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A GENERALISED APPROACH TO COMMON BEST PROXIMITY POINTS IN INTUITIONISTIC FUZZY METRIC SPACES

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Abstract. This paper introduces intuitionistic fuzzy type of well-known iterative mappings. Additionally, we present specific requirements for real-valued functions $S, J : (0, 1] \to (-\infty, \infty)$ for the existence of the best proximity point (BPp) of generalized $IF_{(S,J)}$-iterative mappings within the context of intuitionistic fuzzy metric spaces (IFMS). Moreover, we employ intuitionistic fuzzy type of (S, J)-proximal contraction to investigate common best proximity (CBP) points in intuitionistic fuzzy metric spaces. The paper concludes with several non-trivial examples and our results are supported by an application.

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1. Introduction

FP theory stands out as a captivating field of study, particularly focusing on techniques for solving nonlinear equations of the form $Uu = u$, where U represents a self-mapping. The discussions within FP theory delve into various strategies for determining solutions. Despite the depth of exploration, instances arise where a singular solution is non-existent. The resolution to such challenges often involves leveraging best approximation theorem and best proximity point theorem. These theorems, which have undergone diverse generalizations by numerous researchers, prove instrumental in deriving approximate optimal solutions.

The year 1968 marked a pivotal moment when Kannan [7] introduced a novel type of contraction for discontinuous mappings, yielding several FP results. This innovation provided researchers with an alternative avenue to tackle FP problems. Subsequent advancements, such as the iterative Kannan Mier-type contractions introduced by Karapinar [8], further established Rus Reich Ciric-type contractions

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using simulation functions and Hardy Rogers-type interpolative contractions (see in the references [8–11]).

FP theory has been extensively generalized in various fuzzy and soft metric spaces, demonstrating its broad applicability. Recent studies, such as those by Gupta et al. [4], establish fixed point results in modified intuitionistic fuzzy soft metric spaces, highlighting their utility in mathematical modeling. Similarly, Mani et al. [12] explore fuzzy b-metric spaces under different t-norms, revealing how alternative triangular norms influence fixed point outcomes. Shukla et al. [21] further extend this framework to complex-valued fuzzy metric spaces, enriching the theory of contractive mappings. Most recently, Shukla et al. [24] introduce vector-valued fuzzy metric spaces, unifying classical results with novel axiomatic approaches.

Altun et al. [1, 2] contributed the best proximity point consequences for proximal contractions, extending these results to interpolative proximal contractions, see also [17]. Shazad et al. [20] presented CBP point results. Also, Deep and Betra [3] introduced additional results in this field. The investigation expanded into proximal F-contraction, where Mondal and Dey [13] established results on CBP points in complete MS. Shayanpour and Nematizadeh [19] made contributions within the field of complete FMS. Hierro [6] presented Proinov-type FP results in FMS, later refined by Zhou et al. [26], also [5, 22, 23].

Uddin et al [23] offered integral equation solutions utilizing IFb metric like space [6] presented findings in a new extension of IFM-like spaces. While Nazam et al. [14] studied generalized interpolative contractions, Hussain et al. [6] looked into FPs in FMS. The analytical application of fractional delay differential equations was the main emphasis of Naseem et al. [15].

2. Preliminaries

Definition 2.1. [18] Consider a metric space $(Z, \mathbb{H})$. The mappings $G : \mathring{P} \to \mathring{Q}$ and $U : \mathring{P} \to \mathring{Q}$ are considered to commute proximally if they satisfy the following condition:

$$[\mathbb{H}(a^{\varpi}, Gu`) = \mathbb{H}(e^{\flat}, Uu`) = \mathbb{H}(\mathring{P}, \mathring{Q})] \Rightarrow Ge^{\flat} = Ua^{\varpi},$$

for all $u`, a^{\varpi}, e^{\flat}$ in $\mathring{P}$.

Definition 2.2. [18] Suppose $(Z, \mathbb{H})$ be a metric space. A mapping $U : \mathring{P} \to \mathring{Q}$ proximally dominates another mapping $G : \mathring{P} \to \mathring{Q}$ if, $\exists$ a non-negative number $\alpha < 1$ then

$$\mathbb{H}(a_1^{\varpi}, Gu`_1) = \mathbb{H}(\mathring{P}, \mathring{Q}) = \mathbb{H}(e_1^{\flat}, Uu`_1),$$

$$\mathbb{H}(a_2^{\varpi}, Gu`_2) = \mathbb{H}(\mathring{P}, \mathring{Q}) = \mathbb{H}(e_2^{\flat}, Uu`_2),$$

$$\mathbb{H}(\varpi_1, \varpi_2) \leq \alpha \mathbb{H}(\flat_1, \flat_2),$$

for all $e_1^{\flat}, a_1^{\varpi}, e_2^{\flat}, u`_1, a_2^{\varpi}, u`_2 \in \mathring{P}$.

Definition 2.3. [26] A binary operation $* : [0,1]^2 \to [0,1]$ is said to be a continuous t-norm (ctn) if it satisfies the below axioms:

(1) $\varpi_1 * \varpi_2 = \varpi_1 * \varpi_2$ and $\varpi_1 * (\varpi_2 * \varpi_3) = (\varpi_1 * \varpi_2) * \varpi_3$ for all $\varpi_1, \varpi_2, \varpi_3 \in [0,1]$;

(2) $*$ is continuous;

(3) $\varpi_1 * 1 = s$ for all $\varpi_1 \in [0,1]$;

(4) $\varpi_1 * \varpi_2 \leq \varpi_3 * \varpi_4$ when $\varpi_1 \leq \varpi_3$ and $\varpi_2 \leq \varpi_4$, with $\varpi_1, \varpi_2, \varpi_3, \varpi_4 \in [0,1]$.

