinated	Individuals who are susceptible to the disease but are vaccinated, which provides partial protection against infection.
E	Exposed	Individuals who have been infected with the disease but are yet to become infectious (i.e., disease is still in latency).
I	Infectious	Individuals who have been infected with the disease and have become infectious.
Q	Quarantined	Individuals who have quarantined (e.g., in a medical facility or self-isolating at home) and are thus assumed no longer capable of infecting others with the disease.
R	Recovered	Individuals who have recovered from the disease and are assumed to have received lasting immunity to the disease.

TABLE 2. Parameters of the SVEIQR Model

Parameter	Description
$\beta(t)$	Time-varying transmission rate
Λ	Recruitment rate (i.e., number of individuals who enter S per unit time)
α	Progression rate (i.e., reciprocal of the average length of the latency period)
q	Rate for an infectious individual to go into quarantine
v	Vaccination rate
ρ	Vaccine effectiveness parameter
μ_1	Natural death rate (i.e., not due to the disease) for the entire population
μ_2	Disease-caused death rate for individuals in I
μ_3	Disease-caused death rate for individuals in Q
σ_1	Recovery rate for individuals in I
σ_2	Recovery rate for individuals in Q

2.2. **System of Differential Equations and Initial Value Problem.** At time t, denote by $S(t), V(t), E(t), I(t), Q(t), R(t)$ the number of individuals in the compartments S, V, E, I, Q, R, respectively. The system of differential equations corresponding to the SVEIQR Model is then given by the following:

$$S'(t) = \Lambda - \beta(t)S(t)I(t) - (\mu_1 + v)S(t), \tag{1}$$

$$V'(t) = vS(t) - \beta(t)\rho V(t)I(t) - \mu_1 V(t), \tag{2}$$

$$E'(t) = \beta(t)[S(t) + \rho V(t)]I(t) - (\mu_1 + \alpha)E(t), \tag{3}$$

$$I'(t) = \alpha E(t) - (\mu_1 + \mu_2 + q + \sigma_1)I(t), \tag{4}$$

$$Q'(t) = qI(t) - (\mu_1 + \mu_3 + \sigma_2)Q(t), \tag{5}$$

$$R'(t) = \sigma_1 I(t) + \sigma_2 Q(t) - \mu_1 R(t), \tag{6}$$

with equations (1)-(6) required to hold for all t in some interval that corresponds to the relevant period of interest.

We introduce the *biologically feasible region* $\mathcal{D}$ for the system (1)-(6):

$$\mathcal{D} = \left\{ (S, V, E, I, Q, R) \in \mathbb{R}^6_{\geq 0} : S + V + E + I + Q + R \leq \frac{\Lambda}{\mu_1} \right\}.$$

Moving forward, we consider only initial value problems whose initial condition is in $\mathcal{D}$. Now, let $t_0 \geq 0$ and $(S(t_0), V(t_0), E(t_0), I(t_0), Q(t_0), R(t_0)) \in \mathcal{D}$. By designating t_0 as the initial time, the system (1)-(6) and the initial condition $(S(t_0), V(t_0), E(t_0), I(t_0), Q(t_0), R(t_0)) = (S_{t_0}, V_{t_0}, E_{t_0}, I_{t_0}, Q_{t_0}, R_{t_0})$ form an initial value problem, which we refer to as IVP $(\text{SVEIQR})_{t_0}$.

3. Properties of Solutions

3.1. **Continuous Time-varying Transmission Rate.** Before we consider a switching transmission rate, we first establish properties of solutions when $\beta(t)$ is an arbitrary continuous function of t. More precisely, we assume that:

(C1) $\beta(t)$ is continuous on $[t_0, +\infty)$;

(C2) there are nonnegative constants L, M such that $L \leq \beta(t) \leq M$ on $[t_0, \infty)$.

Thus, our focus in this section is the initial value problem $(\text{SVEIQR})_{t_0}$ with the additional constraint that $\beta(t)$ must satisfy (C1)-(C2). For convenience, let us refer to this new IVP as $(\text{SVEIQR})_{t_0}^{\text{cont}}$. Note that any solution to this IVP, if it exists, must be continuously differentiable on $[t_0, \infty)$.

We now state the existence, uniqueness, and boundedness properties for solutions to the IVP $(\text{SVEIQR})_{t_0}^{\text{cont}}$. These results have appeared in [8] but they are presented here for completeness and for their relevance to the current work. We begin with the following local existence and uniqueness result.

Proposition 1 ([8]). *For some $\delta > 0$, there is a unique solution to $(\text{SVEIQR})_{t_0}^{\text{cont}}$ on $[t_0, t_0 + \delta]$.*

Proposition 1 implies the following expected property pertaining to the trivial case where the system begins with zero individuals in both E and I compartments.

Lemma 2 ([8]). *Suppose $E(t_0) = I(t_0) = 0$. Then $(\text{SVEIQR})_{t_0}^{\text{cont}}$ has a unique solution $X(t) = (S(t), V(t), E(t), I(t), Q(t), R(t))$ on $[t_0, \infty)$. Moreover, $E(t) = I(t) = 0$ for all $t \geq t_0$.*

Next, we establish the nonnegativity of solutions to $(\text{SVEIQR})_{t_0}^{\text{cont}}$. In the following proposition, we use $X(t) = (S(t), V(t), E(t), I(t), Q(t), R(t))$ to denote a solution to $(\text{SVEIQR})_{t_0}^{\text{cont}}$ on $[t_0, \infty)$.

Proposition 3 ([8]). *If $X(t)$ is a solution to $(\text{SVEIQR})_{t_0}^{\text{cont}}$ on $[t_0, \infty)$, then $X(t) \geq 0$ on $[t_0, \infty)$.*

We also establish that any solution to $(\text{SVEIQR})_{t_0}^{\text{cont}}$ is bounded above by Λ/μ_1. More precisely, we have the following result.

Proposition 4 ([8]). *If $X(t)$ is a solution to $(\text{SVEIQR})_{t_0}^{\text{cont}}$ on $[t_0, \infty)$, then $X(t) \in \mathcal{D}$ for all $t \geq t_0$. In other words, $\mathcal{D}$ is positively invariant with respect to $(\text{SVEIQR})_{t_0}^{\text{cont}}$.*

Finally, using a similar approach as for Proposition 1 and applying Proposition 4 and Theorem 3.3 from [10, p. 94], we obtain the following global existence and uniqueness result for $(\text{SVEIQR})_{t_0}^{\text{cont}}$.

Corollary 5. *The initial value problem $(\text{SVEIQR})_{t_0}^{\text{cont}}$ has a unique solution on $[t_0, +\infty)$.*

3.2. **Piecewise Continuous Time-varying Transmission Rate.** We now turn our attention to the case when $\beta(t)$ is a piecewise continuous function of t. We consider $(\text{SVEIQR})_{t_0}$ with the following additional conditions on $\beta(t)$:

(PC1) $\beta(t)$ can be characterized using a sequence $\{t_k\}_{k\geq 0}$ of real numbers and a sequence $\{b_k(t)\}_{k\geq 0}$ of continuous functions, where $t_0 < t_1 < t_2 < \cdots$ with $t_k \to \infty$, and for each $k = 0, 1, 2, \ldots$, we have $\beta(t) = b_k(t)$ for all $t \in [t_k, t_{k+1})$;

(PC2) there are nonnegative constants L, M such that $L \leq \beta(t) \leq M$ on $[t_0, \infty)$.

An example of a function $\beta(t)$ satisfying (PC1)-(PC2) is illustrated in Figure 2.

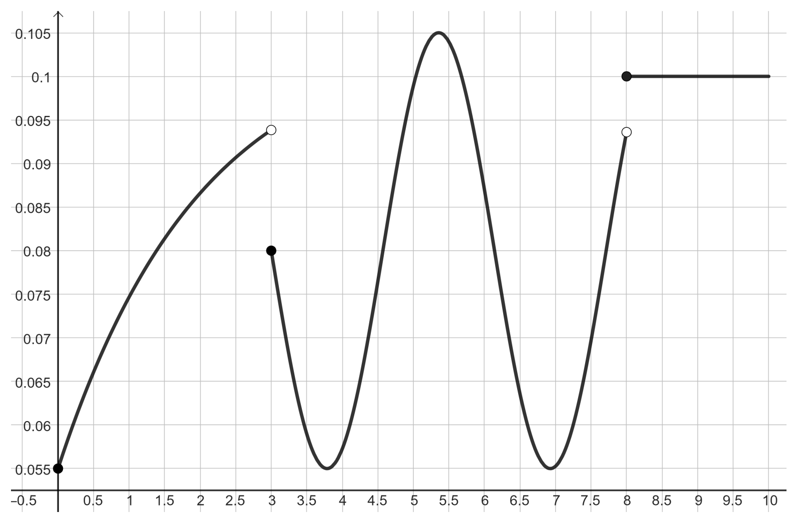


Figure 2. An example of a function $y = \beta(t)$ satisfying assumptions (PC1)-(PC2).

Let us denote by $(\text{SVEIQR})_{t_0}^{\text{pc-cont}}$ the IVP $(\text{SVEIQR})_{t_0}$ with the additional conditions (PC1)-(PC2). In this case, a solution to the IVP, if it exists, must be piecewise continuously differentiable on $[t_0, \infty)$. By applying the properties for the IVP $(\text{SVEIQR})_{t_0}^{\text{cont}}$, we can obtain the following result.

Theorem 6. *The initial value problem* $(\text{SVEIQR})_{t_0}^{\text{pc-cont}}$ *has a unique solution* $X(t)$ *on* $[t_0, +\infty)$. *Moreover,* $X(t) \in \mathcal{D}$ *for all* $t \geq t_0$.

Proof. We construct a sequence $\{X_k(t)\}_{k\geq 0}$ of vector-valued functions. First, consider the IVP $(\text{SVEIQR})_{t_0}^{\text{cont}}$ with initial condition x_{t_0} at $t = t_0$, where the continuous transmission rate $\beta(t)$ is taken to be $b_0(t)$; let us call this IVP $\mathcal{I}_0(t_0, x_{t_0}, b_0(t))$ (or simply, $\mathcal{I}_0$). By Corollary 5, $\mathcal{I}_0$ has a unique solution on $[t_0, +\infty)$; let $X_0(t)$ be this solution. By Proposition 4, $X_0(t) \in \mathcal{D}$ for all $t \geq t_0$.

