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ITERATIVE APPROXIMATION OF FIXED POINTS USING A NEW ITERATION SCHEME

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Abstract. We present iterative approximation outcomes for a new iterative scheme for finding fixed points(FPs) of a nonexpansive map in a uniformly convex Banach space (UCB-space), which are new/generalized to some recently published results of the literature.

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1. Introduction

Functional and integral equations arise from a wide range of engineering and applied science problems. These equations may be transformed into FP theorems using a simple procedure. Furthermore, as shown in [1–4], the presence and distinctiveness of solutions to such integral and differential equations have been established using the FP theorem.

Some schematic algorithms determine FPs by investigating their characteristics. The iterative procedures have been tweaked to get approximate results. An initial iterative procedure for approximating the FP of a contraction mapping is the Picard iterative process. Several prominent mathematicians have lately devised iterative algorithms that accelerate approximation to the FP [5–10].

The study of new iterative techniques for FP and functional equation solutions is a popular area of study with many useful applications (see, for example, [11–17] and others). For the nonexpansive mappings, we thus aim to develop a novel iterative approach. Additionally, using examples, we demonstrate that this novel iterative procedure provides better estimates than other approaches.

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The new iteration approach is defined by:

$$r_n = \mathcal{T}((1-\mathfrak{m}_n)\omega_n + \mathfrak{m}_n\mathcal{T}\omega_n$$
$$s_n = \mathcal{T}((1-v_n)r_n + v_n\mathcal{T}r_n) \tag{1}$$
$$\omega_{n+1} = \mathcal{T}(\mathcal{T}s_n)$$

for all $n \geq 0$ where $\mathfrak{m}_n, v_n, s_n \in (0,1)$.

2. Prelimenris

Throughout this work, $\Upsilon(\mathcal{T})$ indicates the collection of each FP of $\mathcal{T}$, and X is a Banach space (briefly BN-space).

Definition 2.1 ([18]). *Let $\mathcal{Z} \subseteq X$. A mapping $\mathcal{T} : \mathcal{Z} \to \mathcal{Z}$ is termed as nonexpansive (Nonexp-Map) if for any $x, y \in \mathcal{Z}$,*

$$\|\mathcal{T}(x) - \mathcal{T}(y)\| \leq \|x - y\| \tag{2}$$

Lemma 2.2 ([19]). *Let X be a UCB-space and (x_n) be a real sequence with $0 < a \leq x_n \leq b < 1$. Let (u_n) and (v_n) be any two sequences of X where*

$$\lim_{n\to\infty} \sup \|u_n\| \leq \tau, \quad \lim_{n\to\infty} \sup \|v_n\| \leq \tau,$$

and

$$\lim_{n\to\infty} \sup \|x_n u_n + (1-x_n)v_n\| = \tau,$$

hold for some $\tau \geq 0$. Then,

$$\lim_{n\to\infty} \sup \|u_n - v_n\| = 0.$$

Proposition 2.3 ([20]). *Let $\mathcal{T} : \mathcal{Z} \to \mathcal{Z}$ be a map. Then*

i. If $\mathcal{T}$ is Nonexp-Map, then $\mathcal{T}$ is Suzuki's generalized Nonexp-Map

ii. If $\mathcal{T}$ is Suzuki's generalized Nonexp-Map and posseses an FP, then $\mathcal{T}$ is a quasi Nonexp-Map

iii. If $\mathcal{T}$ is a Suzuki's generalized Nonexp-Map, then
$\|p - \mathcal{T}q\| \leq 3\|\mathcal{T}p - p\| + \|p - q\|$, for all $p, q \in \mathcal{Z}$

Definition 2.4 ([19]). *Let $\mathcal{Z} \subseteq X$ ($\mathcal{Z}$ a non-empty closed convex). Suppose (ω_n) is a bounded in X. We put $r(q, (\omega_n)) = \lim_{n\to\infty} \sup \|\omega_n - q\|$ for $q \in X$.*

The asymptotic radius of (ω_n) relative to $\mathcal{Z}$ is given by:
$r(q, (\omega_n)) = \inf\{r(q, (\omega_n)) : q \in \mathcal{Z}\}$ and the asymptotic center of (ω_n) relative to $\mathcal{Z}$ is given by the following set:

$$\mathcal{U}(\mathcal{Z}, (\omega_n)) = \{q \in \mathcal{Z} : r(q, (\omega_n)) = r(\mathcal{Z}, (\omega_n))\} \tag{3}$$

3. Main Results

In this study part, we provide some convergence findings for Nonexp-Map using the newly proposed approach Eq.1.