Definition 2.4. [14] Let $\diamond : [0,1]^2 \to [0,1]$, is categorized as a continuous triangular co-norm if it meets the following requirements:

(i) The operation $\diamond$ exhibits associativity, commutativity and continuity;

(ii) $\varpi_1 \diamond 0 = \varpi_1$, for all $\varpi_1 \in [0,1]$;

(iii) $\varpi_1 \diamond \varpi_2 \leq \varpi_3 \diamond \varpi_4$, whenever $\varpi_1 \leq \varpi_3$ and $\varpi_2 \leq \varpi_4$ $\forall \varpi_1, \varpi_2, \varpi_3, \varpi_4 \in [0,1]$.

For example $\varpi_1 \diamond \varpi_2 = min(\varpi_1 + \varpi_2, 1), \varpi_1 \diamond \varpi_2 = \varpi_1 + \varpi_2 - \varpi_1\varpi_2, \varpi_1 \diamond \varpi_2 = \max(\varpi_1, \varpi_2)$.

Definition 2.5. [18] Let Z denote a non-empty set. Consider a continuous triangular norm represented by the symbol $*$ and a fuzzy set $\mathbb{H}$ defined on $Z \times Z \times (0, \infty)$. The combination of set Z, fuzzy set $\mathbb{H}$, and the operation $*$ is referred to as a fuzzy metric space if it fulfills the following conditions. For any $s, \bar{\xi} \geq 0$, and for all $\varpi_1, \varpi_2, \varpi_3 \in Z$:

(FMS1) $\mathbb{H}(\varpi_1, \varpi_2, \bar{\xi}) > 0$;

(FMS2) $\mathbb{H}(\varpi_1, \varpi_2, \bar{\xi}) = 1$ if and only if $\varpi_1 = \varpi_2$;

(FMS3) $\mathbb{H}(\varpi_1, \varpi_2, \bar{\xi}) = \mathbb{H}(\varpi_2, \varpi_1, \bar{\xi})$;

(FMS4) $\mathbb{H}(\varpi_1, \varpi_2, \bar{\xi}) * \mathbb{H}(\varpi_2, \varpi_3, s) \leq \mathbb{H}(\varpi_1, \varpi_3, \bar{\xi} + s)$;

(FMS5) $\mathbb{H}(\varpi_1, \varpi_2, .) : (0, \infty) \to (0, 1]$ is continuous.

Definition 2.6. [18] Let Z is an arbitrary set, $*$ and $\diamond$ are continuous t-norm and continuous t-conorm respectively. Let $\mathbb{H}$ and $\mathbb{O}$ be fuzzy sets on $Z \times Z \times (0, \infty)$ satisfying the below conditions for every $\varpi_1, \varpi_2, \varpi_3 \in Z$:

(a) $\mathbb{H}(\varpi_1, \varpi_2, \bar{\xi}) + \mathbb{O}(\varpi_1, \varpi_2, \bar{\xi}) \leq 1$;

(b) $\mathbb{H}(\varpi_1, \varpi_2, \bar{\xi}) > 0$;

(c) $\mathbb{H}(\varpi_1, \varpi_2, \bar{\xi}) = 1$, iff $\varpi_1 = \varpi_2$ for all $\bar{\xi} > 0$;

(d) $\mathbb{H}(\varpi_1, \varpi_2, \bar{\xi}) = \mathbb{H}(\varpi_2, \varpi_1, \bar{\xi})$, for all $\bar{\xi} > 0$;

(e) $\mathbb{H}(\varpi_1, \varpi_3, (\bar{\xi} + s)) \geq \mathbb{H}(\varpi_1, \varpi_2, \bar{\xi}) * \mathbb{H}(\varpi_2, \varpi_3, s)$, for all $\bar{\xi}, s > 0$;

(f) $\mathbb{H}(\varpi_1, \varpi_2, .) : [0, \infty) \to [0, 1]$ are continuous;

(g) $\mathbb{O}(\varpi_1, \varpi_2, \bar{\xi}) > 0$;

(h) $\mathbb{O}(\varpi_1, \varpi_2, \bar{\xi}) = 0$, for all $\bar{\xi} > 0$ iff $\varpi_1 = \varpi_2$;

(i) $\mathbb{O}(\varpi_1, \varpi_2, \bar{\xi}) = \mathbb{O}(\varpi_2, \varpi_1, \bar{\xi})$, for all $\bar{\xi} > 0$;

(j) $\mathbb{O}(\varpi_1, \varpi_3, (\bar{\xi} + s)) \leq \mathbb{O}(\varpi_1, \varpi_2, \bar{\xi}) \diamond \mathbb{O}(\varpi_2, \varpi_3, s)$, for all $\bar{\xi}, s > 0$;

(k) $\mathbb{O}(\varpi_1, \varpi_2, .) : [0, \infty) \to [0, 1]$ are continuous map,

where $\mathbb{O}(\varpi_1, \varpi_2, \bar{\xi})$ and $\mathbb{H}(\varpi_1, \varpi_2, \bar{\xi})$ represent the closeness between ϖ_1 and ϖ_2 with respect to $\bar{\xi}$, respectively. Also, $\lim_{\bar{\xi}\to\infty} \mathbb{H}(\varpi_1, \varpi_2, \bar{\xi}) = 1$ and $\lim_{\bar{\xi}\to\infty} \mathbb{O}(\varpi_1, \varpi_2, \bar{\xi}) = 0$. Then a 5-tuple $(Z, \mathbb{H}, \mathbb{O}, *, \diamond)$ is called an IFMS.

