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APPROXIMATING COMMON FIXED POINT OF THE SUZUKI GENERALIZED NONEXPANSIVE MAPPINGS

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Abstract. In this paper, we construct an iteration scheme involving a hybrid pair of the Suzuki generalized nonexpansive single-valued and multi-valued mappings in a complete CAT(0) spaces. In process, we remove a restricted condition (called end-point condition) in Akkasriworn and Sokhuma's results [2] in Banach spaces and utilize the same to prove some convergence theorems. The results in this paper, are analogs of the results of Akkasriworn et al. [3] in Banach spaces.

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1. Introduction

Fixed point theory in a CAT(0) space was first studied by Kirk [10, 11]. He showed that every nonexpansive mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then the existence problem of fixed point and the $\Delta-$convergence problem of iterative sequences to a fixed point for nonexpansive mappings, Suzuki generalized nonexpansive mappings in a CAT(0) space have been rapidly developed and have appeared in many papers.

Let (X, d) be a geodesic metric space. We denote by 2^K the family of nonempty subsets of K, by $FB(K)$ the collection of all nonempty closed bounded subsets of K, by $KC(K)$ the collection of all nonempty compact convex subsets of K.

A subset K of X is called proximinal if for each $x \in X$, there exists an element $k \in K$ such that

$$d(x, k) = \text{dist}(x, K) = \inf\{d(x, y) : y \in K\}.$$

We denote by $PB(K)$, the collection of all nonempty bounded proximinal subsets of K.

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Let H be the Hausdorff metric with respect to d, that is,

$$H(A,B) = \max\{\sup_{x\in A} \text{dist}(x,B), \sup_{y\in B} \text{dist}(y,A)\}, \quad A,B \in FB(X),$$

where $\text{dist}(x,B) = \inf\{d(x,y) : y \in B\}$ is the distance from the point x to the subset B.

A mapping $t : K \to K$ is said to be *nonexpansive* if

$$d(tx,ty) \le d(x,y) \text{ for all } x,y \in K.$$

A mapping $t : K \to K$ is said to be *Suzuki generalized nonexpansive* if

$$\frac{1}{2}d(x,tx) \le d(x,y) \Rightarrow d(tx,ty) \le d(x,y) \text{ for all } x,y \in K.$$

A point x is called a fixed point of t if $tx = x$.

A multi-valued mapping $T : K \to FB(K)$ is said to be *nonexpansive* if

$$H(Tx,Ty) \le d(x,y) \text{ for all } x,y \in K.$$

In 2010, Abkar and Eslamian [1] mentioned the Suzuki generalized multi-valued nonexpansive mapping as follows:

A multi-valued mapping $T : K \to FB(K)$ is said to be a *Suzuki generalized multi-valued nonexpansive mapping* if

$$\frac{1}{2}\text{dist}(x,Tx) \le d(x,y) \Rightarrow H(Tx,Ty) \le d(x,y) \text{ for all } x,y \in K.$$

Let $T : K \to PB(K)$ be a multi-valued mapping and define the mapping P_T for each x by

$$P_T(x) := \{y \in Tx : d(x,y) = \text{dist}(x,Tx)\}.$$

A point x is called a fixed point for a multi-valued mapping T if $x \in Tx$.

We use the notation $Fix(T)$ stands for the set of fixed points of a mapping T and $Fix(t) \cap Fix(T)$ stands for the set of common fixed points of t and T. Precisely, a point x is called a common fixed point of t and T if $tx = x \in Tx$.

In 2009, Agarwal et al. [6] introduced the S-iteration following well-known iteration. For E a convex subset of a linear space X and t a mapping of E into itself, the iterative sequence $\{x_n\}$ of the S-iteration process is generated from $x_1 \in E$ and is defined by

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n t x_n \\ x_{n+1} &= (1-\alpha_n)tx_n + \alpha_n y_n, \end{aligned}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$ satisfying the condition:

$$\sum_{n=1}^{\infty} \alpha_n \beta_n (1-\beta_n) = \infty.$$

In 2013, Sokhuma [15] proved the convergence theorem for a common fixed point in CAT(0) spaces as follows:

Theorem 1.1. *Let K be a nonempty compact convex subset of a complete CAT(0) space X, and $t : K \to K$ and $T : K \to FC(K)$ a single-valued nonexpansive mapping and a multi-valued nonexpansive mapping, respectively, and $Fix(t) \cap Fix(T) \neq \emptyset$ satisfying $Tw = \{w\}$ for all $w \in Fix(t) \cap Fix(T)$. Let the iterative sequence $\{x_n\}$ is generated by $x_1 \in K$,*

$$y_n = (1 - \beta_n)x_n \oplus \beta_n z_n,$$
$$x_{n+1} = (1 - \alpha_n)x_n \oplus \alpha_n t y_n,$$

for all $n \in \mathbb{N}$, where $z_n \in Tx_n$ and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences of positive numbers satisfying $0 < a \leq \alpha_n, \beta_n \leq b < 1$. Then the sequence $\{x_n\}$ converges strongly to a common fixed point of t and T.

In 2015, Akkasriworn and Sokhuma [2] proved the convergence theorem for a common fixed point in a complete CAT(0) spaces as follow:

Theorem 1.2. *Let K be a nonempty bounded closed convex subset of a complete CAT(0) space X, $t : K \to K$ and $T : K \to FB(K)$ an asymptotically nonexpansive mapping and a multi-valued nonexpansive mapping, respectively. Assume that t and T are commuting and Fix$(t) \cap$ Fix$(T) \neq \emptyset$ satisfying $Tw = \{w\}$ for all $w \in$ Fix$(t) \cap$ Fix(T) and $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. Let $\{x_n\}$ be the sequence of the modified Ishikawa iteration defined by*

$$y_n = (1 - \beta_n)x_n \oplus \beta_n z_n,$$
$$x_{n+1} = (1 - \alpha_n)x_n \oplus \alpha_n t^n y_n,$$

for all $n \in \mathbb{N}$, where $z_n \in T(t^n x_n)$ and $\{\alpha_n\}$, $\{\beta_n\} \in [0, 1]$. Then $\{x_n\}$ is $\Delta-$convergent to a common fixed point of t and T.

In 2019, Sokhuma [16] proved the convergence theorem for a common fixed point in CAT(0) spaces as follow:

Theorem 1.3. *Let K be a nonempty bounded closed convex subset of a complete CAT(0) space X, $t : K \to K$ be a single-valued asymptotically nonexpansive mapping, and $T : K \to PB(K)$ be a multi-valued nonexpansive mapping and*

$$P_T(x) = \{y \in Tx : d(x, y) = dist(x, Tx)\}.$$

For fixed $x_1 \in K$. The sequence $\{x_n\}$ of the Ishikawa iteration is defined by

$$y_n = (1 - \beta_n)x_n \oplus \beta_n z_n,$$
$$x_{n+1} = (1 - \alpha_n)x_n \oplus \alpha_n t^n y_n,$$

for all $n \in \mathbb{N}$, where $z_n \in P_T(t^n x_n)$ and $\{\alpha_n\}$, $\{\beta_n\} \in (0, 1)$. Then $\{x_n\}$ is $\Delta-$convergent to a common fixed point of t and T.

In 2011, Espinola et al. [9] proved the theorem for a common fixed point in CAT(0) spaces as follows:

Theorem 1.4. *Let X be a complete uniformly convex space with convex metric and K be a bounded closed convex subset of X. Suppose $t : K \to K$ and $T : K \to KC(X)$ be a Suzuki generalized nonexpansive single-valued and a multi-valued mapping, respectively. If t and T are commute, then there exists $w \in K$ such that $tw = w \in Tw$.*

The purpose of this paper is to study the iterative process, called the Ishikawa iteration method with respect to the Suzuki generalized nonexpansive single-valued and multi-valued mapping in a complete CAT(0) spaces.

We also establish the convergence theorem of a sequence from such process in a nonempty bounded closed convex subset of a complete CAT(0) spaces. We remove a restricted condition (called end-point condition) in Akkasriworn et al. [3] and expand the results of Sokhuma [16] results.

Now, we introduce an iteration method modifying the above ones and call it the S-iteration method.