Now, set $x_{t_1} = X_0(t_1)$ and consider the IVP $(\text{SVEIQR})_{t_1}^{\text{cont}}$ with initial condition x_{t_1} at $t = t_1$ and $\beta(t)$ taken to be $b_1(t)$. We refer to this IVP as $\mathcal{I}_1 = \mathcal{I}_1(t_1, x_{t_1}, b_1(t))$. We can then define $X_1(t)$ to be the unique solution of $\mathcal{I}_1$ on $[t_1, \infty)$. Similarly, $X_1(t) \in \mathcal{D}$ for all $t \geq t_1$. By repeating this process, we are able to construct the sequence $\{X_k(t)\}_{k\geq 0}$, with each $X_k(t)$ being the unique solution to the IVP $\mathcal{I}_k(t_k, x_{t_k}, b_k(t))$ and possessing the property that $X_k(t) \in \mathcal{D}$ for all $t \geq t_k$.

Define $X(t)$ to be the piecewise function such that for each $k \geq 0$, we have $X(t) = X_k(t)$ on $[t_k, t_{k+1})$. Then $X(t)$ is continuous on $[t_0, \infty)$ and $X(t_0) = X_0(t_0) = x_{t_0}$. Moreover, in each interval $[t_k, t_{k+1})$, $X(t) = X_k(t)$ while, from (PC1), $\beta(t) = b_k(t)$; thus, $X(t)$ satisfies $x'(t) = f(t, x(t))$ on each such interval. Therefore, $X(t)$ is a solution to $(\text{SVEIQR})_{t_0}^{\text{pc-cont}}$ on $[t_0, \infty)$. The uniqueness of $X(t)$ follows immediately from the uniqueness of solution to each of the IVPs $\mathcal{I}_1, \mathcal{I}_2, \ldots$. Finally, it is clear that, by construction, $X(t) \in \mathcal{D}$ for all $t \geq t_0$. □

4. Analysis of the SVEIQR Model with Switching Transmission Rate

4.1. The IVP $(\textbf{SVEIQR})_{t_0}^{\textbf{switch}}$.

We now turn our attention to the SVEIQR Model with a switching transmission rate $\beta(t)$. We use definitions from [7]. A switched system can be defined as $x'(t) = \sum_{p=1}^{N} u_p(t) f_p(x)$, where each $f_p(x)$ is a distinct continuous vector field of $\mathbb{R}^n$ and $u = (u_1, u_2, \ldots, u_N) : [0, +\infty) \to U$, with U denoting the set of all canonical basis vectors of $\mathbb{R}^N$. A switched solution is a continuous function $\varphi(t) : [0, +\infty) \to \mathbb{R}^n$ for which there is a positive divergent sequence of times $0 = \tau_0 < \tau_1 < \tau_2 < \cdots$ (called switching times) and a sequence of indices p_k, each one in P, such that $\varphi'(t) = f_{p_k}(x)$ on (τ_k, τ_{k+1}) (equivalently, this means that on (τ_k, τ_{k+1}), $u(t)$ is equal to the vector with a 1 in the p_kth position and 0 in all other positions).

Thus, we can say that in solving a switched system, we consider N possible modes (i.e., $f_1, f_2, \ldots, f_N$) and only exactly one of these is active at each time t as dictated by the switching signal $u(t)$. The function $u(t)$ can be equivalently described by the function $\sigma(t) : [0, +\infty) \to P$ defined as follows: $\sigma(t) = p$ if and only if the vector $u(t)$ has a 1 in the pth position and 0 in all other positions. The function $\sigma(t)$ is typically assumed to be right continuous.

For our study, the switching happens at the level of the parameter $\beta(t)$. Thus, given an initial time $t_0 \geq 0$, we simply assume that $\beta(t)$ is a piecewise-constant function on $[t_0, +\infty)$. We then make the following additional assumptions:

- (S1) $\beta(t)$ can be characterized using two sequences $\{t_k\}_{k\geq 0}$ and $\{\beta_k\}_{k\geq 0}$, where $t_k \to \infty$, $t_0 < t_1 < t_2 < \cdots$, and for each $k = 1, 2, \ldots$, we have $\beta(t) = \beta_k$ for all $t \in [t_k, t_{k+1})$;
- (S2) the range of $\beta(t)$ is $\mathcal{B} := \{B_1, B_2, \ldots, B_m\}$, where $2 \leq m < \infty$ and each B_k is a positive constant;
- (S3) there exists an $h > 0$ such that $\inf_{k\geq 1}(t_k - t_{k-1}) \geq h$.

An example of a function $\beta(t)$ satisfying the discussed assumptions (S1)-(S3) is illustrated in Figure 3. In light of assumption (S2), which implies that $\beta(t)$ only has m possible values, we can say that our model has m distinct *modes*, each one corresponding to the different possible values of the transmission rate $\beta(t)$. For any time interval I, we also say that our model is in mode p when $\beta(t) = B_p$ for all $t \in I$.

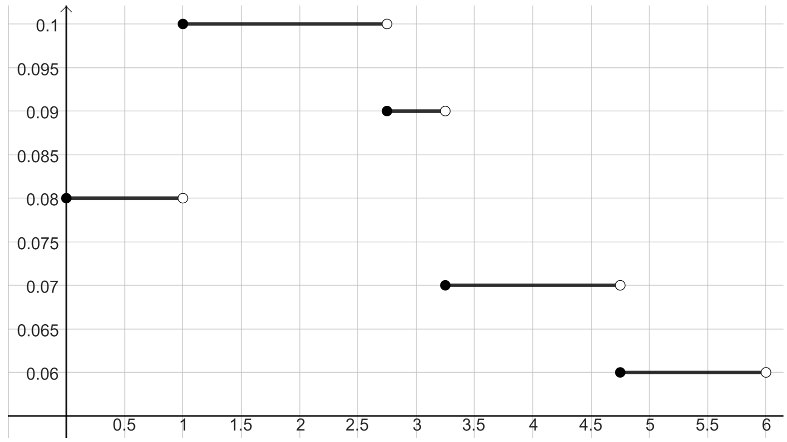


FIGURE 3. An example of a function $y = \beta(t)$ satisfying assumptions (S1)-(S3)

To summarize, our study will now focus on IVP $(\text{SVEIQR})_{t_0}^{\text{switch}}$, which is simply the IVP $(\text{SVEIQR})_{t_0}$ but with $\beta(t)$ required to satisfy conditions (S1)-(S3).

To represent IVP $(\text{SVEIQR})_{t_0}^{\text{switch}}$ as a switched system, let $x = (S, V, E, I, Q, R)$ be a variable in $\mathbb{R}^6$ and for each mode $p \in \{1, 2, ..., m\}$, define $f_p : \mathbb{R}^6 \to \mathbb{R}^6$ by

$$f_p(x) = \begin{pmatrix} \Lambda - B_p SI - (\mu_1 + v)S \\ vS - B_p \rho VI - \mu_1 V \\ B_p(S + \rho V)I - (\mu_1 + \alpha)E \\ \alpha E - (\mu_1 + \mu_2 + q + \sigma_1)I \\ qI - (\mu_1 + \mu_3 + \sigma_2)Q \\ \sigma_1 I + \sigma_2 Q - \mu_1 R \end{pmatrix}.$$

Moreover, for each $t \in [t_0, +\infty)$, define $u(t)$ as the unit vector $(u_1(t), ..., u_m(t))$ with a 1 in the pth position if $\beta(t) = B_p$. Then IVP $(\text{SVEIQR})_{t_0}^{\text{switch}}$ can be written as

$$\begin{cases} x'(t) = \displaystyle\sum_{p=1}^{m} \left[u_p(t) \cdot f_p(x(t))\right], \quad t \in [t_0, +\infty), \\ x(t_0) = (S_{t_0}, V_{t_0}, E_{t_0}, I_{t_0}, Q_{t_0}, R_{t_0}). \end{cases}$$

Note that when we fix a $\beta(t)$ satisfying (S1)-(S3), a solution $X(t)$ to $(\text{SVEIQR})_{t_0}^{\text{switch}}$ is trivially a switched solution of the corresponding switched system. In light of (S3), we can say further that $X(t)$ is a switched solution with a nonvanishing dwell time [7]. Note that a transmission rate $\beta(t)$ satisfying (S1)-(S3) also satisfies (PC1)-(PC2). Thus, Theorem 6 immediately implies the following.

Corollary 7. *The initial value problem* $(\text{SVEIQR})_{t_0}^{\text{switch}}$ *has a unique solution* $X(t)$ *on* $[t_0, +\infty)$. *Moreover,* $X(t) \in \mathcal{D}$ *for all* $t \geq t_0$.

Following standard methods (such as those done in [8,9]), we find that $(\text{SVEIQR})_{t_0}^{\text{switch}}$ has a unique equilibrium solution that corresponds to the DFE; this is given by

$$\text{DFE} = \left(\frac{\Lambda}{\mu_1 + v}, \frac{v\Lambda}{\mu_1(\mu_1 + v)}, 0, 0, 0, 0\right). \tag{7}$$

The uniqueness of this equilibrium point is due to $\beta(t)$ being non-constant (by (S2)).

We will investigate asymptotic properties of the DFE. In the autonomous case, such studies are typically done in relation to a threshold parameter such as the basic reproduction number. If $\beta(t)$ is replaced by a constant parameter, for instance, then the basic reproduction number of the resulting system can easily be obtained, as in [9]. Particularly, using the next-generation matrix (NGM) approach in [11], each mode p of our model has basic reproduction number

$$\mathcal{R}_0^p = \frac{B_p\Lambda(\mu_1 + \rho v)\alpha}{\mu_1(\mu_1 + v)(\mu_1 + \alpha)(\mu_1 + \mu_2 + q + \sigma_1)}. \tag{8}$$

However, as $(\text{SVEIQR})_{t_0}^{\text{switch}}$ involves a nonautonomous system, there is no general method for computing its basic reproduction number. Thus, we will carry out our analysis with respect to the following quantities: (1) the maximum reproduction number $\mathcal{R}^{\max}$, and (2) the time-weighted average $\mathcal{R}^{\text{ave}}$ of the basic reproduction numbers of the system's modes.

4.2. **Analysis with respect to $\mathcal{R}^{\max}$.** The quantity $\mathcal{R}^{\max}$ is obtained by replacing B_p in (8) by the maximum possible value of $\beta(t)$; that is,

$$\mathcal{R}^{\max} = \frac{\max\{B_1, B_2, ..., B_m\} \cdot \Lambda(\mu_1 + \rho v)\alpha}{\mu_1(\mu_1 + v)(\mu_1 + \alpha)(\mu_1 + \mu_2 + q + \sigma_1)}. \tag{9}$$

The condition $\mathcal{R}^{\max} < 1$ is then equivalent to $\mathcal{R}_0^p < 1$ for all $p \in \{1, 2, ..., m\}$. Thus, under these equivalent conditions, we expect that for any initial condition $x_{t_0} \in \mathcal{D}$, the unique solution to $(\text{SVEIQR})_{t_0}^{\text{switch}}$ will converge to the DFE. We establish exactly this fact in this section.