Definition 2.7. [19] A sequence $\{\varpi_n\}$ in an IFMS $(Z, \mathbb{H}, \mathbb{O}, *, \diamond)$ is said to be convergent to ϖ if every $\zeta \in (0, 1)$ and $\epsilon > 0$, $\exists\, a_0 \in \mathbb{N}$ such that

$$\mathbb{H}(\varpi, \varpi_n, \bar{\xi}) > 1 - \zeta \text{ and } \mathbb{O}(\varpi, \varpi_n, \bar{\xi}) < \zeta \text{ for all } n \geq a_0$$

i.e,

$$\lim_{n\to\infty} \mathbb{H}(\varpi, \varpi_n, \bar{\xi}) = 1 \text{ and } \lim_{n\to\infty} \mathbb{O}(\varpi, \varpi_n, \bar{\xi}) = 0 \text{ for all } \bar{\xi} > 0.$$

Definition 2.8. [19] A sequence $\{\varpi_n\}$ in an IFMS $(Z, \mathbb{H}, \mathbb{O}, *, \diamond)$ is said to be Cauchy if for each ϵ and ζ, $\exists\, a_0 \in \mathbb{N}$ such that

$$\mathbb{H}(\varpi_n, \varpi_{n+p}, \bar{\xi}) > 1 - \zeta \text{ and } \mathbb{O}(\varpi_n, \varpi_{n+p}, \bar{\xi}) < \zeta$$

for every p, $n \geq a_0$ where $\epsilon > 0$, $\zeta \in (0, 1)$.

$$i.e, \lim_{n\to\infty} \mathbb{H}(\varpi_n, \varpi_{n+p}, \bar{\xi}) = 1 \text{ and } \lim_{n\to\infty} \mathbb{O}(\varpi_n, \varpi_{n+p}, \bar{\xi}) = 0 \text{ for all } \bar{\xi} > 0.$$

In addition to, an IFMS$(Z, \mathbb{H}, \mathbb{O}, *, \diamond)$ is said to be complete iff every Cauchy sequence is convergent.

Definition 2.9. [19] Let $(Z, \mathbb{H}, *)$ be an FMS and $\mathring{P}, \mathring{Q} \subset Z$. Consider

$$\mathbb{H}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \sup_{\varpi_1 \in \mathring{P}, \varpi_2 \in \mathring{Q}} \mathbb{H}(\varpi_1, \varpi_2, \bar{\xi}), \quad \bar{\xi} > 0,$$

Then the distance between $\mathring{P}$ and $\mathring{Q}$ is called fuzzy distance.

Definition 2.10. [19] Let $(Z, \mathbb{H}, *)$ be an FMS, $\mathring{P}, \mathring{Q} \subseteq Z$, and $U, G : \mathring{P} \to \mathring{Q}$ be two mappings. A point $\varpi \in \mathring{P}$ is called a CBP point of the mappings U and G, if

$$\mathbb{H}(\varpi, U\varpi, \bar{\xi}) = \mathbb{H}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{H}(\varpi, G\varpi, \bar{\xi}).$$

Definition 2.11. [19] Suppose $(Z, \mathbb{H}, *)$ be an FMS and $\mathring{P}, \mathring{Q} \subset Z$. The following sets are defined by us:

$$\mathring{P}_0 = \{\varpi_1 \in \mathring{P} : \exists \varpi_2 \in \mathring{Q} \text{ s.t. } \forall \bar{\xi} > 0, \mathbb{H}(\varpi_1, \varpi_2, \bar{\xi}) = \mathbb{H}(\mathring{P}, \mathring{Q}, \bar{\xi})\},$$

$$\mathring{Q}_0 = \{\varpi_2 \in \mathring{Q} : \exists \varpi_1 \in \mathring{P} \text{ s.t. } \forall \bar{\xi} > 0, \mathbb{H}(\varpi_1, \varpi_2, \bar{\xi}) = \mathbb{H}(\mathring{P}, \mathring{Q}, \bar{\xi})\}.$$

Definition 2.12. [19] Let $(Z, \mathbb{H}, *)$ be an FMS, $\mathring{P}, \mathring{Q} \subseteq Z$, and $U, G : \mathring{P} \to \mathring{Q}$ be two mappings are said to be commute proximally if

$$\mathbb{H}(\varpi_1, U\varpi, \bar{\xi}) = \mathbb{H}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{H}(\varpi_2, G\varpi, \bar{\xi}), \quad \forall \bar{\xi} > 0,$$

then $U\varpi_2 = G\varpi_1$, where $\varpi, \varpi_1, \varpi_2 \in \mathring{P}$.

Definition 2.13. [19] Let $(Z, \mathbb{H}, *)$ be an FMS, $\mathring{P}, \mathring{Q} \subseteq Z$, and $U, G : \mathring{P} \to \mathring{Q}$ be the mappings then the mapping U is to dominate G proximally if

$$\mathbb{H}(\varpi_1, Uh_1, \bar{\xi}) = \mathbb{H}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{H}(b_1, Gh_2, \bar{\xi}),$$
$$\mathbb{H}(\varpi_2, Uh_1, \bar{\xi}) = \mathbb{H}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{H}(b_2, Gh_2, \bar{\xi}),$$

for all $\bar{\xi} > 0$, then $\exists\, \alpha \in (0,1)$ such that,

$$\mathbb{H}(\varpi_1, \varpi_2, \alpha\bar{\xi}) \geq \mathbb{H}(b_1, b_2, \bar{\xi}),$$

where $\varpi_1, \varpi_2, b_1, b_2$ and $h_1, h_2 \in \mathring{P}$.