Theorem 3.1. *Let $\mathcal{Z} \subseteq X$ ($\mathcal{Z}$ a non-empty closed convex) and $\mathcal{T} : \mathcal{Z} \to \mathcal{Z}$ is a Nonexp-Map with $\Upsilon(\mathcal{T}) \neq \phi$. For $\omega_0 \in \mathcal{Z}$ the sequence (ω_0) generated by Eq.1. Then $\lim_{n\to\infty} \|\omega_n - q\|$ exists for all $q \in \Upsilon(\mathcal{T})$.*

Proof.
Consider $q \in \Upsilon(\mathcal{T})$. From Proposition 2.3 (ii) and Definition 2.1, we have

$$\begin{aligned}
\|r_n - q\| &= \|\mathcal{T}((1-\xi_n)\omega_n + \xi_n\mathcal{T}\omega_n) - q\| \\
&\leq \|(1-\xi_n)\omega_n + \xi_n\mathcal{T}\omega_n - q\| \\
&\leq (1-\xi_n)\|\omega_n - q\| + \xi_n\|\mathcal{T}\omega_n - q\| \\
&\leq (1-\xi_n)\|\omega_n - q\| + \xi_n\|\omega_n - q\| \\
&= \|\omega_n - q\|
\end{aligned} \tag{4}$$

$$\begin{aligned}
\|s_n - q\| &= \|\mathcal{T}((1-v_n)r_n + v_n\mathcal{T}r_n) - q\| \\
&\leq \|(1-v_n)r_n + v_n\mathcal{T}r_n - q\| \\
&\leq (1-v_n)\|r_n - q\| + v_n\|\mathcal{T}r_n - q\| \\
&\leq (1-v_n)\|r_n - q\| + v_n\|r_n - q\| \\
&= \|r_n - q\| \\
&\leq \|\omega_n - q\|
\end{aligned} \tag{5}$$

$$\begin{aligned}
\|\omega_{n+1} - q\| &= \|\mathcal{T}((1-\mathfrak{m}_n)\mathcal{T}r_n + \mathfrak{m}_n\mathcal{T}s_n) - q\| \\
&\leq \|(1-\mathfrak{m}_n)\mathcal{T}r_n + \mathfrak{m}_n\mathcal{T}s_n - q\| \\
&\leq (1-\mathfrak{m}_n)\|\mathcal{T}r_n - q\| + \mathfrak{m}_n\|\mathcal{T}s_n - q\| \\
&\leq (1-\mathfrak{m}_n)\|r_n - q\| + \mathfrak{m}_n\|s_n - q\| \\
&\leq (1-\mathfrak{m}_n)\|\omega_n - q\| + \mathfrak{m}_n\|\omega_n - q\| \\
&= \|\omega_n - q\|
\end{aligned} \tag{6}$$

Hence $\lim_{n\to\infty} \|\omega_n - q\|$ exists for all $q \in \Upsilon(\mathcal{T})$.

Theorem 3.2. *Let $\mathcal{Z} \subseteq X$ ($\mathcal{Z}$ a non-empty closed convex) and $\mathcal{T} : \mathcal{Z} \to \mathcal{Z}$ is a Nonexp-Map. For $\omega_n \in \mathcal{Z}$, the sequence (ω_n) is generated by Eq.1. Then $\Upsilon(\mathcal{T}) \neq \phi$; if and only if $\lim_{n\to\infty} \|\mathcal{T}\omega_n - \omega_n\| = 0$.*

Proof.

Assume $\Upsilon(\mathcal{T}) \neq \phi$ and $q \in \Upsilon(\mathcal{T})$. By theorem 3.1, $\lim_{n\to\infty} \|\omega_n - q\|$ exists and (ω_n) is bounded. Let

$$\lim_{n\to\infty} \|\omega_n - q\| = c \tag{7}$$

Utilizing Eq.1 and Eq.7, we get

$$\lim_{n\to\infty} \sup \|r_n - q\| \leq \lim_{n\to\infty} \sup \|\omega_n - q\| = c \tag{8}$$

According to Proposition 2.3(ii),

$$\lim_{n\to\infty} \sup \|\mathcal{T}\omega_n - q\| \leq \lim_{n\to\infty} \sup \|\omega_n - q\| = c \tag{9}$$