We begin by considering the subsystem $(\text{SVEI})_{t_0}$ given by

$$y'(t) = \sum_{p=1}^{m} \left[u_p(t) \cdot g_p(x(t))\right], \quad t \in [t_0, +\infty),$$

where, for each mode $p \in \{1, 2, ..., m\}$, $g_p : \mathbb{R}^4 \to \mathbb{R}^4$ is defined as

$$g_p(x) = \begin{pmatrix} \Lambda - B_pSI - (\mu_1 + v)S \\ vS - B_p\rho VI - \mu_1 V \\ B_p(S + \rho V)I - (\mu_1 + \alpha)E \\ \alpha E - (\mu_1 + \mu_2 + q + \sigma_1)I \end{pmatrix}.$$

The biologically feasible region associated with $(\text{SVEI})_{t_0}$ is

$$\mathcal{D}_{\text{SVEI}} = \left\{(S, V, E, I) \in \mathbb{R}^4_{\geq 0} : S + V + E + I \leq \frac{\Lambda}{\mu_1}\right\}.$$

Corollary 7 implies that the IVP defined by $(\text{SVEI})_{t_0}$ and with initial condition $y(t_0) = (S_{t_0}, V_{t_0}, E_{t_0}, I_{t_0}) \in \mathcal{D}_{\text{SVEI}}$ has a unique solution that lies entirely in $\mathcal{D}_{\text{SVEI}}$. Moreover, the unique equilibrium of $(\text{SVEI})_{t_0}$ is given by $\left(\frac{\Lambda}{\mu_1+v}, \frac{v\Lambda}{\mu_1(\mu_1+v)}, 0, 0\right)$, which corresponds to the DFE.

We now write $\mathcal{D}_{\text{SVEI}} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3 \cup \mathcal{D}_4$, where

$$\mathcal{D}_1 = \left\{(S, V, E, I) \in \mathcal{D}_{\text{SVEI}} : S \le \frac{\Lambda}{\mu_1 + v}, V \le \frac{v\Lambda}{\mu_1(\mu_1 + v)}\right\},$$
$$\mathcal{D}_2 = \left\{(S, V, E, I) \in \mathcal{D}_{\text{SVEI}} : S \ge \frac{\Lambda}{\mu_1 + v}, V \le \frac{v\Lambda}{\mu_1(\mu_1 + v)}\right\},$$
$$\mathcal{D}_3 = \left\{(S, V, E, I) \in \mathcal{D}_{\text{SVEI}} : S \le \frac{\Lambda}{\mu_1 + v}, V \ge \frac{v\Lambda}{\mu_1(\mu_1 + v)}\right\},$$
$$\mathcal{D}_4 = \left\{(S, V, E, I) \in \mathcal{D}_{\text{SVEI}} : S \ge \frac{\Lambda}{\mu_1 + v}, V \ge \frac{v\Lambda}{\mu_1(\mu_1 + v)}\right\}.$$

In the next two lemmas, we establish the attractivity of the DFE for all initial conditions in $\mathcal{D}_1$ or $\mathcal{D}_2$. In both lemmas, we apply the invariance principle for nonlinear switched systems developed in [7].

Lemma 8. *Suppose $\mathcal{R}^{\max} < 1$. Then every solution $Y(t)$ to $(\text{SVEI})_{t_0}$ for which $Y(t_0) \in \mathcal{D}_1$ converges to the DFE.*

Proof. We perform the change of variables given by $(\bar{S}, \bar{V}, \bar{E}, \bar{I}) = \left(\frac{\Lambda}{\mu_1+v} - S, \frac{v\Lambda}{\mu_1(\mu_1+v)} - V, E, I\right)$. Thus, the subsystem $(\text{SVEI})_{t_0}$ is equivalent to system $(\overline{\text{SVEI}})_{t_0}^{\mathcal{D}_1}$ given by the following:

$$y'(t) = \sum_{p=1}^{m} \left[u_p(t) \cdot \bar{g}_p(x(t))\right], \quad t \in [t_0, +\infty),$$

where, for each mode $p \in \{1, 2, ..., m\}$, $\bar{g}_p : \mathbb{R}^4 \to \mathbb{R}^4$ is defined as

$$\bar{g}_p(x) = \begin{pmatrix} B_p\left(\frac{\Lambda}{\mu_1+v} - \bar{S}\right)\bar{I} - (\mu_1 + v)\bar{S} \\ v\bar{S} + B_p\left(\frac{\rho v\Lambda}{\mu_1(\mu_1+v)} - \rho\bar{V}\right)\bar{I} - \mu_1\bar{V} \\ B_p\left(\frac{\Lambda(\mu_1+\rho v)}{\mu_1(\mu_1+v)} - \bar{S} - \rho\bar{V}\right)\bar{I} - (\mu_1 + \alpha)\bar{E} \\ \alpha\bar{E} - (\mu_1 + \mu_2 + q + \sigma_1)\bar{I} \end{pmatrix}.$$

Note that the corresponding feasible region for the converted system is given by

$$\overline{\mathcal{D}}_1 := \left\{(\bar{S}, \bar{V}, \bar{E}, \bar{I}) \in \mathbb{R}^4_{\ge 0} : \bar{S} \le \frac{\Lambda}{\mu_1 + v}, \bar{V} \le \frac{v\Lambda}{\mu_1(\mu_1 + v)}, \bar{E} + \bar{I} \le \bar{S} + \bar{V}\right\}.$$

Moreover, we note that $\bar{E}^0 = (0, 0, 0, 0)$ is the unique equilibrium solution of $(\overline{\text{SVEI}})_{t_0}^{\mathcal{D}_1}$. Now, define $\mathscr{V} : \overline{\mathcal{D}}_1 \to [0, +\infty)$ as

$$\mathscr{V}(\bar{S}, \bar{V}, \bar{E}, \bar{I}) = \frac{1 - \mathcal{R}^{\max}}{\mathcal{R}^{\max}} \cdot \alpha(\bar{S} + \bar{V}) + \alpha\bar{E} + (\mu_1 + \alpha)\bar{I}.$$

Clearly, $\mathscr{V}$ is of class C^1 (i.e., it is continuously differentiable) and is positive definite (i.e., $\mathscr{V} > 0$ on $\overline{\mathcal{D}}_1 - (0,0,0,0)$). Furthermore, for each $p \in \{1, 2, ..., m\}$ and $x = (\bar{S}, \bar{V}, \bar{E}, \bar{I}) \in \overline{\mathcal{D}}_1$, we have

$$\begin{aligned}
\nabla\mathscr{V}(x)\cdot\bar{g}_p(x) &= -\frac{1-\mathcal{R}^{\max}}{\mathcal{R}^{\max}}\alpha\mu_1(\bar{S}+\bar{V}) - \frac{1}{\mathcal{R}^{\max}}\alpha B_p\left(\bar{S}+\rho\bar{V}\right)\bar{I} \\
&\quad + \left[\frac{1}{\mathcal{R}^{\max}}\cdot\frac{B_p\Lambda(\mu_1+\rho v)\alpha}{\mu_1(\mu_1+v)} - (\alpha+\mu_1)(\mu_1+\mu_2+q+\sigma_1)\right]\bar{I} \\
&= -\frac{1-\mathcal{R}^{\max}}{\mathcal{R}^{\max}}\alpha\mu_1(\bar{S}+\bar{V}) - \frac{1}{\mathcal{R}^{\max}}\alpha B_p\left(\bar{S}+\rho\bar{V}\right)\bar{I} \qquad (10)\\
&\quad - (\alpha+\mu_1)(\mu_1+\mu_2+q+\sigma_1)\left(\frac{\mathcal{R}^{\max}-\mathcal{R}_0^p}{\mathcal{R}^{\max}}\right)\bar{I} \qquad (11)\\
&\le 0
\end{aligned}$$

since $\mathcal{R}_0^p \le \mathcal{R}^{\max} < 1$. Thus, $\mathscr{V}$ is a common weak Liapunov function, following Definition 3 in [7, p. 1110].

Consider the set

$$Z = \left\{x \in \overline{\mathcal{D}}_1 : \exists p \in \{1,2,...,m\} \text{ such that } \nabla\mathscr{V}(x)\cdot\bar{g}_p(x) = 0\right\}.$$

Note that $\nabla\mathscr{V}(x)\cdot\bar{g}_p(x)$ is given by (10)-(11). We consider two cases depending on the value of p. First, for values of p for which $\mathcal{R}_0^p < \mathcal{R}^{\max}$, all the coefficients in (10)-(11) are strictly negative; thus, $\mathscr{V}(x)\cdot\bar{g}_p(x)$ if and only if $\bar{S} = \bar{V} = \bar{I} = 0$. For the second case, we consider values of p for which $\mathcal{R}_0^p = \mathcal{R}^{\max}$. Then

$$\nabla\mathscr{V}(x)\cdot\bar{g}_p(x) = -\frac{\alpha\mu_1}{\mathcal{R}^{\max}}\left[1-\mathcal{R}^{\max}+\frac{B_p}{\mu_1}\bar{I}\right]\bar{S} - \frac{\alpha\mu_1}{\mathcal{R}^{\max}}\left[1-\mathcal{R}^{\max}+\rho\frac{B_p}{\mu_1}\bar{I}\right]\bar{V}.$$

Since $\mathcal{R}^{\max} < 1$, the entire expression multiplied to $\bar{S}$ (resp. $\bar{V}$) is strictly negative. Thus, in this case, $\mathscr{V}(x)\cdot\bar{g}_p(x) = 0$ if and only if $\bar{S} = \bar{V} = 0$. Combining the results of the two cases, we therefore have $Z = \left\{x \in \overline{\mathcal{D}}_1 : \bar{S} = \bar{V} = 0\right\} = \{(0,0,0,0)\}$, where the last equality follows from the definition of $\overline{\mathcal{D}}_1$.

We now choose a large enough l so that $\Omega_l := \{x \in \overline{\mathcal{D}}_1 : \mathscr{V}(x) < l\} = \overline{\mathcal{D}}_1$, which is connected and bounded. Let M be the union of all the compact, weakly invariant sets that are contained in $Z \cap \Omega_l$. Since $Z = \{(0,0,0,0)\} = \{\bar{E}^0\}$, it follows that $M = \{\bar{E}^0\}$ as well. By Theorem 1 in [7, p. 1112], we conclude that if $\varphi(t)$ is a solution of $(\overline{\text{SVEI}})_{t_0}^{\mathcal{D}_1}$ such that φ has nonvanishing dwell time and $\varphi(t_0) \in \Omega_l = \overline{\mathcal{D}}_1$, then $\varphi(t)$ is attracted by $M = \{\bar{E}^0\}$.