Definition 2.14. [26] Let $\mathcal{L}$ represent the collection of pairs (J, S), where J and S are functions defined on $(0,1] \to \mathbb{R}$ and satisfy the specified properties outlined below:

(1) $S(\varpi) > J(\varpi)$ for any $\varpi \in (0,1)$;
(2) J is non-decreasing;
(3) $\lim_{\varpi \to T^-} \inf S(\varpi) > \lim_{s \to T^-} \inf J(\varpi)$ for any $0 < T^- < 1$;
(4) if $\varpi \in (0,1)$ is such that $S(\varpi) \geq J(1)$ then $\varpi = 1$.

Lemma 2.1. [16] *Suppose $J : (0,1] \to (-\infty, \infty)$ then the below conditions are mutually similar:*

(I) $\inf_{\bar{\xi} > \epsilon} J(\bar{\xi}) > -\infty$ *for all ϵ, where $0 < \epsilon < 1$;*
(II) *For any $\epsilon \in (0,1)$, $\lim_{\bar{\xi} \to \epsilon^-} \inf J(\bar{\xi}) > -\infty$;*
(III) $\lim_{n\to\infty} J(\bar{\xi}_n) = -\infty$ *implies that* $\lim_{n\to\infty} \bar{\xi}_n = 1$.

3. Main Results

Definition 3.1. Let $(Z, \mathbb{H}, \mathbb{O}, *, \diamond)$ be an IFMS and $\mathring{P}, \mathring{Q} \subset Z$. Consider

$$\mathbb{H}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \sup_{\varpi_1 \in \mathring{P}, \varpi_2 \in \mathring{Q}} \mathbb{H}(\varpi_1, \varpi_2, \bar{\xi}),$$

$$\mathbb{O}(\mathring{P}, \mathring{Q}, \bar{\xi}) = 1 - \sup_{\varpi_1 \in \mathring{P}, \varpi_2 \in \mathring{Q}} \mathbb{O}(\varpi_1, \varpi_2, \bar{\xi}), \quad \bar{\xi} > 0,$$

Then the distance between $\mathring{P}$ and $\mathring{Q}$ is called intuitionistic fuzzy distance.

Definition 3.2. Let $(Z, \mathbb{H}, \mathbb{O}, *, \diamond)$ be an IFMS and $\mathring{P}, \mathring{Q} \subset Z$. The following sets defined by us:

$$\mathring{P}_0 = \{\varpi_1 \in \mathring{P} : \exists \varpi_2 \in \mathring{Q} \text{ s.t. } \forall \bar{\xi} > 0, \mathbb{H}(\varpi_1, \varpi_2, \bar{\xi}) = \mathbb{H}(\mathring{P}, \mathring{Q}, \bar{\xi})\},$$

$$\mathring{Q}_0 = \{\varpi_2 \in \mathring{Q} : \exists \varpi_1 \in \mathring{P} \text{ s.t. } \forall \bar{\xi} > 0, \mathbb{H}(\varpi_1, \varpi_2, \bar{\xi}) = \mathbb{H}(\mathring{P}, \mathring{Q}, \bar{\xi})\},$$

$$\mathring{P}_1 = \{\varpi_1 \in \mathring{P} : \exists \varpi_2 \in \mathring{Q} \text{ s.t. } \forall \bar{\xi} > 0, \mathbb{O}(\varpi_1, \varpi_2, \bar{\xi}) = \mathbb{O}(\mathring{P}, \mathring{Q}, \bar{\xi})\},$$

$$\mathring{Q}_1 = \{\varpi_2 \in \mathring{Q} : \exists \varpi_1 \in \mathring{P} \text{ s.t. } \forall \bar{\xi} > 0, \mathbb{O}(\varpi_1, \varpi_2, \bar{\xi}) = \mathbb{O}(\mathring{P}, \mathring{Q}, \bar{\xi})\}.$$

Definition 3.3. Let $(Z, \mathbb{H}, \mathbb{O}, *, \diamond)$ be an IFMS, $\mathring{P}, \mathring{Q} \subseteq Z$, and $U, G : \mathring{P} \to \mathring{Q}$ be two mappings. A point $\varpi \in \mathring{P}$ is called a CBP point of the mappings U and G, if

$$\mathbb{H}(\varpi, U\varpi, \bar{\xi}) = \mathbb{H}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{H}(\varpi, G\varpi, \bar{\xi}),$$

$$\mathbb{O}(\varpi, U\varpi, \bar{\xi}) = \mathbb{O}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{O}(\varpi, G\varpi, \bar{\xi}).$$

Definition 3.4. Let $(Z, \mathbb{H}, \mathbb{O}, *, \diamond)$ be an IFMS, $\mathring{P}, \mathring{Q} \subseteq Z$, and $U, G : \mathring{P} \to \mathring{Q}$ be two mappings are said to be commute proximally if

$$\mathbb{H}(\varpi_1, U\varpi, \bar{\xi}) = \mathbb{H}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{H}(\varpi_2, G\varpi, \bar{\xi}),$$

$$\mathbb{O}(\varpi_1, U\varpi, \bar{\xi}) = \mathbb{O}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{O}(\varpi_2, G\varpi, \bar{\xi}), \quad \forall \bar{\xi} > 0,$$

then $U\varpi_2 = G\varpi_1$, where $\varpi, \varpi_1, \varpi_2 \in \mathring{P}$.