Definition 1.5. *Let K be a nonempty bounded closed convex subset of a complete CAT(0) space X, $t : K \to K$ and $T : K \to PB(K)$ be a Suzuki generalized nonexpansive single-valued and a multi-valued mapping, respectively and*

$$P_T(x) = \{y \in Tx : d(x,y) = dist(x,Tx)\}.$$

For fixed $x_1 \in K$. The sequence $\{x_n\}$ of the S-iteration is defined by

$$\begin{aligned} y_n &= (1-\beta_n)x_n \oplus \beta_n z_n, \\ x_{n+1} &= (1-\alpha_n)z_n \oplus \alpha_n t y_n, \end{aligned} \tag{1.1}$$

for all $n \in \mathbb{N}$, where $z_n \in P_T(tx_n)$ and $\{\alpha_n\}$, $\{\beta_n\} \in (0,1)$.

2. Preliminaries

With a view to make, our presentation self contained, we collect some relevant basic definitions, results and iterative methods which will be used frequently in the text later.

Let (X,d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ is a map c from a closed interval $[0,s] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(s) = y$, and $d(c(t), c(u)) = |t-u|$ for all $t, u \in [0,s]$. In particular, c is an isometry and $d(x,y) = s$. The image α of c is called a geodesic (or metric) segment joining x and y. When it is unique this geodesic segment is denoted by $[x,y]$. The space (X,d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be convex if Y includes every geodesic segment joining any two of its points.

A geodesic triangle $\Delta(x_1,x_2,x_3)$ in a geodesic metric space (X,d) consists of three points x_1,x_2,x_3 in X (the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ). A comparison triangle for the geodesic triangle $\Delta(x_1,x_2,x_3)$ in (X,d) is a triangle $\overline{\Delta}(x_1,x_2,x_3) := \Delta(\overline{x}_1,\overline{x}_2,\overline{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\overline{x}_i,\overline{x}_j) = d(x_i,x_j)$ for $i,j \in \{1,2,3\}$.

A geodesic space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom.

Let Δ be a geodesic triangle in X and let $\overline{\Delta}$ be a comparison triangle for Δ. Then Δ is said to satisfy the CAT(0) inequality if for all $x, y \in \Delta$ and all comparison points $\overline{x}, \overline{y} \in \overline{\Delta}$, $d(x,y) \leq d_{\mathbb{E}^2}(\overline{x}, \overline{y})$. If x, y_1, y_2 are points in a CAT(0) space and $y_0 = \frac{1}{2}y_1 \oplus \frac{1}{2}y_2$, then the CAT$(0)$ inequality implies that

$$d(x,y_0)^2 \leq \frac{1}{2}d(x,y_1)^2 + \frac{1}{2}d(x,y_2)^2 - \frac{1}{4}d(y_1,y_2)^2. \tag{2.1}$$

This is the (CN) inequality of Bruhat and Tits [5]. In fact, a geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality [4].

The following results and methods deal with the concept of asymptotic centers. Let K be a nonempty closed convex subset of a CAT(0) space X and $\{x_n\}$ be a bounded sequence in X. For $x \in X$, define the asymptotic radius of $\{x_n\}$ at x as the number

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x_n, x).$$

Let $r \equiv r(K, \{x_n\}) := \inf \{r(x, \{x_n\}) : x \in K\}$ and $A \equiv A(K, \{x_n\}) := \{x \in K : r(x, \{x_n\}) = r\}$.

The number r and the set A are called the asymptotic radius and asymptotic center of $\{x_n\}$ relative to K, respectively.

It is easy to know that if X is a complete CAT(0) spaces and K is a closed convex subset of X, then $A(K, \{x_n\})$ consists of exactly one point. A sequence $\{x_n\}$ in CAT(0) space X is said to be $\Delta-$convergent to $x \in X$ if x is the unique asymptotic center of every subsequence of $\{x_n\}$. A bounded sequence $\{x_n\}$ is said to be regular with respect to K if for every subsequence $\{x'_n\}$, we get

$$r(K, \{x_n\}) = r(K, \{x'_n\}).$$

We now give the definition of $\Delta-$convergence.