On the other hand

$$\begin{aligned}
\|\omega_{n+1} - q\| &= \|\mathcal{T}((1-\mathfrak{m}_n)\mathcal{T}r_n + \mathfrak{m}_n\mathcal{T}s_n) - q\| \\
&\leq \|(1-\mathfrak{m}_n)\mathcal{T}r_n + \mathfrak{m}_n\mathcal{T}s_n - q\| \\
&\leq (1-\mathfrak{m}_n)\|r_n - q\| + \mathfrak{m}_n\|s_n - q\| \\
&\leq (1-\mathfrak{m}_n)\|r_n - q\| + \mathfrak{m}_n\|r_n - q\| \\
&\leq (1-\mathfrak{m}_n)\|\omega_n - q\| + \mathfrak{m}_n\|r_n - q\|
\end{aligned}$$

This implies that

$$\begin{aligned}
\frac{\|\omega_{n+1} - q\| - \|\omega_n - q\|}{\mathfrak{m}_n} &\leq [\|r_n - q\| - \|\omega_n - q\|] \\
\|\omega_{n+1} - q\| - \|\omega_n - q\| \leq \frac{\|\omega_{n+1} - q\| - \|\omega_n - q\|}{\mathfrak{m}_n} & \\
&\leq [\|r_n - q\| - \|\omega_{n+1} - q\|] \\
\|\omega_{n+1} - q\| &\leq \|r_n - q\|
\end{aligned}$$

$$c \leq \lim_{n\to\infty} \inf \|r_n - q\| \tag{10}$$

From Eq.8 and Eq.3, we get

$$\begin{aligned}
c &= \lim_{n\to\infty} \|r_n - q\| \\
&= \lim_{n\to\infty} \|(1-\xi_n)\omega_n + \xi_n\mathcal{T}\omega_n - q\|
\end{aligned}$$

$$= \lim_{n\to\infty} \|(1-\xi_n)(\omega_n - q) + \xi_n(\mathcal{T}\omega_n - q)\| \tag{11}$$

Based on Lemma 2.2 and Eq.7, Eq.9, Eq.11 one get

$\lim_{n\to\infty} \|\mathcal{T}\omega_n - \omega_n\| = 0$

Conversely, suppose (ω_n) is bounded and $\lim_{n\to\infty} \|\mathcal{T}\omega_n - \omega_n\| = 0$

Consider $q \in \Upsilon(\mathcal{T})$. By Proposition 2.3(ii) ,we get

$$\begin{aligned} r(\mathcal{T}q,(\omega_n)) &= \lim_{n\to\infty} \sup \|\omega_n - \mathcal{T}q\| \\ &\le \lim_{n\to\infty} \sup[3\|\mathcal{T}\omega_n - \omega_n\| + \|\omega_n - q\|] \\ &\le \lim_{n\to\infty} \sup \|\omega_n - q\| \\ &= r(q,(\omega_n)) \end{aligned}$$

This shows that $\mathcal{T}q \in \mathcal{U}(\mathcal{Z}(\omega_n))$. Because X is uniformly convex, $\mathcal{U}(\mathcal{Z}(\omega_n))$ is a singleton. Thus, $\mathcal{T}q = q$, i.e. $\Upsilon(\mathcal{T}) \neq \phi$.

Now, to confirm the convergence result mentioned in the previous results, some examples are presented. Numerical and graphical examination of the examples shows that the iterative method given in Eq.1 converges to FP faster than the iterations of Agarwal [21], Abbas [22], and Thakur [23].

Example 3.1. *Consider* $X = R$ *and* $\mathcal{Z} = [1, 30]$, $\mathcal{T} : \mathcal{Z} \to \mathcal{Z}$ *be a mapping defined by* $\mathcal{T}\omega = \sqrt{\omega^2 - 9\omega + 27}$ *for all* $\omega \in \mathcal{Z}$.

For $\omega_0 = 5$, *and* $\mathfrak{m}_n = \upsilon_n = \xi_n = 0.75$.

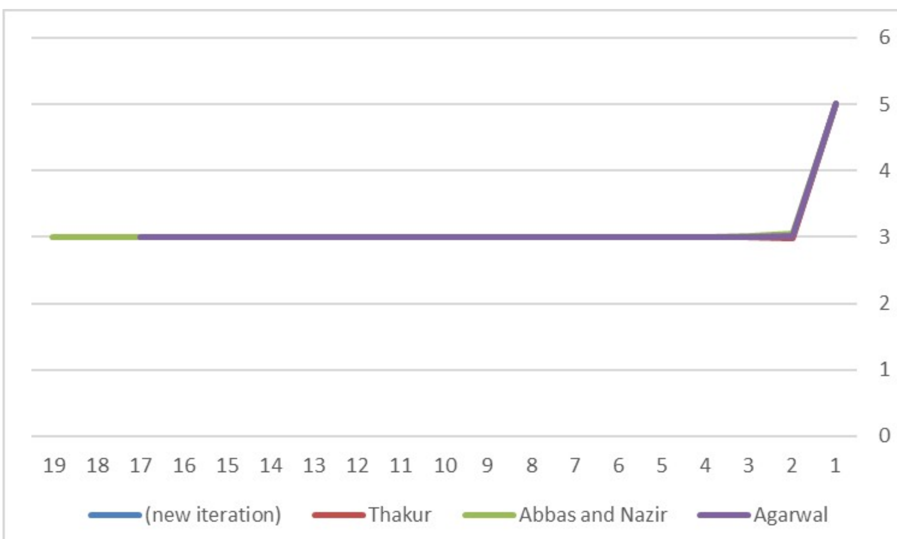

Figure 1. Graphic illustration of the convergence of iterative approaches for example 3.1