Definition 3.5. Let $(Z, \mathbb{H}, \mathbb{O}, *, \diamond)$ be an IFMS, $\mathring{P}, \mathring{Q} \subseteq Z$, and $U, G : \mathring{P} \to \mathring{Q}$ be two mappings then U is to dominate G proximally if

$$\mathbb{H}(\varpi_1, Uh_1, \bar{\xi}) = \mathbb{H}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{H}(b_1, Gh_2, \bar{\xi}),$$

$$\mathbb{H}(\varpi_2, Uh_1, \bar{\xi}) = \mathbb{H}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{H}(b_2, Gh_2, \bar{\xi})$$

and

$$\mathbb{O}(\varpi_1, Uh_1, \bar{\xi}) = \mathbb{O}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{O}(b_1, Gh_2, \bar{\xi}),$$

$$\mathbb{O}(\varpi_2, Uh_1, \bar{\xi}) = \mathbb{O}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{O}(b_2, Gh_2, \bar{\xi}),$$

for all $\bar{\xi} > 0$, then $\exists\, \alpha \in (0, 1)$ such that,

$$\mathbb{H}(\varpi_1, \varpi_2, \alpha\bar{\xi}) \geq \mathbb{H}(b_1, b_2, \bar{\xi}),$$

$$\mathbb{O}(\varpi_1, \varpi_2, \alpha\bar{\xi}) \leq \mathbb{O}(b_1, b_2, \bar{\xi}),$$

where $\varpi_1, \varpi_2, b_1, b_2$ and $h_1, h_2 \in \mathring{P}$.

Definition 3.6. Assume $\mathring{P}$ and $\mathring{Q}$ be subsets of $(Z, \mathbb{H}, \mathbb{O}, *, \diamond)$. The mappings $U, G : \mathring{P} \to \mathring{Q}$ are called intuitionistic fuzzy (S, J)-proximal $(IF_{(S,J)} - proximal)$ if

$$\begin{aligned} \mathbb{H}(a_1^{\varpi}, Gu_1^{\varpi}, \bar{\xi}) &= \mathbb{H}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{H}(e_1^{\flat}, Uu_1^{\flat}, \bar{\xi}), \\ \mathbb{H}(a_2^{\varpi}, Gu_2^{\varpi}, \bar{\xi}) &= \mathbb{H}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{H}(e_2^{\flat}, Uu_2^{\flat}, \bar{\xi}), \\ J(\mathbb{H}(\varpi_1, \varpi_2, \bar{\xi})) &\geq S(\mathbb{H}(\flat_1, \flat_2, \bar{\xi})) \end{aligned} \tag{1}$$

and

$$\begin{aligned} \mathbb{O}(a_1^{\varpi}, Gu_1^{\varpi}, \bar{\xi}) &= \mathbb{O}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{O}(e_1^{\flat}, Uu_1^{\flat}, \bar{\xi}), \\ \mathbb{O}(a_2^{\varpi}, Gu_2^{\varpi}, \bar{\xi}) &= \mathbb{O}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{O}(e_2^{\flat}, Uu_2^{\flat}, \bar{\xi}), \\ J(\mathbb{O}(\varpi_1, \varpi_2, \bar{\xi})) &\leq S(\mathbb{O}(\flat_1, \flat_2, \bar{\xi})), \end{aligned} \tag{2}$$

for all $e_1^\flat, a_1^\varpi, e_2^\flat, u_1^\flat, a_2^\varpi, u_2^\flat \in \mathring{P}$ and $\bar{\xi} > 0$.

Example 3.1. Consider $(Z, \mathbb{H}, \mathbb{O}, *, \diamond)$ be an IFMS and $\mathbb{H}, \mathbb{O}$ defined by

$$\mathbb{H}(u^\flat, n^\varpi, \bar{\xi}) = e^{\frac{-|u^\flat - n^\varpi|}{\bar{\xi}}}$$

$$\mathbb{O}(u^\flat, n^\varpi, \bar{\xi}) = 1 - e^{\frac{-|u^\flat - n^\varpi|}{\bar{\xi}}}.$$

Let $\mathring{Q} = \{1, 3, 5, 7, 9, 11\}$ and $\mathring{P} = \{0, 2, 4, 6, 8, 10\}$. The mappings defined $G : \mathring{P} \to \mathring{Q}$ and $U : \mathring{P} \to \mathring{Q}$ as

$$U(0) = 3, U(2) = 5, U(4) = 7, U(6) = 3, U(8) = 9, U(10) = 11,$$

$$G(0) = 3, G(2) = 1, G(4) = 9, G(6) = 7, G(8) = 5, G(10) = 11.$$

Then, $\mathbb{H}(\mathring{P}, \mathring{Q}, \bar{\xi}) = e^{\frac{-1}{\bar{\xi}}}$, and $\mathbb{O}(\mathring{P}, \mathring{Q}, \bar{\xi}) = 1 - e^{\frac{-1}{\bar{\xi}}}$, $\mathring{P}_0 = \mathring{P}$, $\mathring{Q}_0 = \mathring{Q}$ and $\mathring{P}_1 = \mathring{P}$, $\mathring{Q}_1 = \mathring{Q}$. Clearly, $G(\mathring{P}_0) \subseteq \mathring{Q}_0$, $U(\mathring{P}_0) \subseteq \mathring{Q}_0$ and $G(\mathring{P}_1) \subseteq \mathring{Q}_1$, $U(\mathring{P}_1) \subseteq \mathring{Q}_1$.

Define the functions J, S by

$$S(\bar{\xi}) = \begin{cases} \frac{1}{\ln \bar{\xi}^2} & \text{if } \bar{\xi} \in (0,1), \\ 2 & \text{if } \bar{\xi} = 1. \end{cases}$$

$$J(\bar{\xi}) = \begin{cases} \frac{1}{\ln \bar{\xi}} & \text{if } \bar{\xi} \in (0,1), \\ 1 & \text{if } \bar{\xi} = 1, \end{cases}$$

To show that G and U are $IF_{(S,J)} - proximal$ in IFMS. Let $\varpi_1 = 0$, $\varpi_2 = 8$, $\flat_1 = 4$, $\flat_2 = 6$, $u_1^\flat = 2$, $u_2^\flat = 4$, and $\bar{\xi} = 1$. Then,

$$\mathbb{H}(0, G(2), 1) = \mathbb{H}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{H}(4, U(2), 1),$$

$$\mathbb{H}(8, G(4), 1) = \mathbb{H}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{H}(6, U(4), 1),$$

and

$$\mathbb{O}(0, G(2), 1) = \mathbb{O}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{O}(4, U(2), 1),$$

$$\mathbb{O}(8, G(4), 1) = \mathbb{O}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{O}(6, U(4), 1).$$