Definition 2.1. [11,13] *A sequence $\{x_n\}$ in a CAT(0) space X is said to be $\Delta-$convergent to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta - \lim_{n\to\infty} x_n = x$ and call x the $\Delta-$limit of $\{x_n\}$.*

We now collect some elementary facts about CAT(0) spaces which will be used in the proofs of our main results. The following lemma can be found in [7,8,11].

Lemma 2.2. [7] *If K is a closed convex subset of a complete $CAT(0)$ space and $\{x_n\}$ is a bounded sequence in K, then the asymptotic center of $\{x_n\}$ is in K.*

Lemma 2.3. [8] *Let (X, d) be a $CAT(0)$ space.*

(i) *For $x, y \in X$ and $u \in [0,1]$, there exists a unique point $z \in [x,y]$ such that*

$$d(x,z) = ud(x,y) \quad \textit{and} \quad d(y,z) = (1-u)d(x,y). \tag{2.2}$$

We use the notation $(1-u)x \oplus ty$ for the unique point z satisfying (2.2).

(ii) *For $x, y, z \in X$ and $u \in [0,1]$, we have*

$$d((1-u)x \oplus uy, z) \leq (1-u)d(x,z) + ud(y,z).$$

Lemma 2.4. [11] *Every bounded sequence in a complete $CAT(0)$ space has a $\Delta-$convergent subsequence.*

We now collect some basic properties of Suzuki generalized nonexpansive mapping. Although the proofs follow the idea of the proofs in [17]. The following two propositions are very easy to verify.

Proposition 2.5. *Let K be a nonempty subset of a $CAT(0)$ space X and $t : K \to K$ be a nonexpansive mapping. Then t is a Suzuki generalized nonexpansive mapping.*

Proposition 2.6. *Let K be a nonempty subset of a $CAT(0)$ space X. Suppose $t : K \to K$ is a Suzuki generalized nonexpansive mapping and has a fixed point. Then t is a quasi-nonexpansive mapping.*

Lemma 2.7. *Let K be a nonempty subset of a CAT(0) space X. Suppose $t : K \to K$ is a Suzuki generalized nonexpansive mapping. Then*

$$d(x, ty) \leq 3d(tx, x) + d(x, y)$$

holds for all $x, y \in K$.

The existence of fixed points for generalized Suzuki nonexpansive mappings in CAT(0) spaces was proved by Nanjaras et al. [14] as the following result.

Theorem 2.8. *Let K be a nonempty bounded closed convex subset of a complete CAT(0) space X. Suppose $t : K \to K$ is a Suzuki generalized nonexpansive mappings. Then t has a fixed point in K.*

Lemma 2.9. *Let K be a closed and convex subset of a complete $CAT(0)$ space X and let $t : K \to X$ be a generalized Suzuki nonexpansive mappings. Let $\{x_n\}$ be a bounded sequence in K such that $\lim_{n\to\infty} d(tx_n, x_n) = 0$ and $\Delta - \lim_{n\to\infty} x_n = w$. Then $tw = w$.*

The following fact is well-known [12].

Lemma 2.10. *Let X be a complete $CAT(0)$ space and let $x \in X$. Suppose $\{\alpha_n\}$ is a sequence in $[a,b]$ for some $a, b \in (0,1)$ and $\{x_n\}, \{y_n\}$ are sequences in X such that $\limsup_{n\to\infty} d(x_n, x) \leq r$, $\limsup_{n\to\infty} d(y_n, x) \leq r$, and $\lim_{n\to\infty} d((1-\alpha_n)x_n \oplus \alpha_n y_n, x) = r$ for some $r \geq 0$. Then $\lim_{n\to\infty} d(x_n, y_n) = 0$.*

Lemma 2.11. *Let X be a CAT(0) space, K be a nonempty compact convex subset of X and $\{x_n\}$ be a sequence in K. Then,*

$$dist(y, Ty) \leq d(y, x_n) + dist(x_n, Tx_n) + H(Tx_n, Ty)$$

where $y \in K$ and T be a multi-valued mapping from K in to $FB(K)$.

3. Main Results

We first prove the following lemmas which play very important roles in this section.