Returning to the original subsystem $(\text{SVEI})_{t_0}$, we find that every solution $Y(t)$ for which $Y(t_0) \in \mathcal{D}_1$ converges to the DFE $\left(\frac{\Lambda}{\mu_1+v}, \frac{v\Lambda}{\mu_1(\mu_1+v)}, 0, 0\right)$. □

Lemma 9. *Suppose $\mathcal{R}^{\max} < 1$. Then every solution $Y(t)$ to $(\text{SVEI})_{t_0}$ for which $Y(t_0) \in \mathcal{D}_2$ converges to the DFE.*

Proof. We proceed similarly as in the proof of Lemma 8. We first perform the change of variables given by $(\bar{\bar{S}}, \bar{\bar{V}}, \bar{\bar{E}}, \bar{\bar{I}}) = \left(S - \frac{\Lambda}{\mu_1+v}, \frac{v\Lambda}{\mu_1(\mu_1+v)} - V, E, I\right)$. Thus, the subsystem $(\text{SVEI})_{t_0}$ is equivalent to system

$(\overline{\text{SVEI}})_{t_0}^{\mathcal{D}_2}$ given by the following:

$$y'(t) = \sum_{p=1}^{m} \left[u_p(t) \cdot \bar{\bar{g}}_p(x(t))\right], \quad t \in [t_0, +\infty),$$

where, for each mode $p \in \{1, 2, ..., m\}$, $\bar{\bar{g}}_p : \mathbb{R}^4 \to \mathbb{R}^4$ is defined as

$$\bar{\bar{g}}_p(x) = \begin{pmatrix} -B_p\left(\bar{\bar{S}} + \frac{\Lambda}{\mu_1+v}\right)\bar{\bar{I}} - (\mu_1+v)\bar{\bar{S}} \\ -v\bar{\bar{S}} + B_p\rho\left(\frac{v\Lambda}{\mu_1(\mu_1+v)} - \bar{\bar{V}}\right)\bar{\bar{I}} - \mu_1\bar{\bar{V}} \\ B_p\left(\frac{\Lambda(\mu_1+\rho v)}{\mu_1(\mu_1+v)} + \bar{\bar{S}} - \rho\bar{\bar{V}}\right)\bar{\bar{I}} - (\mu_1+\alpha)\bar{\bar{E}} \\ \alpha\bar{\bar{E}} - (\mu_1+\mu_2+q+\sigma_1)\bar{\bar{I}} \end{pmatrix}.$$

Note that the corresponding feasible region for the converted system is given by

$$\overline{\mathcal{D}}_2 := \left\{(\bar{\bar{S}}, \bar{\bar{V}}, \bar{\bar{E}}, \bar{\bar{I}}) \in \mathbb{R}^4_{\geq 0} : \bar{\bar{S}} \leq \frac{v\Lambda}{\mu_1(\mu_1+v)}, \bar{\bar{V}} \leq \frac{v\Lambda}{\mu_1(\mu_1+v)}, \bar{\bar{E}} + \bar{\bar{I}} \leq \bar{\bar{V}} - \bar{\bar{S}}\right\}.$$

Moreover, we note that $\bar{\bar{E}}^0 = (0,0,0,0)$ is the unique equilibrium solution of $(\overline{\text{SVEI}})_{t_0}^{\mathcal{D}_2}$. Now, define $\mathscr{V}_2 : \overline{\mathcal{D}}_2 \to [0, +\infty)$ as

$$\mathscr{V}_2(\bar{\bar{S}}, \bar{\bar{V}}, \bar{\bar{E}}, \bar{\bar{I}}) = \alpha\bar{\bar{S}} + \frac{1-\mathcal{R}^{\max}}{\mathcal{R}^{\max}}\alpha\bar{\bar{V}} + \alpha\bar{\bar{E}} + (\mu_1+\alpha)\bar{\bar{I}}.$$

Clearly, $\mathscr{V}_2$ is of class C^1 and is positive definite. Furthermore, for each $p \in \{1, 2, ..., m\}$ and each $x = (\bar{\bar{S}}, \bar{\bar{V}}, \bar{\bar{E}}, \bar{\bar{I}}) \in \overline{\mathcal{D}}_2$, we have

$$\begin{aligned} \nabla\mathscr{V}_2(x) \cdot \bar{\bar{g}}_p(x) = & -\alpha(\mu_1+v)\bar{\bar{S}} - \frac{1-\mathcal{R}^{\max}}{\mathcal{R}^{\max}}\alpha v\bar{\bar{S}} - \frac{1-\mathcal{R}^{\max}}{\mathcal{R}^{\max}}\alpha\mu_1\bar{\bar{V}} \qquad (12) \\ & - \frac{1}{\mathcal{R}^{\max}}\left[\frac{\Lambda}{\mu_1+v} + \rho\bar{\bar{V}}\right]\alpha B_p\bar{\bar{I}} \qquad (13) \\ & - (\mu_1+\alpha)(\mu_1+\mu_2+q+\sigma_1)\left(\frac{\mathcal{R}^{\max} - \mathcal{R}_0^p}{\mathcal{R}^{\max}}\right)\bar{\bar{I}} \qquad (14) \\ \leq & \ 0 \end{aligned}$$

since $\mathcal{R}_0^p \leq \mathcal{R}^{\max} < 1$. Thus, $\mathscr{V}_2$ is a common weak Liapunov function.

Consider the set

$$Z_2 = \left\{x \in \overline{\mathcal{D}}_2 : \exists p \in \{1, 2, ..., m\} \text{ such that } \nabla\mathscr{V}_2(x) \cdot \bar{\bar{g}}_p(x) = 0\right\}.$$

Noting that $\nabla\mathscr{V}_2(x) \cdot \bar{\bar{g}}_p(x)$ is given by (12)-(14), we find that $Z_2 = \left\{x \in \overline{\mathcal{D}}_2 : \bar{\bar{S}} = \bar{\bar{V}} = \bar{\bar{I}} = 0\right\} = \{(0,0,0,0)\}$.

We now choose a large enough l so that $\Omega_{2,l} := \{x \in \overline{\mathcal{D}}_2 : \mathscr{V}_2(x) < l\} = \overline{\mathcal{D}}_2$, which is connected and bounded. Let M_2 be the union of all the compact, weakly invariant sets that are contained in $Z_2 \cap \Omega_{2,l}$. Since $Z_2 = \{(0,0,0,0)\} = \{\bar{\bar{E}}^0\}$, it follows that $M_2 = \{\bar{\bar{E}}^0\}$ as well. By Theorem 1 in [7, p. 1112], we conclude that if $\varphi(t)$ is a solution of $(\overline{\text{SVEI}})_{t_0}^{\mathcal{D}_2}$ such that φ has nonvanishing dwell time and $\varphi(t_0) \in \Omega_{2,l} = \overline{\mathcal{D}}_2$, then $\varphi(t)$ is attracted by $M_2 = \{\bar{\bar{E}}^0\}$.

Returning to the original subsystem $(\text{SVEI})_{t_0}$, we find that every solution $Y(t)$ for which $Y(t_0) \in \mathcal{D}_2$ converges to the DFE $\left(\frac{\Lambda}{\mu_1+v}, \frac{v\Lambda}{\mu_1(\mu_1+v)}, 0, 0\right)$. □

We are left to consider $(\text{SVEI})_{t_0}$ when the initial condition $y(t_0) = (S_{t_0}, V_{t_0}, E_{t_0}, I_{t_0})$ is in $\mathcal{D}_3$ or $\mathcal{D}_4$. We need the following lemma.

Lemma 10. *If $S_{t_0} + V_{t_0} < \frac{\Lambda}{\mu_1}$ and $S_{t_0} \le \frac{\Lambda}{\mu_1+v}$, then there exists $t^* \ge t_0$ such that the unique solution $Y(t)$ to $(\text{SVEI})_{t_0}$ is in $\mathcal{D}_1$ for all $t \ge t^*$.*

Proof. First, note that equations (1) and (2) are respectively equivalent to

$$S(t) = \frac{\Lambda}{\mu_1+v} + \left[S_{t_0} - \frac{\Lambda}{\mu_1+v}\right] e^{(\mu_1+v)(t_0-t)} - \frac{\displaystyle\int_{t_0}^{t} \beta(\theta)S(\theta)I(\theta)e^{(\mu_1+v)\theta} d\theta}{e^{(\mu_1+v)t}}, \tag{15}$$

$$V(t) = V_{t_0} e^{\mu_1(t_0-t)} + \frac{\displaystyle\int_{t_0}^{t} [vS(\theta) - \beta(\theta)\rho V(\theta)I(\theta)]e^{\mu_1\theta}\, d\theta}{e^{\mu_1 t}}. \tag{16}$$

Since S, I, and β are nonnegative everywhere, (15) implies

$$S(t) \le \frac{\Lambda}{\mu_1+v} + \left[S_{t_0} - \frac{\Lambda}{\mu_1+v}\right] e^{(\mu_1+v)(t_0-t)}, \quad t \in [t_0, +\infty). \tag{17}$$

Given that $S_{t_0} \le \frac{\Lambda}{\mu_1+v}$, we have $S(t) \le \frac{\Lambda}{\mu_1+v}$ for all $t \ge t_0$, as desired.

We are left to establish that $V(t) \le \frac{v\Lambda}{\mu_1(\mu_1+v)}$ on $[t^*, +\infty)$, for some $t^* \ge t_0$. Using (17) and again applying the nonnegativity of S, I, and β, (16) implies

$$\begin{aligned} V(t) &\le V_{t_0} e^{\mu_1(t_0-t)} + \frac{1}{e^{\mu_1 t}} \int_{t_0}^{t} v \left[\frac{\Lambda}{\mu_1+v} + \left[S_{t_0} - \frac{\Lambda}{\mu_1+v}\right] e^{(\mu_1+v)(t_0-\theta)}\right] e^{\mu_1\theta}\, d\theta \\ &= \frac{v\Lambda}{\mu_1(\mu_1+v)} + \left[V_{t_0} - \frac{v\Lambda}{\mu_1(\mu_1+v)}\right] e^{\mu_1(t_0-t)} \\ &\qquad + \left[S_{t_0} - \frac{\Lambda}{\mu_1+v}\right] e^{\mu_1(t_0-t)}(1 - e^{v(t_0-t)}), \quad t \in [t_0, +\infty). \end{aligned} \tag{18}$$

Since $S_{t_0} < \frac{\Lambda}{\mu_1+v}$, it follows that if $V_{t_0} \le \frac{v\Lambda}{\mu_1(\mu_1+v)}$, then the desired inequality follows for all $t \ge t_0$.