This implies that,

$$J(\mathbb{H}(\varpi_1, \varpi_2, \bar{\xi})) \geq S(\mathbb{H}(\flat_1, \flat_2, \bar{\xi})),$$

$$J(\mathbb{H}(0, 8, 1)) \geq S(\mathbb{H}(4, 6, 1)),$$

$$-0.1233 \geq -0.2500,$$

and

$$J(\mathbb{O}(\varpi_1, \varpi_2, \bar{\xi})) \leq S(\mathbb{O}(\flat_1, \flat_2, \bar{\xi})),$$

$$J\left(1 - e^{\frac{-|0-8|}{1}}\right) \leq S\left(1 - e^{\frac{-|4-6|}{1}}\right),$$

$$J\,(0.9996646) \leq S\,(0.86466471676)\,,$$

$$-2981.01458 \leq 47.2923386627,$$

Therefore the mappings G and U are $IF_{(S,J)} - proximal$. After that, the following shows that G and U are not proximal. We know that

$$\mathbb{H}(0, G(2), 1) = \mathbb{H}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{H}(4, U(2), 1),$$
$$\mathbb{H}(8, G(4), 1) = \mathbb{H}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{H}(6, U(4), 1),$$

and

$$\mathbb{O}(0, G(2), 1) = \mathbb{O}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{O}(4, U(2), 1),$$
$$\mathbb{O}(8, G(4), 1) = \mathbb{O}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{O}(6, U(4), 1).$$

If we take, $\alpha = 0.5 \in (0, 1)$, then

$$\mathbb{H}(\varpi_1, \varpi_2, \alpha\bar{\xi}) \geq \mathbb{H}(\flat_1, \flat_2, \bar{\xi}),$$
$$\mathbb{H}(0, 8, (0.5)1) \geq \mathbb{H}(4, 6, 1),$$
$$0.00000 \leq 0.1353,$$

and

$$\mathbb{O}(\varpi_1, \varpi_2, \alpha\bar{\xi}) \leq \mathbb{O}(\flat_1, \flat_2, \bar{\xi}),$$
$$\mathbb{O}(0, 8, (0.5)1) \leq \mathbb{O}(4, 6, 1),$$
$$1 \geq 0.86467.$$

Hence a contradiction (see equation (1-2)). Therefore, mappings G, U are not IF-proximal.

Example 3.2. Consider $(Z, \mathbb{H}, \mathbb{O}, *, \diamond)$ be an IFMS defined by

$$\mathbb{H}(u^\flat, n, \bar{\xi}) = e^{\frac{-e^{|u_1^\flat - u_2^\flat| + |n_1 - n_2|}}{\bar{\xi}}}$$

and

$$\mathbb{O}(u^\flat, n, \bar{\xi}) = 1 - e^{\frac{-e^{|u_1^\flat - u_2^\flat| + |n_1 - n_2|}}{\bar{\xi}}}$$

with a continuous t-norm (ctn) and t-conorm as $s * \bar{\xi} = s\bar{\xi}$, $s \diamond \bar{\xi} = \min\{s, \bar{\xi}\}$. Suppose $\mathring{Q} = \{(1, m) \mid m \in \mathbb{R}\}$ and $\mathring{P} = \{(0, m) \mid m \in \mathbb{R}\}$. The mappings defined $G, U : \mathring{P} \to \mathring{Q}$ as

$$U(0, m) = \left(1, \frac{m}{3}\right)$$

and

$$G(0, m) = \left(1, \frac{m}{2}\right).$$

Then, $\mathbb{H}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{H}(u^\flat, n, \bar{\xi}) = e^{\frac{-1}{\bar{\xi}}}$, and $\mathbb{O}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{O}(u^\flat, n, \bar{\xi}) = 1 - e^{\frac{-1}{\bar{\xi}}}$, $\mathring{P}_0 = \mathring{P}_1 = \mathring{P}$, $\mathring{Q}_0 = \mathring{Q}_1 = \mathring{Q}$. Clearly, $G(\mathring{P}_0) \subseteq \mathring{Q}_0$, $U(\mathring{P}_0) \subseteq \mathring{Q}_0$ and $G(\mathring{P}_1) \subseteq \mathring{Q}_1$, $U(\mathring{P}_1) \subseteq \mathring{Q}_1$. The functions defined J, S by

$$S(\bar{\xi}) = \begin{cases} \frac{1}{2^{\ln\bar{\xi}}}, \ if & \bar{\xi} \in (0,1), \\ 2, \ if & \bar{\xi} = 1, \end{cases}$$

$$J(\bar{\xi}) = \begin{cases} \frac{1}{2^{\ln(2\bar{\xi})}}, \ if & \bar{\xi} \in (0,1), \\ 1, \ if & \bar{\xi} = 1. \end{cases}$$

Therefore the mappings G and U are $IF_{(S,J)} - proximal$. After that, we prove that G and U are not IF-proximal. let,

$$\mathbb{H}((0,3), G(0,6), 1) = \mathbb{H}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{H}((0,2), U(0,6), 1),$$

$$\mathbb{H}((0,0), G(0,0), 1) = \mathbb{H}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{H}((0,0), U(0,6), 1),$$

and

$$\mathbb{O}((0,0), G(0,0), 1) = \mathbb{O}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{O}((0,0), U(0,6), 1),$$

$$\mathbb{O}((0,3), G(0,6), 1) = \mathbb{O}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{O}((0,2), U(0,6), 1).$$