Lemma 3.1. *Let $T : K \to PB(K)$ be a multi-valued mapping and*
$P_T(x) = \{y \in Tx : d(x, y) = dist(x, Tx)\}$. Then the followings are equivalent:

(1) *$x \in Fix(T)$, that is $x \in Tx$;*

(2) *$P_T(x) = \{x\}$, that is $x = y$ for each $y \in P_T(x)$;*

(3) *$x \in Fix(P_T)$, that is $x \in P_T(x)$.*

Further, $Fix(T) = Fix(P_T)$.

Proof. $(1) \Rightarrow (2)$. Since $x \in Tx$, $d(x, Tx) = 0$. Therefore, for any $y \in P_T(x)$, $d(x, y) = \text{dist}(x, Tx) = 0$ and so $x = y$. That is, $P_T(x) = \{x\}$.

$(2) \Rightarrow (3)$. Since $P_T(x) = \{x\}$, $x \in Fix(P_T)$ and we get $x \in P_T(x)$.

$(3) \Rightarrow (1)$. Since $x \in Fix(P_T)$, $x \in P_T(x)$. Therefore, $d(x, x) = \text{dist}(x, Tx) = 0$ and so $x \in Tx$ by the closedness of Tx. This implies that $Fix(T) = Fix(P_T)$. □

Lemma 3.2. *Let K be a nonempty bounded closed convex subset of a complete CAT(0) space X, $t : K \to K$ and $T : K \to PB(K)$ be a single-valued and a multi-valued of Suzuki generalized nonexpansive mapping, respectively, with $Fix(t) \cap Fix(T) \neq \emptyset$ such that P_T is nonexpansive. Let $\{x_n\}$ be the sequence of S-iteration defined by (1.1). Then $\lim_{n\to\infty} d(x_n, w)$ exists for all $w \in Fix(t) \cap Fix(T)$.*

Proof. Let $x_1 \in K$ and $w \in Fix(t) \cap Fix(T)$, in view of Lemma 3.1 we have $w \in P_T(w) = \{w\}$. Since, $\frac{1}{2}d(tw, w) = 0 \leq d(x_n, w), d(tx_n, tw) \leq d(x_n, w)$. Similarly, we obtain $\frac{1}{2}d(tw, w) = 0 \leq d(y_n, w)$, and then we get $d(ty_n, tw) \leq d(y_n, w)$. Now consider,

$$\begin{aligned}
d(y_n, w) &= d((1-\beta_n)x_n \oplus \beta_n z_n, w) \\
&\leq (1-\beta_n)d(x_n, w) + \beta_n d(z_n, w) \\
&= (1-\beta_n)d(x_n, w) + \beta_n \text{dist}(z_n, P_T(w)) \\
&\leq (1-\beta_n)d(x_n, w) + \beta_n H(P_T(tx_n), P_T(w)) \\
&\leq (1-\beta_n)d(x_n, w) + \beta_n d(tx_n, w) \\
&\leq (1-\beta_n)d(x_n, w) + \beta_n d(x_n, w) \\
&= d(x_n, w).
\end{aligned}$$

Hence, $d(y_n, w) \leq d(x_n, w)$. It implied that

$$\begin{aligned} d(x_{n+1}, w) &= d((1-\alpha_n)z_n \oplus \alpha_n t y_n, w) \\ &\leq (1-\alpha_n)d(z_n, w) + \alpha_n d(ty_n, w) \\ &\leq (1-\alpha_n)\text{dist}(z_n, P_T(w)) + \alpha_n d(y_n, w) \\ &\leq (1-\alpha_n)H(P_T(tx_n), P_T(w)) + \alpha_n d(y_n, w) \\ &\leq (1-\alpha_n)d(x_n, w) + \alpha_n d(x_n, w) \\ &= d(x_n, w). \end{aligned}$$

Since $\{d(x_n, w)\}$ is bounded below and decreasing sequence, we obtain the limit of $\{d(x_n, w)\}$. □