We now assume that $V_{t_0} > \frac{v\Lambda}{\mu_1(\mu_1+v)}$. Then $V(t) \le \frac{v\Lambda}{\mu_1(\mu_1+v)}$ holds whenever

$$\left[V_{t_0} - \frac{v\Lambda}{\mu_1(\mu_1+v)}\right] e^{\mu_1(t_0-t)} + \left[S_{t_0} - \frac{\Lambda}{\mu_1+v}\right] e^{\mu_1(t_0-t)}(1 - e^{v(t_0-t)}) \le 0,$$

which is equivalent to

$$t \ge t_0 - \frac{1}{v} \ln\left[\frac{\frac{\Lambda}{\mu_1} - S_{t_0} - V_{t_0}}{\frac{\Lambda}{\mu_1+v} - S_{t_0}}\right].$$

Note that the argument of $\ln$ above is positive due to the assumption that $S_{t_0} + V_{t_0} < \frac{\Lambda}{\mu_1}$. Therefore, by choosing $t^* = \max\left\{t_0, t_0 - \frac{1}{v}\ln\left[\frac{\frac{\Lambda}{\mu_1} - S_{t_0} - V_{t_0}}{\frac{\Lambda}{\mu_1+v} - S_{t_0}}\right]\right\}$, the desired inequality for $V(t)$ holds on $[t^*, +\infty)$. □

Now, suppose $\mathcal{R}^{\max} < 1$. If the initial condition $y(t_0) = (S_{t_0}, V_{t_0}, E_{t_0}, I_{t_0})$ is in $\mathcal{D}_3$, then we have two cases:

(i) If $S_{t_0} + V_{t_0} = \frac{\Lambda}{\mu_1}$, then $E_{t_0} = I_{t_0} = 0$. Following similar arguments as in Lemma 2, it can be shown that $E(t) = I(t) = 0$ for all $t \geq t_0$. We can then easily verify, using (15) and (16), that the solution $Y(t)$ to $(\text{SVEI})_{t_0}$ approaches the DFE.

(ii) If $S_{t_0} + V_{t_0} < \frac{\Lambda}{\mu_1}$, then we can apply Lemma 10; that is, for some $t^* \geq t_0$, the solution $Y(t)$ is in $\mathcal{D}_1$ for $t \geq t^*$. By applying Lemma 8 to the IVP defined by $(\text{SVEI})_{t^*}$ and the initial condition $y(t^*) = Y(t^*)$, we see that $Y(t)$ must also converge to the DFE.

Lastly, noting the definition of $\mathcal{D}_{\text{SVEI}}$, we see that $\mathcal{D}_4$ is simply equal to the singleton $\left\{\left(\frac{\Lambda}{\mu_1+v}, \frac{v\Lambda}{\mu_1(\mu_1+v)}, 0, 0\right)\right\}$. Hence, the case when $y(t_0) \in \mathcal{D}_4$ is equivalent to the case addressed in (i) above. We have thus proven the following.

Lemma 11. *Suppose $\mathcal{R}^{\max} < 1$. Then every solution $Y(t)$ to $(\text{SVEI})_{t_0}$ for which $Y(t_0) \in \mathcal{D}_{\text{SVEI}}$ converges to the DFE.*

With the preceding lemma, we are now ready to establish this section's main result—the global attractivity of the DFE of $(\text{SVEIQR})_{t_0}^{\text{switch}}$ when $\mathcal{R}^{\max} < 1$.

Theorem 12. *Suppose $\mathcal{R}^{\max} < 1$. Then every solution $X(t)$ to $(\text{SVEIQR})_{t_0}^{\text{switch}}$ for which $X(t_0) \in \mathcal{D}$ converges to the DFE given in (7); that is, the DFE is globally attractive.*

Proof. In light of Lemma 11, we have $S(t) \to \frac{\Lambda}{\mu_1+v}$, $V(t) \to \frac{v\Lambda}{\mu_1(\mu_1+v)}$, $E(t) \to 0$, and $I(t) \to 0$ as $t \to \infty$. Thus, we are left to prove that $Q(t) \to 0$ and $R(t) \to 0$ as $t \to \infty$. Using (5) and (6), we find that

$$Q(t) = \frac{Q(t_0)e^{(\mu_1+\mu_3+\sigma_2)t_0}}{e^{(\mu_1+\mu_3+\sigma_2)t}} + \frac{1}{e^{(\mu_1+\mu_3+\sigma_2)t}} \int_{t_0}^{t} e^{(\mu_1+\mu_3+\sigma_2)\theta} qI(\theta)\, d\theta$$

and

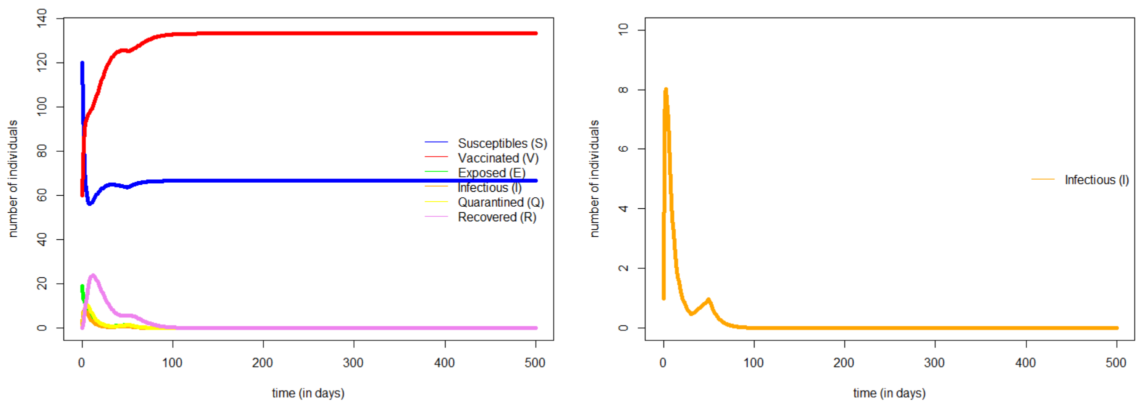
$$R(t) = \frac{R(t_0)e^{\mu_1 t_0}}{e^{\mu_1 t}} + \frac{1}{e^{\mu_1 t}} \int_{t_0}^{t} e^{\mu_1 \theta} [\sigma_1 I(\theta) + \sigma_2 Q(\theta)]\, d\theta.$$

It can be easily verified that $\lim_{t\to\infty} Q(t) = 0$, which leads to $\lim_{t\to\infty} R(t) = 0$ as desired. □

4.3. **Analysis with respect to $\mathcal{R}^{\text{ave}}$.** Consider the following periodic switching transmission rate:

$$\beta(t) = \begin{cases} 0.0125, & \text{if } t \in \bigcup_{k=0}^{\infty} [150k, 30+150k) \\ 0.016, & \text{if } t \in \bigcup_{k=0}^{\infty} [30+150k, 50+150k) \\ 0.0115, & \text{if } t \in \bigcup_{k=0}^{\infty} [50+150k, 90+150k) \\ 0.012, & \text{if } t \in \bigcup_{k=0}^{\infty} [90+150k, 115+150k) \\ 0.0115, & \text{if } t \in \bigcup_{k=0}^{\infty} [115+150k, 150+150k) \end{cases}. \tag{19}$$

Moreover, set $t_0 = 0$ and $\Lambda = 20$, $\rho = 0.15$, $v = 0.2$, $\alpha = 0.7$, $q = 0.7$, $\sigma_1 = 0.2$, $\sigma_2 = 0.3$, $\mu_1 = 0.1$, $\mu_2 = 0.075$, and $\mu_3 = 0.05$. Using (8), we find that the basic reproduction numbers of the system's five modes are 0.8817829, 1.128682, 0.8112403, 0.8465116, and 0.8112403; thus, by (9), we have $\mathcal{R}^{\max} = 1.128682$. Thus, Theorem 12 does not apply. Solving $(\text{SVEIQR})^{\text{switch}}_{t_0}$ numerically given the above parameters, we obtain the solution illustrated in Fig. 4.



(A) (B)

FIGURE 4. A numerical solution to $(\text{SVEIQR})^{\text{switch}}_{t_0}$ given a set of model parameters for which $\mathcal{R}^{\max} > 1$. In (A), all compartments are plotted while in (B), a zoomed-in plot for $I(t)$ is presented. It can be observed that the solution approaches the DFE.

The preceding example implies that the condition $\mathcal{R}^{\max} < 1$, while sufficient for the global attractivity of the DFE of $(\text{SVEIQR})^{\text{switch}}_{t_0}$, is not necessary for the convergence of a solution to the DFE. This opens exploration for other sufficient conditions. In this section, we provide such a sufficient condition in the case that $\beta(t)$ is also periodic; that is, aside from satisfying assumptions (S1)-(S3), $\beta(t)$ is also assumed to satisfy:

(S4) $\beta(t)$ is a periodic function with a period of length ω, i.e. $\beta(t+\omega) = \beta(t)$.

We now define $\tau_k = t_k - t_{k-1}$ for each $k \geq 0$, where $\{t_k\}_{k\geq 0}$ is the sequence of times defined in (S1). Since $\beta(t)$ also satisfies (S4), then there exists $\ell \in \mathbb{N}$ such that $\tau_{k+\ell} = \tau_k$ for all k and $\sum_{i=1}^{\ell} \tau_i = \omega$. This implies that we can represent $\beta(t)$ as

$$\beta(t) = \begin{cases} \beta_1, & \text{if } t \in \bigcup_{k=0}^{\infty} [t_{k\ell}, t_{k\ell+1}) = [t_0, t_1) \cup [t_\ell, t_{\ell+1}) \cup \cdots, \\ \beta_2, & \text{if } t \in \bigcup_{k=0}^{\infty} [t_{k\ell+1}, t_{k\ell+2}) = [t_1, t_2) \cup [t_{\ell+1}, t_{\ell+2}) \cup \cdots, \\ \vdots & \\ \beta_\ell, & \text{if } t \in \bigcup_{k=0}^{\infty} [t_{(k+1)\ell-1}, t_{(k+1)\ell}) = [t_{\ell-1}, t_\ell) \cup [t_{2\ell-1}, t_{2\ell}) \cup \cdots. \end{cases}$$

It should be noted that $\beta_1, \beta_2, \ldots, \beta_\ell$ are not necessarily distinct and that $\beta_k \in \mathcal{B} := \{B_1, B_2, \ldots, B_m\}$, where $\mathcal{B}$ is as defined in (S2). Hence, $\ell \geq m$. Moreover, $\beta_{k+\ell} = \beta_k$ for all k. Given $\beta(t)$ satisfying (S1)-(S4), denote the corresponding IVP as $(\text{SVEIQR})_{t_0}^{\text{periodic}}$.