TABLE 1. Comparison of convergence rates for various iteration approaches for example 3.1

step	*Agarwal* [21]	*Abbas* [22]	*Thakur* [23]	(*new iteration*)
1	5	5	5	5
2	3.0196029628	3.0540525027	2.9869096817	3.0026320501
3	3.0020978431	3.0081163308	3.0004940741	3.0000201368
4	3.0002254108	3.0012323501	2.9999814126	3.0000001541
5	3.0000242304	3.0001874173	3.0000006993	3.0000000011
6	3.0000026047	3.0000285096	2.9999999736	3.0000000010
7	3.0000002800	3.0000043369	3.0000000009	3.0000000008
8	3.0000000301	3.0000006597	2.9999999999	3.0000000007
9	3.0000000032	3.0000001003	3.0000000000	3.0000000004
10	3.0000000029	3.0000000152	2.9999999999	3.0000000001
11	3.0000000022	3.0000000023	3.0000000002	3
12	3.0000000017	3.0000000019	3	
13	3.0000000011	3.0000000014		
14	3.0000000009	3.0000000010		
15	3.0000000004	3.0000000007		
16	3.0000000001	3.0000000004		
17	3	3.0000000003		
18		3.0000000001		
19		3		

Example 3.2. *Let* $X = R$, $\mathcal{Z} = [1, 30]$ *and* $\mathcal{T} : \mathcal{Z} \to \mathcal{Z}$ *be a mapping defined by*

$$\mathcal{T}\omega = \begin{cases} 4 - \omega & if \;\; \omega \in [1, \frac{9}{7}) \\ \frac{\omega + 12}{7} & if \;\; \omega \in [\frac{9}{7}, 3] \end{cases}$$

for all $\omega \in \mathcal{Z}$.

Consider $\omega_0 = 1$, *and* $\mathfrak{m}_n = \upsilon_n = \xi_n = 0.8$.

TABLE 2. Comparison of convergence rates for various iteration approaches for example 3.2

step	*Agarwal* [21]	*Abbas* [22]	*Thakur* [23]	(*new iteration*)
1	1	1	1	1
2	2.0762448979	1.7849795918	2.0386938775	2.0005497709
2	2.0762448979	1.7849795918	2.0386938775	2.0005497709
3	2.0028790784	1.9511985168	2.0007842560	2.0000001583
4	2.0001087166	1.9889239129	2.0000158954	2.0000000004
5	2.0000041052	1.9974861480	2.0000003221	2.0000000001
6	2.0000001550	1.9994294508	2.0000000065	2
7	2.0000000058	1.9998705069	2.0000000033	
8	2.0000000032	1.9999706099	2.0000000034	
9	2.0000000028	1.9999933295	2.0000000019	
10	2.0000000017	1.9999984860	2.0000000011	
11	2.0000000006	1.9999996563	2.0000000004	
12	2.0000000002	1.9999999220	2	
13	2	1.9999999823		
14		1.9999999959		
15		1.9999999960		
16		1.9999999981		
17		1.9999999992		
18		1.9999999994		
19		1.9999999999		
20		1.9999999999		
21		2		