Then, there exists $\lambda = 0.2$ such that

$$\mathbb{H}(\varpi_1, \varpi_2, \lambda\bar{\xi}) \geq \mathbb{H}(\flat_1, \flat_2, \bar{\xi}),$$

$$\mathbb{H}((0,0), (0,3), (0.2)1) \geq \mathbb{H}((0,0), (0,2), 1),$$

$$0.0000 \geq 0.1353,$$

and

$$\mathbb{O}(\varpi_1, \varpi_2, \lambda\bar{\xi}) \leq \mathbb{O}(\flat_1, \flat_2, \bar{\xi}),$$

$$\mathbb{O}((0,0), (0,3), (0.2)1) \leq \mathbb{O}((0,0), (0,2), 1),$$

$$1 \leq 0.8647,$$

this is a contradiction. Therefore, G and U are not IF-proximal.

Lemma 3.1. *Suppose $(Z, \mathbb{H}, \mathbb{O}, *, \diamond)$ be an IFMS and $\varpi_n \subset Z$ be a sequence satisfying:*

$$\lim_{n\to\infty} \mathbb{H}(\varpi_n, \varpi_{n+1}, \bar{\xi}) = 1,$$

$$\lim_{n\to\infty} \mathbb{O}(\varpi_n, \varpi_{n+1}, \bar{\xi}) = 0.$$

If the sequence $\{\varpi_n\}$ does not form a Cauchy sequence, then there exist sub-sequences $\varpi_{n_{\bar{\xi}}}$, $\varpi_{q_{\bar{\xi}}}$, and $x, x' > 0$ such that

$$\lim_{\bar{\xi}\to\infty} \mathbb{H}(\varpi_{n_{\bar{\xi}}+1}, \varpi_{q_{\bar{\xi}}+1}, \bar{\xi}) = x, \tag{3}$$

$$\lim_{\bar{\xi}\to\infty} \mathbb{H}(\varpi_{n_{\bar{\xi}}}, \varpi_{n_{q_{\bar{\xi}}}}, \bar{\xi}) = \lim_{\bar{\xi}\to\infty} \mathbb{H}(\varpi_{n_{\bar{\xi}}+1}, \varpi_{q_{\bar{\xi}}}, \bar{\xi}) = \lim_{\bar{\xi}\to\infty} J(\varpi_{n_{\bar{\xi}}}, \varpi_{q_{\bar{\xi}}+1}, \bar{\xi}) = x. \tag{4}$$

$$\lim_{\bar{\xi}\to\infty} \mathbb{O}(\varpi_{n_{\bar{\xi}}+1}, \varpi_{q_{\bar{\xi}}+1}, \bar{\xi}) = x', \tag{5}$$

$$\lim_{\bar{\xi}\to\infty} \mathbb{O}(\varpi_{n_{\bar{\xi}}}, \varpi_{n_{q_{\bar{\xi}}}}, \bar{\xi}) = \lim_{\bar{\xi}\to\infty} \mathbb{O}(\varpi_{n_{\bar{\xi}}+1}, \varpi_{q_{\bar{\xi}}}, \bar{\xi}) = \lim_{\bar{\xi}\to\infty} J(\varpi_{n_{\bar{\xi}}}, \varpi_{q_{\bar{\xi}}+1}, \bar{\xi}) = x'. \tag{6}$$

Lemma 3.2. *The mappings $U, G : \mathring{P} \to \mathring{Q}$ satisfying equation (1-2). Assume $\{\varpi_n\}$ is a sequence such that*

$$\lim_{n\to\infty} \mathbb{H}(\varpi_n, \varpi_{n+1}, \bar{\xi}) = 1, \ \textit{and} \ \lim_{n\to\infty} \mathbb{O}(\varpi_n, \varpi_{n+1}, \bar{\xi}) = 0 \textit{ for any } \epsilon > 0.$$

If the functions $S, J : (0,1] \to \mathbb{R}$ with

$$\limsup_{\bar{\xi}\to\epsilon+} S(\bar{\xi}) < J(\epsilon+),$$

and

$$\lim_{\bar{\xi}\to\epsilon+} (1 - \sup S(\bar{\xi})) > J(\epsilon+).$$

Then $\{\varpi_n\}$ is a Cauchy sequence.

Proof. Suppose $\{\varpi_n\}$ is not a Cauchy sequence, then, by using Lemma 3.1, $\exists$ two sub-sequences $\{\varpi_{n_{\bar{\xi}}}\}$, $\{\varpi_{q_{\bar{\xi}}}\}$ of $\{\varpi_n\}$ such that the Equations (3-6) hold. From Equation (3) and (5), we have,

$$\mathbb{H}(\varpi_{n_{\bar{\xi}}+1}, \varpi_{q_{\bar{\xi}}+1}, \bar{\xi}) > \epsilon, \ \textit{and} \ \mathbb{O}(\varpi_{n_{\bar{\xi}}+1}, \varpi_{q_{\bar{\xi}}+1}, \bar{\xi}) < \epsilon.$$