We next investigate the asymptotic properties of the DFE corresponding to $(\text{SVEIQR})_{t_0}^{\text{periodic}}$ by defining the threshold parameter $\mathcal{R}^{\text{ave}}$. Let $\mathcal{R}_0^k$ represent the basic reproduction number of IVP $(\text{SVEIQR})_{t_0}$ where $\beta(t)$ is replaced by a particular β_k (note that $\beta(t)$ is constant in this case). Then by (8), we have

$$\mathcal{R}_0^k = \frac{\beta_k \Lambda(\mu_1 + \rho v)\alpha}{\mu_1(\mu_1 + v)(\mu_1 + \alpha)(\mu_1 + \mu_2 + q + \sigma_1)}.$$

We can now define the threshold parameter $\mathcal{R}^{\text{ave}}$ of $(\text{SVEIQR})_{t_0}^{\text{periodic}}$ as

$$\mathcal{R}^{\text{ave}} = \frac{1}{\omega} \sum_{k=1}^{\ell} \tau_k \mathcal{R}_0^k; \tag{20}$$

that is, $\mathcal{R}^{\text{ave}}$ is defined as the weighted average of $\mathcal{R}_0^1, \mathcal{R}_0^2, \ldots, \mathcal{R}_0^\ell$ with weights $\frac{\tau_1}{\omega}, \frac{\tau_2}{\omega}, \ldots, \frac{\tau_\ell}{\omega}$, respectively.

We will establish a sufficient condition, in relation to $\mathcal{R}^{\text{ave}}$, for the global attractivity of the DFE. But before this, we establish the following result.

Lemma 13. *For any solution $X(t)$ to $(\text{SVEIQR})_{t_0}^{\text{periodic}}$ with $X(t_0) \in D$ and $S_{t_0} + V_{t_0} < \frac{\Lambda}{\mu_1}$, there exists $t^* \geq t_0$ such that*

$$S(t) + \rho V(t) \leq \frac{\Lambda(\mu_1 + \rho v)}{\mu_1(\mu_1 + v)}, \quad \forall t \geq t^*.$$

Proof. From inequalities (17) and (18) in the proof of Lemma 10, we have

$$S(t) + \rho V(t) \leq \frac{\Lambda(\mu_1 + \rho v)}{\mu_1(\mu_1 + v)} + (1 - \rho)\left(S_{t_0} - \frac{\Lambda}{\mu_1 + v}\right) e^{(\mu_1 + v)(t_0 - t)} + \rho\left(S_{t_0} + V_{t_0} - \frac{\Lambda}{\mu_1}\right) e^{\mu_1(t_0 - t)}.$$

Case 1: Suppose $S_{t_0} \leq \frac{\Lambda}{\mu_1 + v}$. Since $S_{t_0} + V_{t_0} < \frac{\Lambda}{\mu_1}$ by assumption, then the desired inequality immediately follows for all $t \geq t_0$. Thus, we can simply choose $t^* = t_0$.

Case 2: Suppose $S_{t_0} > \frac{\Lambda}{\mu_1 + v}$. Note that $S(t) + \rho V(t) \leq \frac{\Lambda(\mu_1 + \rho v)}{\mu_1(\mu_1 + v)}$ holds whenever

$$(1 - \rho)\left(S_{t_0} - \frac{\Lambda}{\mu_1 + v}\right) e^{(\mu_1 + v)(t_0 - t)} + \rho\left(S_{t_0} + V_{t_0} - \frac{\Lambda}{\mu_1}\right) e^{\mu_1(t_0 - t)} \leq 0.$$

which is equivalent to

$$t \geq t_0 - \frac{1}{v} \ln\left(\frac{\rho\left[\frac{\Lambda}{\mu_1} - S_{t_0} - V_{t_0}\right]}{(1 - \rho)\left(S_{t_0} - \frac{\Lambda}{\mu_1 + v}\right)}\right).$$

Note that the argument of the $\ln$ above is positive due to the assumption that $S_{t_0} + V_{t_0} < \frac{\Lambda}{\mu_1}$. Therefore, the desired inequality holds on $[t^*, +\infty)$ by setting

$$t^* = \max\left\{t_0, t_0 - \frac{1}{v} \ln\left(\frac{\rho\left[\frac{\Lambda}{\mu_1} - S_{t_0} - V_{t_0}\right]}{(1 - \rho)\left(S_{t_0} - \frac{\Lambda}{\mu_1 + v}\right)}\right)\right\}.$$

□

For the succeeding discussion, we utilize the periodic function λ defined by

$$\lambda(t) = \beta(t) \cdot \frac{\Lambda(\mu_1 + \rho v)}{\mu_1(\mu_1 + v)} - (\mu_1 + \mu_2 + q + \sigma_1).$$

In addition, we denote by λ_k the value of $\lambda(t)$ whenever $\beta(t)$ assumes the value β_k. Since λ also has a period of length ω, it follows that $\lambda_{k+\ell} = \lambda_k$ for all k. Moreover, the integral of λ over any interval of length ω has a fixed value; specifically,

$$\int_a^b \lambda(\theta)\, d\theta = \sum_{k=1}^{\ell} \lambda_k \tau_k$$

whenever $b - a = \omega$.

Lemma 14. *Suppose* $\mathcal{R}^{ave} \cdot \frac{\mu_1+\alpha}{\alpha} < 1$ *and* $\lambda(t) \geq -\mu_1$ *for all* $t \geq t_0$. *Every solution* $X(t) = (S(t), V(t), E(t), I(t), Q(t), R(t))$ *to* $(\mathrm{SVEIQR})_{t_0}^{\mathrm{periodic}}$ *for which* $X(t_0) \in D$ *is such that* $\lim_{t\to\infty} (E(t) + I(t)) = 0$.

Proof. Let $U(t) = E(t) + I(t)$. It follows that $U'(t) = E'(t) + I'(t)$ and

$$U'(t) = [\beta(t)(S(t) + \rho V(t)) - (\mu_1 + \mu_2 + q + \sigma_1)]\, I(t) - \mu_1 E(t).$$

From Lemma 13, there exists $T^* = t_0 + r\omega$ for some sufficiently large positive integer r such that $S(t) + \rho V(t) \leq \frac{\Lambda(\mu_1+\rho v)}{\mu_1(\mu_1+v)}$ for all $t \geq T^*$. Thus, for all $t \geq T^*$, we have

$$\begin{aligned} U'(t) &\leq \left[\beta(t) \cdot \frac{\Lambda(\mu_1 + \rho v)}{\mu_1(\mu_1 + v)} - (\mu_1 + \mu_2 + q + \sigma_1)\right] I(t) - \mu_1 E(t) \\ &= \lambda(t)\, I(t) - \mu_1 E(t) \\ &\leq \max\{\lambda(t), -\mu_1\}\,(I(t) + E(t)). \end{aligned}$$

Since $\lambda(t) \geq -\mu_1$ for all $t \geq t_0$ by assumption, then $U'(t) \leq \lambda(t)\, U(t)$ for all $t \geq T^*$.

Let $w(t) = \exp\left(\int_{T^*}^t \lambda(\theta) d\theta\right)$, then $w(T^*) = 1$, $w(t) > 0$ for all $t \geq T^*$, and $w'(t) = w(t)\lambda(t)$. Thus, for all $t \geq T^*$, we have

$$\frac{d}{dt}\left(\frac{U(t)}{w(t)}\right) = \frac{w(t)U'(t) - U(t)w'(t)}{(w(t))^2} = \frac{U'(t) - U(t)\lambda(t)}{w(t)} \leq 0$$

since $w(t) > 0$ and $U'(t) - U(t)\lambda(t) \leq 0$. Since $\frac{d}{dt}\left(\frac{U(t)}{w(t)}\right)$ is always nonpositive, then $\frac{U(t)}{w(t)}$ is a nonincreasing function, and hence is bounded above by its initial value at $t = T^*$. Thus, we have

$$\frac{U(t)}{w(t)} \leq \frac{U(T^*)}{w(T^*)} = \frac{U(T^*)}{1} = U(T^*).$$

Hence, the resulting inequality for $U(t)$ is given by

$$U(t) \leq U(T^*) \exp\left(\int_{T^*}^t \lambda(\theta)\, d\theta\right). \tag{21}$$

Now consider the sum $\sum_{k=1}^{\ell} \lambda_k \tau_k$, which can be expressed in terms of $\mathcal{R}^{\text{ave}}$ as

$$\sum_{k=1}^{\ell} \lambda_k \tau_k = \omega(\mu_1 + \mu_2 + q + \sigma_1) \left(\mathcal{R}^{\text{ave}} \cdot \frac{\mu_1 + \alpha}{\alpha} - 1 \right).$$

Since $\mathcal{R}^{\text{ave}} \cdot \dfrac{\mu_1 + \alpha}{\alpha} < 1$ by assumption, then it follows that $\eta := \exp \left[\sum_{k=1}^{\ell} \lambda_k \tau_k \right] < 1$. Now suppose $t = T^* + h\omega + \epsilon$, where $h \in \mathbb{N}$ and for any $\epsilon \in [0, \omega)$. Applying inequality (21), we have

$$U(T^* + h\omega + \epsilon) \leq U(T^*) \exp \left(\int_{T^*}^{T^* + h\omega + \epsilon} \lambda\left(\theta\right) d\theta \right).$$

But since $T^* = t_0 + r\omega$ for some $r \in \mathbb{N}$, then

$$\begin{aligned}
U(T^* + h\omega + \epsilon) &\leq U(T^*) \exp \left(\int_{t_0}^{t_0 + h\omega + \epsilon} \lambda\left(\theta\right) d\theta \right) \\
&= U(T^*) \exp \left(\sum_{i=1}^{h} \int_{t_0 + (i-1)\omega}^{t_0 + i\omega} \lambda(\theta)\, d\theta \right) \times \exp \left(\int_{t_0 + h\omega}^{t_0 + h\omega + \epsilon} \lambda\left(\theta\right) d\theta \right) \\
&= U(T^*) \left[\exp \left(\sum_{i=1}^{h} \left(\sum_{k=1}^{\ell} \lambda_k \tau_k \right) \right) \right] \exp \left(\int_{t_0}^{t_0 + \epsilon} \lambda\left(\theta\right) d\theta \right) \\
&= U(T^*) \eta^h \exp \left(\int_{t_0}^{t_0 + \epsilon} \lambda\left(\theta\right) d\theta \right) \\
&\leq U(T^*) \eta^h C
\end{aligned}$$

where $C := \exp \left(\int_{t_0}^{t_0 + \omega} \left| \lambda\left(\theta\right) \right| d\theta \right)$. Note that C is finite as $\lambda\left(t\right)$ is bounded. Since C is independent of ϵ, the inequality $U(T^* + h\omega + \epsilon) \leq U(T^*)\eta^h C$ holds for any $\epsilon \in [0, \omega)$. Now, observe that the sequence $\left\{ U(T^*)\eta^h C \right\}_{h \geq 0}$ approaches 0 as $h \to \infty$. It now follows that $\lim_{t \to \infty} U\left(t\right) = \lim_{t \to \infty} \left(E\left(t\right) + I\left(t\right)\right) = 0$. ◻

We are now ready to present and prove the following sufficient condition for the global attractivity of the DFE for IVP $(\text{SVEIQR})_{t_0}^{\text{periodic}}$.