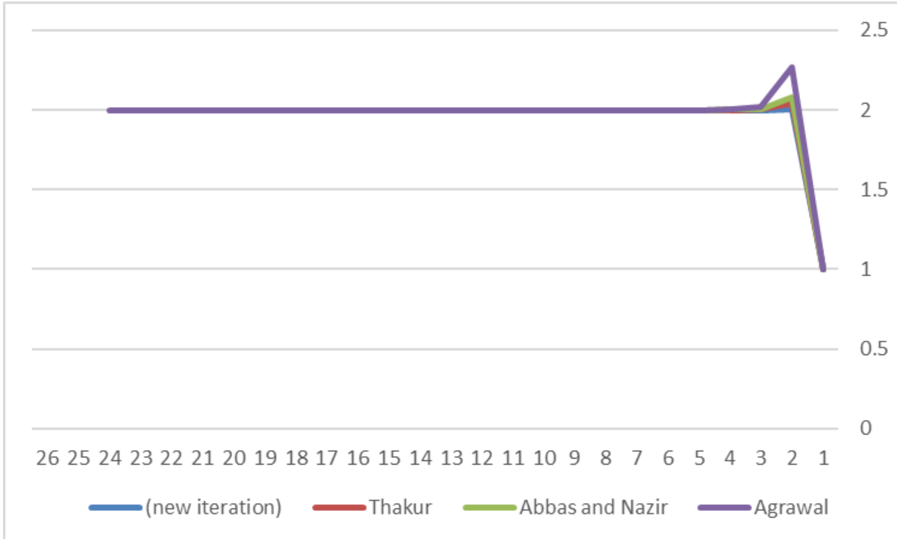

FIGURE 2. Graphic illustration of the convergence of iterative approaches for example 3.2

Example 3.3. *Let* $X = R$ *and* $\mathcal{Z} = [1, 30]$ *and* $\mathcal{T} : \mathcal{Z} \to \mathcal{Z}$ *defined by* $\mathcal{T}\omega = \frac{2\omega+1}{4}$ *for all* $\omega \in \mathcal{Z}$. *Consider* $\omega_0 = 0.5$*, and* $\mathfrak{m}_n = \upsilon_n = \xi_n = 0.7$.

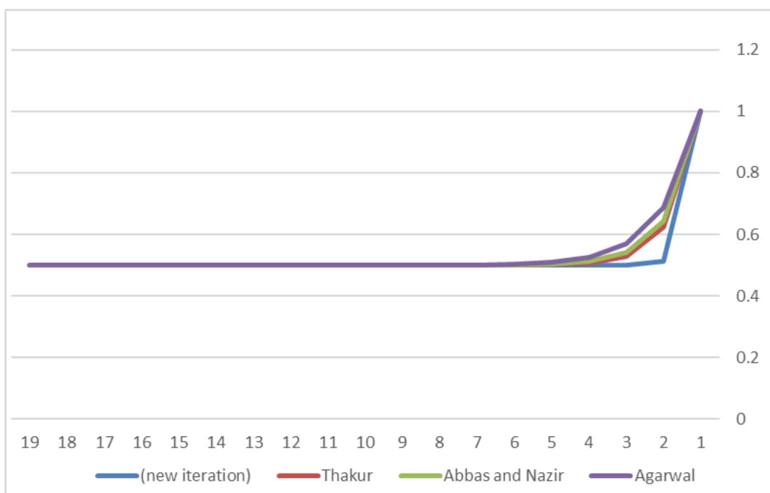

FIGURE 3. Graphic illustration of the convergence of iterative approaches for example 3.3

TABLE 3. Comparison of convergence rates for various iteration approaches for example 3.3

step	*Agarwal* [21]	*Abbas* [22]	*Thakur* [23]	*new iteration*
0	1	1	1	1
1	0.6887500000	0.6420625000	0.6226875000	0.51320312500
2	0.5712531250	0.5403635078	0.5301044453	0.5003486450
3	0.5268980546	0.5114682816	0.5073868782	0.5000092064
4	0.5101540156	0.5032584255	0.5018125552	0.5000002431
5	0.5038331409	0.5009258001	0.5004447557	0.5000000064
6	0.5014470106	0.5002630429	0.5001091319	0.5000000001
7	0.5005462465	0.5000747370	0.5000267782	0.5
8	0.5002062080	0.5000212346	0.5000065707	
9	0.5000778435	0.5000060333	0.5000016122	
10	0.5000293859	0.5000017142	0.5000003956	
11	0.5000110931	0.5000004870	0.5000000970	
12	0.5000041876	0.5000001383	0.5000000238	
13	0.5000015808	0.5000000393	0.5000000058	
14	0.5000005967	0.5000000111	0.5000000014	
15	0.5000002252	0.5000000031	0.5000000003	
16	0.5000000850	0.5000000009	0.5	
17	0.5000000321	0.5000000002		
18	0.5000000121	0.5		
19	0.5000000045			
20	0.5000000017			
21	0.5000000006			
22	0.5000000002			
23	0.5			

It is clear from the data and graphs for all the above examples that the proposed new method approaches the FP faster than other methods, which indicates the efficiency of the proposed method.

4. Conclusion

In this study, we introduce a novel iterative approach for estimating the FPs of nonexpansive mapping. We have numerically demonstrated that our innovative iterative approach converges more quickly than other popular iterative algorithms. In addition, we proved some convergence results for nonexpansive mapping in UCB-space.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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