Lemma 3.3. *Let K be a nonempty bounded closed convex subset of a complete $CAT(0)$ space X, $t : K \to K$ and $T : K \to PB(K)$ be a single-valued and a multi-valued of Suzuki generalized nonexpansive mapping, respectively, with $Fix(t) \cap Fix(T) \neq \emptyset$ such that P_T is nonexpansive. Let $\{x_n\}$ be the sequence of S-iteration defined by (1.1). If $0 < a \leq \alpha_n \leq b < 1$ for some $a, b \in \mathbb{R}$. Then $\lim_{n\to\infty} d(ty_n, z_n) = 0$.*

Proof. Let $x_1 \in K$ and $w \in Fix(t) \cap Fix(T)$, in view of Lemma 3.1 we have $w \in P_T(w) = \{w\}$. From Lemma 3.2, we setting $\lim_{n\to\infty} d(x_n, w) = c$. Recall that, $d(tz_n, w) \leq d(z_n, w) \leq d(x_n, w)$. Then we have,

$$\limsup_{n\to\infty} d(tz_n, w) \leq \limsup_{n\to\infty} d(z_n, w) \leq \limsup_{n\to\infty} d(x_n, w) = c. \tag{3.1}$$

Hence, $d(ty_n, w) \leq d(y_n, w) \leq d(x_n, w)$. It implied that

$$\limsup_{n\to\infty} d(ty_n, w) \leq \limsup_{n\to\infty} d(y_n, w) \leq \limsup_{n\to\infty} d(x_n, w) = c. \tag{3.2}$$

Since, $c = \lim_{n\to\infty} d(x_{n+1}, w) = \lim_{n\to\infty} d((1-\alpha_n)z_n \oplus \alpha_n t y_n, w)$, it follows from the condition of α_n and Lemma 2.10 that $\lim_{n\to\infty} d(ty_n, z_n) = 0$. □

Lemma 3.4. *Let K be a nonempty bounded closed convex subset of a complete $CAT(0)$ space X, $t : K \to K$ and $T : K \to PB(K)$ be a single-valued and a multi-valued of Suzuki generalized nonexpansive mapping, respectively, with $Fix(t) \cap Fix(T) \neq \emptyset$ such that P_T is nonexpansive. Let $\{x_n\}$ be the sequence of S-iteration defined by (1.1). If $0 < a \leq \alpha_n \leq b < 1$ for some $a, b \in \mathbb{R}$. Then $\lim_{n\to\infty} d(x_n, z_n) = 0$.*

Proof. Let $x_0 \in K$ and $w \in Fix(t) \cap Fix(T)$, in view of Lemma 3.1 we have $w \in P_T(w) = \{w\}$. Consider,

$$\begin{aligned} d(x_{n+1}, w) &= d((1-\alpha_n)z_n \oplus \alpha_n t y_n, w) \\ &\leq (1-\alpha_n)d(z_n, w) + \alpha_n d(ty_n, w) \\ &\leq (1-\alpha_n)\text{dist}(z_n, P_T(w)) + \alpha_n d(y_n, w) \\ &\leq (1-\alpha_n)H(P_T(tx_n), P_T(w)) + \alpha_n d(y_n, w) \end{aligned}$$

$$\leq (1-\alpha_n)d(tx_n, w) + \alpha_n d(y_n, w)$$
$$\leq (1-\alpha_n)d(x_n, w) + \alpha_n d(y_n, w)$$

and hence

$$\frac{d(x_{n+1}, w) - d(x_n, w)}{\alpha_n} \leq d(y_n, w) - d(x_n, w).$$

Therefore, since $0 < a \leq \alpha_n \leq b < 1$,

$$\left(\frac{d(x_{n+1}, w) - d(x_n, w)}{\alpha_n}\right) + d(x_n, w) \leq d(y_n, w).$$

Thus,

$$\liminf_{n\to\infty} \left\{ \left(\frac{d(x_{n+1}, w) - d(x_n, w)}{\alpha_n}\right) + d(x_n, w) \right\} \leq \liminf_{n\to\infty} d(y_n, w).$$

It follows that $c \leq \liminf_{n\to\infty} d(y_n, w)$. Since, from (3.2), $\limsup_{n\to\infty} d(y_n, w) \leq c$, we have

$$c = \lim_{n\to\infty} d(y_n, w) = \lim_{n\to\infty} d((1-\beta_n)x_n \oplus \beta_n z_n, w). \tag{3.3}$$