Theorem 15. *Suppose $\mathcal{R}^{ave} \cdot \frac{\mu_1 + \alpha}{\alpha} < 1$ and $\lambda\left(t\right) \geq -\mu_1$ for all $t \geq t_0$. Then every solution $X\left(t\right)$ to $(\text{SVEIQR})_{t_0}^{\text{periodic}}$ for which $X\left(t_0\right) \in D$ converges to the DFE given in (7); that is, the DFE is globally attractive.*

Proof. The nonnegativity of $E(t)$ and $I(t)$, and Lemma 14 imply that $\lim_{t \to \infty} E(t) = \lim_{t \to \infty} I(t) = 0$. In addition, as presented in the proof of Theorem 12, we have $\lim_{t \to \infty} Q(t) = 0$ and $\lim_{t \to \infty} R(t) = 0$. Using equations (1) to (6), we get $N'(t) = \Lambda - \mu_2 I(t) - \mu_3 Q(t) - \mu_1 N(t)$. Solving for $N(t)$ yields

$$N(t) = \frac{\Lambda}{\mu_1} + \left[N(t_0) - \frac{\Lambda}{\mu_1} \right] e^{\mu_1 (t_0 - t)} - \frac{1}{e^{\mu_1 t}} \int_{t_0}^{t} [\mu_2 I(\theta) + \mu_3 Q(\theta)] e^{\mu_1 \theta}\, d\theta.$$

It can then be easily shown that

$$\lim_{t\to\infty} N(t) = \frac{\Lambda}{\mu_1} + \lim_{t\to\infty} \frac{\mu_2 I(t) + \mu_3 Q(t)}{\mu_1} = \frac{\Lambda}{\mu_1}.$$

Next we consider the implicit equation of $S(t)$ in (15). Given the boundedness of $S(t)$ by Proposition 4, the boundedness of $\beta(t)$ by assumption, and $\lim_{t\to\infty} I(t) = 0$, it can be shown that $\lim_{t\to\infty} \beta(t)S(t)I(t) = 0$. Thus, taking the limit of (15) as $t \to \infty$, we arrive at $\lim_{t\to\infty} S(t) = \dfrac{\Lambda}{\mu_1 + v}$. Since $V(t) = N(t) - S(t) - E(t) - I(t) - Q(t) - R(t)$, then $\lim_{t\to\infty} V(t) = \dfrac{v\Lambda}{\mu_1(\mu_1 + v)}$. □

To illustrate Theorem 15, consider IVP $(\text{SVEIQR})^{\text{periodic}}_{t_0}$ with $\beta(t)$ given by (19). It can be shown that $\mathcal{R}^{\text{ave}} \cdot \frac{\mu_1+\alpha}{\alpha} = 0.9983 < 1$ and $\lambda(t) \geq -\mu_1$ for all $t \geq t_0$, and as presented in Figure 4, the solution indeed approaches the DFE.

5. Simulations

In this section, we present several simulations to illustrate our results and to explore the rich dynamics of solutions in some scenarios. All simulation figures were made using the software RStudios. We set $t_0 = 0$ and the initial condition as

$$X(t_0) = \left(0.6\frac{\Lambda}{\mu_1}, 0.3\frac{\Lambda}{\mu_1}, 0.095\frac{\Lambda}{\mu_1}, 0.005\frac{\Lambda}{\mu_1}, 0, 0\right).$$

For the first set of simulations, we use the transmission rate

$$\beta(t) = \begin{cases} \beta_1, & \text{if } t \in \bigcup_{k=0}^{\infty} [150k, 30 + 150k), \\ \beta_2, & \text{if } t \in \bigcup_{k=0}^{\infty} [30 + 150k, 50 + 150k), \\ \beta_3, & \text{if } t \in \bigcup_{k=0}^{\infty} [50 + 150k, 90 + 150k), \\ \beta_4, & \text{if } t \in \bigcup_{k=0}^{\infty} [90 + 150k, 115 + 150k), \\ \beta_5, & \text{if } t \in \bigcup_{k=0}^{\infty} [115 + 150k, 150 + 150k), \end{cases}$$

which is a periodic piecewise-constant function. Given the values of $(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5)$ in Table 3, we obtain numerical solutions to $(\text{SVEIQR})^{\text{periodic}}_{t_0}$ using the parameters shown in Table 4, in which the corresponding values for $\mathcal{R}^{\max}$ and $\mathcal{R}^{\text{ave}} \cdot \frac{\mu_1+\alpha}{\alpha}$ are also indicated. Graphs of these numerical solutions are then presented in Figs. 5, 6, and 7. All parameters used are hypothetical.

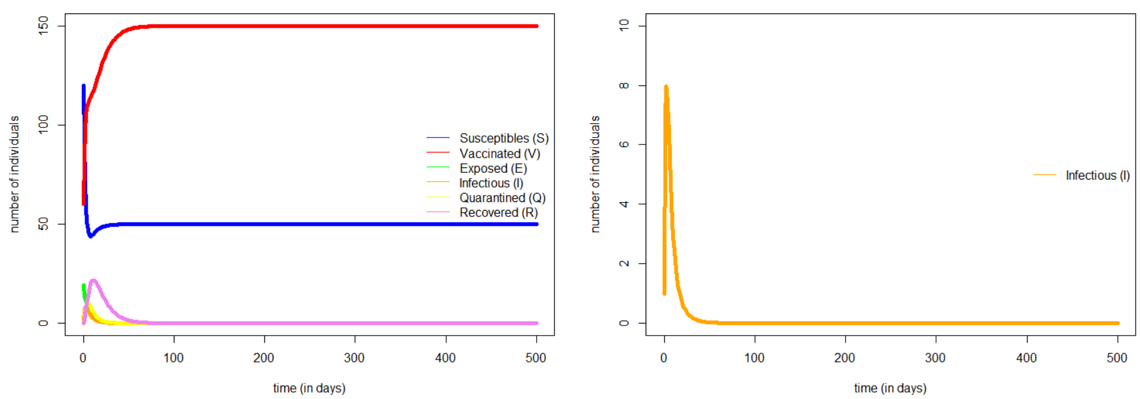
The solution presented in Fig. 5 arises from parameters for which both $\mathcal{R}^{\max}$ and $\mathcal{R}^{\text{ave}} \cdot \frac{\mu_1+\alpha}{\alpha}$ are less than 1, and $\lambda(t) \geq -\mu_1$ for all $t \geq t_0$. Thus, Theorem 12 or Theorem 15 guarantee the convergence of the solution to the DFE and this is evident in the graphs shown in Fig. 5.

TABLE 3. Beta values used for the numerical solutions presented in Figs. 5, 6, and 7.

Fig.	β_1	β_2	β_3	β_4	β_5
5	0.01345	0.01365	0.01515	0.01525	0.01535
6	0.008	0.00975	0.01075	0.00925	0.01025
7	0.008	0.01	0.009	0.01	0.009

TABLE 4. Parameter values used for the numerical solutions presented in Figs. 5–7. Corresponding values for $\mathcal{R}^{\max}$ and $\mathcal{R}^{\text{ave}} \cdot \frac{\mu_1+\alpha}{\alpha}$ are also indicated.

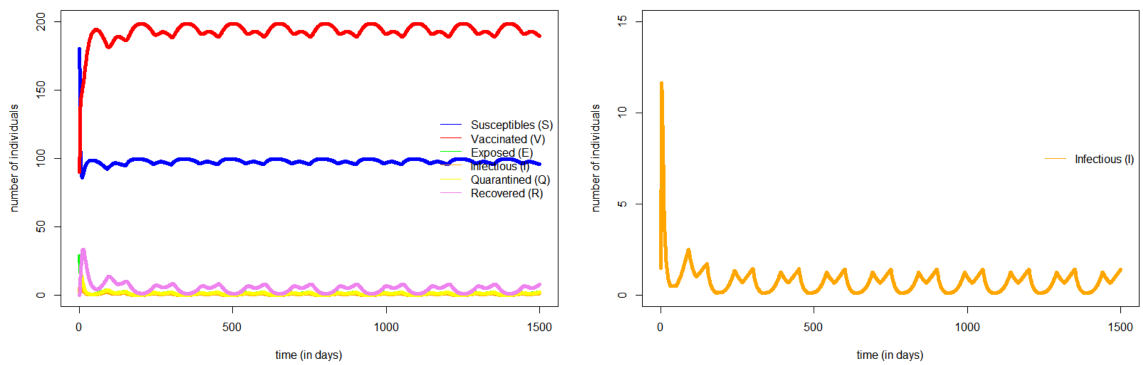
Fig.	$\mathcal{R}^{\max}$	$\mathcal{R}^{\text{ave}} \cdot \frac{\mu_1+\alpha}{\alpha}$	Λ	ρ	v	α	q	σ_1	σ_2	μ_1	μ_2	μ_3
5	0.9058	0.9896	20	0.15	0.3	0.7	0.7	0.2	0.3	0.1	0.075	0.05
6	1.1375	1.173	30	0.15	0.2	0.7	0.7	0.2	0.3	0.1	0.075	0.05
7	1.0581	1.1005	30	0.15	0.2	0.7	0.7	0.2	0.3	0.1	0.075	0.05
8	2.0714	2.4799	40	0.15	0.3	0.5	0.5	0.2	0.3	0.1	0.075	0.05



(A) (B)

FIGURE 5. A numerical solution to $(\text{SVEIQR})_{t_0}^{\text{periodic}}$ given a set of model parameters for which $\mathcal{R}^{\max} < 1$ and $\mathcal{R}^{\text{ave}} \cdot \frac{\mu_1+\alpha}{\alpha} < 1$. In (A), the graphs of all compartments are shown while in (B), a zoomed-in graph for $I(t)$ is presented. It can be observed that the solution approaches the DFE.

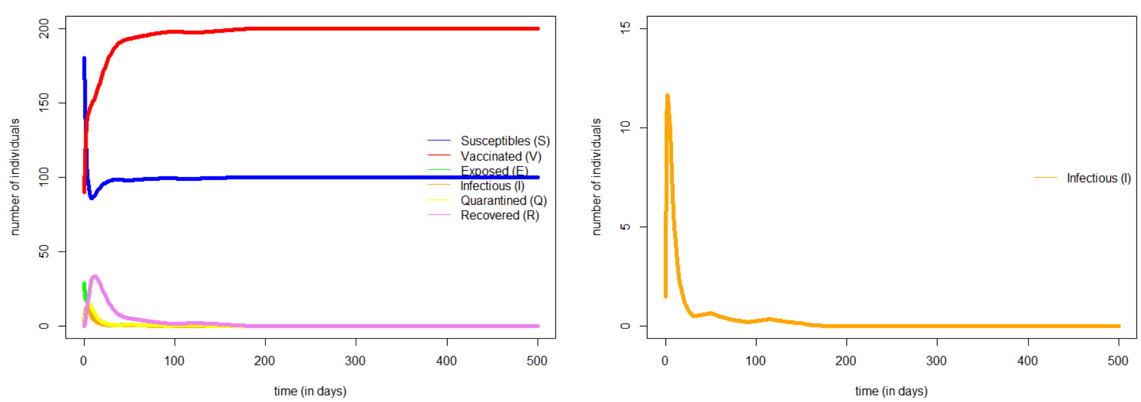
The solution presented in Fig. 6 arises from parameters for which both $\mathcal{R}^{\max}$ and $\mathcal{R}^{\text{ave}} \cdot \frac{\mu_1+\alpha}{\alpha}$ are greater than 1. Thus, Theorem 12 and Theorem 15 do not apply. As evident in Fig. 6, the solution eventually exhibits a periodic nature, with fluctuations that seem to persist over time. Thus, in this case, we anticipate that the solution does not converge to an equilibrium point.



(A) (B)

Figure 6. A numerical solution to $(\text{SVEIQR})_{t_0}^{\text{periodic}}$ given a set of model parameters for which $\mathcal{R}^{\max} > 1$ and $\mathcal{R}^{\text{ave}} \cdot \frac{\mu_1+\alpha}{\alpha} > 1$. In (A), the graphs of all compartments are shown while in (B), a zoomed-in graph for $I(t)$ is presented. It can be observed that the solution eventually enters a periodic (or periodic-like) state, with fluctuations that seem to persist over time.

In the next example (Fig. 7), again we use parameters for which both $\mathcal{R}^{\max}$ and $\mathcal{R}^{\text{ave}} \cdot \frac{\mu_1+\alpha}{\alpha}$ are greater than 1. Thus, Theorem 12 and Theorem 15 do not apply. However, as evident in Fig. 7, the solution still converges to the DFE. This implies that conditions $\mathcal{R}^{\max} < 1$ and $\mathcal{R}^{\text{ave}} \cdot \frac{\mu_1+\alpha}{\alpha} < 1$ are not necessary for the solution to approach the DFE.



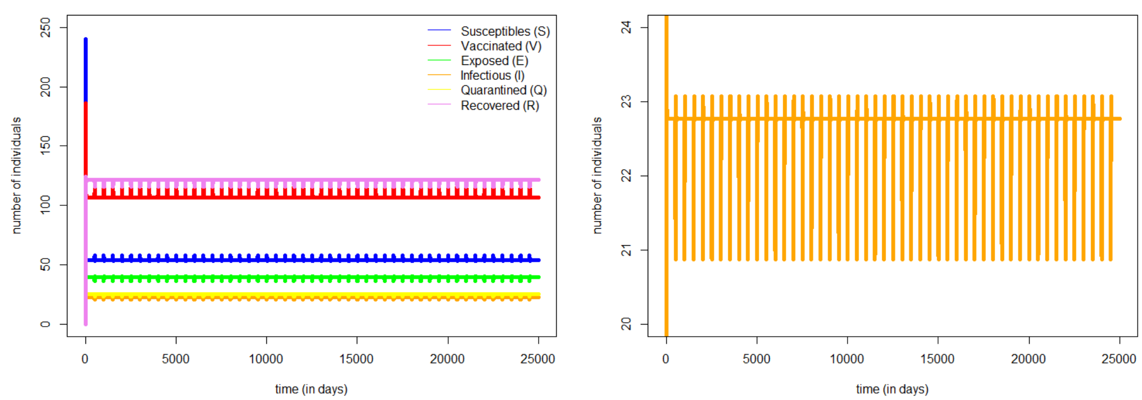
(A) (B)

Figure 7. A numerical solution to $(\text{SVEIQR})_{t_0}^{\text{periodic}}$ given a set of model parameters for which $\mathcal{R}^{\max} > 1$ and $\mathcal{R}^{\text{ave}} \cdot \frac{\mu_1+\alpha}{\alpha} > 1$. In (A), the graphs of all compartments are shown while in (B), a zoomed-in graph for $I(t)$ is presented. It can be observed that the solution approaches the DFE.

For the last example (Fig. 8), a numerical solution is obtained using the corresponding parameter values in Table 4 and the following transmission rate:

$$\beta_2(t) = \begin{cases} 0.014, & \text{if } t \in \bigcup_{k=0}^{\infty} [500k, 10+500k), \\ 0.0145, & \text{if } t \in \bigcup_{k=0}^{\infty} [10+500k, 25+500k), \\ 0.015, & \text{if } t \in \bigcup_{k=0}^{\infty} [25+500k, 500+500k). \end{cases}$$

Notice that $\beta_2(t)$ is again periodic, with period of length 500, and that $\beta_2(t) = 0.015$ for 95% of one period; that is, the system is in the third mode (in which $\beta_2(t) = 0.015$) majority of the time. Despite this deliberate setup for the transmission rate, the graphs in Fig. 8 suggest that the solution still does not converge (e.g., to an equilibrium point of the third mode). Instead, the solution again eventually enters a periodic or periodic-like state in which the system's two other modes cause fluctuations that persist over time. It is worth noting that, in this case, $\mathcal{R}^{\max} > 1$ and $\mathcal{R}^{\text{ave}} \cdot \frac{\mu_1+\alpha}{\alpha} > 1$ as well.



(A) (B)

Figure 8. A numerical solution to $(\text{SVEIQR})_{t_0}^{\text{periodic}}$ given a set of model parameters for which $\mathcal{R}^{\max} > 1$ and $\mathcal{R}^{\text{ave}} \cdot \frac{\mu_1+\alpha}{\alpha} > 1$. In (A), the graphs of all compartments are shown while in (B), a zoomed-in graph for $I(t)$ is presented. It can be observed that the solution eventually enters a periodic (or periodic-like) state, with fluctuations that seem to persist over time.

6. Discussion and Conclusion

This work focused on the mathematical analysis an SVEIQR model with a switching transmission rate. Representing the model as a nonautonomous nonlinear system of differential equations, we discussed fundamental properties of the solutions including existence, uniqueness, boundedness, and the existence of a unique DFE. Mainly using tools from theory of switched systems ([7]), we then established two sufficient conditions for the global attractivity of this DFE. The first condition (Theorem 12) involves the maximum reproduction number $\mathcal{R}^{\max}$, which is simply the maximum among the basic reproduction numbers of the system's different modes. The second condition (Theorem 15), on the other hand, involves the time-weighted average $\mathcal{R}^{\text{ave}}$ of the basic reproduction numbers of the system's modes and applies when the switching rate is periodic. More precisely, under some general assumptions, the conditions $\mathcal{R}^{\max} < 1$ and $\mathcal{R}^{\text{ave}} \cdot \frac{\mu_1+\alpha}{\alpha} < 1$ were established to be sufficient for the global attractivity of the DFE of $(\text{SVEIQR})_{t_0}^{\text{switch}}$ and $(\text{SVEIQR})_{t_0}^{\text{periodic}}$, respectively. The second condition, motivated by the example in Fig. 4, implies that, despite periods of higher transmission rates, the system is still guaranteed to enter the disease-free state eventually as long as the time-weighted average of the modes' reproduction numbers is sufficiently small.

We illustrated our results using simulations and also explored scenarios when the aforementioned sufficient conditions did not hold. In two simulated outputs wherein $\mathcal{R}^{\max} > 1$ and $\mathcal{R}^{\text{ave}} \cdot \frac{\mu_1+\alpha}{\alpha} > 1$, we saw that the numerical solutions, as may be expected, did not converge to an equilibrium solution. However, we observed that the solutions eventually became periodic (or periodic-like), with fluctuations that seemed to persist over time. It is worth-noting that this remained the case even when we deliberately used a transmission rate for which the system spends majority of the time in one mode. From a mathematical perspective, further work on this eventual periodic nature of solutions may prove to be interesting and relevant.

In another simulation, again with $\mathcal{R}^{\max} > 1$ and $\mathcal{R}^{\text{ave}} \cdot \frac{\mu_1+\alpha}{\alpha} > 1$, we saw that the numerical solution converged to the DFE. Thus, the conditions $\mathcal{R}^{\max} < 1$ and $\mathcal{R}^{\text{ave}} \cdot \frac{\mu_1+\alpha}{\alpha} < 1$ are sufficient but not necessary for the convergence of solutions to the DFE.

As switching transmission rates can be used to model public health interventions (such as the imposition of lockdown measures whose stringency level may shift over time), our results provide a better understanding of the effects of such measures on long-term disease transmission dynamics. Particularly, the sufficient condition $\mathcal{R}^{\text{ave}} \cdot \frac{\mu_1+\alpha}{\alpha} < 1$ and the simulation results discussed previously suggest that even if higher transmission rates are allowed over certain periods (i.e., to the point that $\mathcal{R}^{\max} > 1$, as was the case for the example presented in Figure 4), it is still possible for the system to reach a disease-free state eventually. This is a relevant insight to public health decision-makers who have to balance public health outcomes with other considerations such as economic and social costs. However, for these results to be more practical, we hope that future work can establish alternative or stronger sufficient conditions and/or explore necessary conditions for the global attractivity of the DFE.

For future work, we also recommend studies that focus on applications of switched compartmental models for forecasting the prevalence or incidence of infectious cases, and for quantifying the effectiveness of some public health interventions.

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