Recall that

$$d(z_n, w) = \text{dist}(z_n, P_T(w)) \leq H(P_T(tx_n), P_T(w)) \leq d(x_n, w).$$

Hence we have

$$\limsup_{n\to\infty} d(z_n, w) \leq \limsup_{n\to\infty} d(x_n, w) = c.$$

Since $0 < a \leq \alpha_n \leq b < 1$ for some $a, b \in \mathbb{R}$ and (3.3) we obtain $\lim_{n\to\infty} d(x_n, z_n) = 0$. □

Lemma 3.5. *Let K be a nonempty bounded closed convex subset of a complete $CAT(0)$ space X, $t : K \to K$ and $T : K \to PB(K)$ be a single-valued and a multi-valued of Suzuki generalized nonexpansive mapping, respectively, with $Fix(t) \cap Fix(T) \neq \emptyset$ such that P_T is nonexpansive. Let $\{x_n\}$ be the sequence of S-iteration defined by (1.1). If $0 < a \leq \alpha_n, \beta_n \leq b < 1$ for some $a, b \in \mathbb{R}$. Then $\lim_{n\to\infty} d(ty_n, x_n) = 0$ and $\lim_{n\to\infty} d(tx_n, x_n) = 0$.*

Proof. Recall that $d(ty_n, x_n) \leq d(ty_n, z_n) + d(z_n, x_n)$. Hence we have

$$\limsup_{n\to\infty} d(ty_n, x_n) \leq \limsup_{n\to\infty} d(ty_n, z_n) + \limsup_{n\to\infty} d(z_n, x_n).$$

Since, from Lemma 3.3 and Lemma 3.4, we have

$$\lim_{n\to\infty} d(ty_n, x_n) = 0. \tag{3.4}$$

By Lemma 2.7, we obtain

$$d(tx_n, x_n) \leq d(tx_n, y_n) + d(y_n, x_n)$$
$$\leq 3d(ty_n, y_n) + d(x_n, y_n) + d(x_n, y_n)$$
$$= 3d(ty_n, y_n) + 2d(x_n, y_n)$$

$$
\begin{aligned}
&\leq 3d(ty_n, x_n) + 3d(x_n, y_n) + 2d(x_n, y_n)\\
&= 3d(ty_n, x_n) + 5d(x_n, y_n)\\
&= 3d(ty_n, x_n) + 5d(x_n, (1-\beta_n)x_n \oplus \beta_n z_n)\\
&\leq 3d(ty_n, x_n) + 5(1-\beta_n)d(x_n, x_n) + 5\beta_n d(x_n, z_n)\\
&= 3d(ty_n, x_n) + 5\beta_n d(x_n, z_n).
\end{aligned}
$$

Therefore, we have

$$\lim_{n\to\infty} d(tx_n, x_n) \leq \lim_{n\to\infty} 3d(ty_n, x_n) + \lim_{n\to\infty} 5\beta_n d(x_n, z_n).$$

Hence, by (3.4) and Lemma 3.4, $\lim_{n\to\infty} d(tx_n, x_n) = 0$. □

Theorem 3.6. *Let K be a nonempty bounded closed convex subset of a complete $CAT(0)$ space X, $t : K \to K$ and $T : K \to PB(K)$ be a single-valued and a multi-valued of Suzuki generalized nonexpansive mapping, respectively, with Fix$(t) \cap$ Fix$(T) \neq \emptyset$ such that P_T is nonexpansive. Let $\{x_n\}$ be the sequence of S-iteration defined by (1.1). If $0 < a \leq \alpha_n, \beta_n \leq b < 1$ for some $a, b \in \mathbb{R}$, then $\{x_n\}$ is $\Delta-$convergent to y in $Fix(t) \cap Fix(T)$.*

Proof. Since $\{x_n\}$ is $\Delta-$convergent to y, from Lemma 3.5, $\lim_{n\to\infty} d(tx_n, x_n) = 0$. By Lemma 2.9, $y \in K$ and $ty = y$, it follows that $y \in Fix(t)$. By Lemma 2.11, which implies that

$$
\begin{aligned}
\text{dist}(y, P_T(y)) &\leq d(y, x_n) + \text{dist}(x_n, P_T(tx_n)) + H(P_T(tx_n), P_T(y))\\
&\leq d(y, x_n) + d(x_n, z_n) + d(tx_n, y)\\
&\leq d(x_n, y) + d(x_n, z_n) + d(tx_n, x_n) + d(x_n, y) \to 0 \ \text{ as } n \to \infty.
\end{aligned}
$$

It follows that, $y \in Fix(P_T)$ then $y \in Fix(T)$. Therefore $y \in Fix(t) \cap Fix(T)$ as desired. □

Theorem 3.7. *Let K be a nonempty bounded closed convex subset of a complete $CAT(0)$ space X, $t : K \to K$ and $T : K \to PB(K)$ be a single-valued and a multi-valued of Suzuki generalized nonexpansive mapping, respectively, with $Fix(t) \cap Fix(T) \neq \emptyset$ such that P_T is nonexpansive. Let $\{x_n\}$ be the sequence of S-iteration defined by (1.1). If $0 < a \leq \alpha_n, \beta_n \leq b < 1$ for some $a, b \in \mathbb{R}$, then $\{x_n\}$ is $\Delta-$convergent to a common fixed point of t and T.*

Proof. Since Lemma 3.5 guarantees that $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(tx_n, x_n) = 0$. We now let $\omega_w(x_n) := \bigcup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. We claim that $\omega_w(x_n) \subset Fix(t) \cap Fix(T)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 2.2 and Lemma 2.3 there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_{n\to\infty} v_n = v \in K$. Since

$\lim_{n\to\infty} d(tv_n, v_n) = 0$, then $v \in Fix(t)$. Hence,

$$\begin{aligned}
\text{dist}(v, P_T(v)) &\le \text{dist}(v, P_T(tv_n)) + H(P_T(tv_n), P_T(v)) \\
&\le d(v, z_n) + d(tv_n, v) \\
&\le d(v, v_n) + d(v_n, z_n) + d(tv_n, v) \to 0 \ \text{ as } n \to \infty.
\end{aligned}$$

It follows that $v \in Fix(P_T)$, we get $v \in Fix(T)$ by Lemma 3.1. Therefore $v \in Fix(t) \cap Fix(T)$ as desired. We claim that $u = v$. If not, since t is a single-valued Suzuki generalized mapping and $v \in Fix(t) \cap Fix(T)$ such that $\lim_{n\to\infty} d(x_n, v)$ exists by Lemma 3.2, then by the uniqueness of asymptotic centers,

$$\begin{aligned}
\limsup_{n\to\infty} d(v_n, v) &< \limsup_{n\to\infty} d(v_n, u) \\
&\le \limsup_{n\to\infty} d(u_n, u) \\
&< \limsup_{n\to\infty} d(u_n, v) \\
&\le \limsup_{n\to\infty} d(x_n, v) \\
&= \limsup_{n\to\infty} d(v_n, v)
\end{aligned}$$

which is a contradiction, and hence $u = v \in Fix(t) \cap Fix(T)$.

To show that $\{x_n\}$ is $\Delta-$convergent to a common fixed point, it suffices to show that $\omega_w(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$. By Lemma 2.2 and Lemma 2.3 there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_{n\to\infty} v_n = v \in K$. Let $A(\{u_n\}) = \{u\}$ and $A(\{x_n\}) = \{x\}$. We have seen that $u = v$ and $v \in Fix(t) \cap Fix(T)$.

We can complete the proof by showing that $x = v$. If not, since $\lim_{n\to\infty} d(x_n, v)$ exists, by the uniqueness of asymptotic center,

$$\begin{aligned}
\limsup_{n\to\infty} d(v_n, v) &< \limsup_{n\to\infty} d(v_n, x) \\
&\le \limsup_{n\to\infty} d(x_n, x) \\
&< \limsup_{n\to\infty} d(x_n, v) \\
&= \limsup_{n\to\infty} d(v_n, v)
\end{aligned}$$

which is a contradiction, and hence the conclusion follows. $\square$

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