Since, for $\varpi_{n_{\bar{\xi}}}, \varpi_{n_{\bar{\xi}}+1}, \varpi_{q_{\bar{\xi}}}, \varpi_{q_{\bar{\xi}}+1}, u_{n_{\bar{\xi}}}, u_{n_{\bar{\xi}}+1}, u_{q_{\bar{\xi}}}, u_{q_{\bar{\xi}}+1} \in \mathring{P}$, we get,

$$\mathbb{H}(\varpi_{n_{\bar{\xi}}+1}, G(u_{n_{\bar{\xi}}+1}), \bar{\xi}) = \mathbb{H}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{H}(\varpi_{q_{\bar{\xi}}+1}, G_{u_{q_{\bar{\xi}}+1}}, \bar{\xi})$$

$$\mathbb{H}(\varpi_{n_{\bar{\xi}}+1}, U(u_{n_{\bar{\xi}}+1}), \bar{\xi}) = \mathbb{H}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{H}(\varpi_{q_{\bar{\xi}}+1}, U_{u_{q_{\bar{\xi}}+1}}, \bar{\xi}),$$

and

$$\mathbb{O}(\varpi_{n_{\bar{\xi}}+1}, G(u_{n_{\bar{\xi}}+1}), \bar{\xi}) = \mathbb{O}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{O}(\varpi_{q_{\bar{\xi}}+1}, G_{u_{q_{\bar{\xi}}+1}}, \bar{\xi})$$

$$\mathbb{O}(\varpi_{n_{\bar{\xi}}+1}, U(u_{n_{\bar{\xi}}+1}), \bar{\xi}) = \mathbb{O}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{O}(\varpi_{q_{\bar{\xi}}+1}, U_{u_{q_{\bar{\xi}}+1}}, \bar{\xi}).$$

Thus, from Equation (1) and (2), we have

$$J(\mathbb{H}(\varpi_{n_{\bar{\xi}}+1}, \varpi_{q_{\bar{\xi}}+1}), \bar{\xi}) \geq S(\mathbb{H}(\varpi_{n_{\bar{\xi}}}, \varpi_{q_{\bar{\xi}}}, \bar{\xi}))$$

and

$$J(\mathbb{O}(\varpi_{n_{\bar{\xi}}+1}, \varpi_{q_{\bar{\xi}}+1}), \bar{\xi}) \leq S(\mathbb{O}(\varpi_{n_{\bar{\xi}}}, \varpi_{q_{\bar{\xi}}}, \bar{\xi}))$$

for all $\bar{\xi} \geq 1$. Let $q_{\bar{\xi}} = \mathbb{H}(\varpi_{n_{\bar{\xi}}+1}, \varpi_{q_{\bar{\xi}}+1}, \bar{\xi})$ and $q_{\bar{\xi}-1} = \mathbb{H}(\varpi_{n_{\bar{\xi}}}, \varpi_{q_{\bar{\xi}}}, \bar{\xi})$ and $q'_{\bar{\xi}} = \mathbb{O}(\varpi_{n_{\bar{\xi}}+1}, \varpi_{q'_{\bar{\xi}}+1}, \bar{\xi})$ and $q'_{\bar{\xi}-1} = \mathbb{O}(\varpi_{n_{\bar{\xi}}}, \varpi_{q'_{\bar{\xi}}}, \bar{\xi})$. We have

$$J(q_{\bar{\xi}}) \geq S(q_{\bar{\xi}-1}) \textit{ and } J(q'_{\bar{\xi}}) \leq S(q'_{\bar{\xi}-1}) \textit{ for any } \bar{\xi} \geq 1. \tag{7}$$

From using Equations (3-6), we get,

$$\lim_{\bar{\xi}\to\infty} q_{\bar{\xi}} = \epsilon, \lim_{\bar{\xi}\to\infty} q'_{\bar{\xi}} = \epsilon.$$

From Equation (7) we have,

$$J(\epsilon+) = \lim_{\bar{\xi}\to\infty} J(q'_{\bar{\xi}}) \leq \liminf_{\bar{\xi}\to\infty} S(q'_{\bar{\xi}-1}) \leq \liminf_{c\to\bar{\xi}} S(c), \tag{8}$$

and

$$J(\epsilon+) = \lim_{\bar{\xi}\to\infty} J(q_{\bar{\xi}}) \geq \liminf_{\bar{\xi}\to\infty} S(q_{\bar{\xi}-1}) \geq \liminf_{c\to\bar{\xi}} S(c). \tag{9}$$

Hence a contradiction to condition (I). Therefore, the sequence $\{\varpi_n\}$ is a Cauchy. □

Theorem 3.1. *Suppose* $\mathring{P}, \mathring{Q} \subseteq (Z, \mathbb{H}, \mathbb{O}, *, \diamond)$ *in Complete IFMS such that*

$$\lim_{\bar{\xi}\to\infty} \mathbb{H}(\varpi_1, \varpi_2, \bar{\xi}) = 1, \lim_{\bar{\xi}\to\infty} \mathbb{O}(\varpi_1, \varpi_2, \bar{\xi}) = 0$$

and $\mathring{Q}$ *is AC w.r.t* $\mathring{P}$. *Let* $G : \mathring{P} \to \mathring{Q}$ *and* $U : \mathring{P} \to \mathring{Q}$ *satisfy the subsequently axiom:*

(I) *U dominates G and are* $IF_{(J,S)} - proximal$,

(II) $\lim_{\bar{\xi}\to\epsilon+} S(\bar{\xi}) > J(\epsilon+)$ *for any* $\epsilon > 0$ *and J is non-decreasing.*

(III) $G(\mathring{P}_0) \subseteq \mathring{Q}_0$, $G(\mathring{P}_0) \subseteq U(\mathring{P}_0)$ *and* $G(\mathring{P}_1) \subseteq \mathring{Q}_1$, $G(\mathring{P}_1) \subseteq U(\mathring{P}_1)$, *where* $\mathring{P}_1$, $\mathring{Q}_1$, $\mathring{P}_0$, $\mathring{Q}_0 \neq \emptyset$

(IV) *G and U are continuous and compact proximal,*

Then, U and G have a unique $u \in \mathring{P}$ *such that*

$$\mathbb{H}(u, Uu, \bar{\xi}) = \mathbb{H}(\mathring{P}, \mathring{Q}, \bar{\xi}) \text{ and } \mathbb{H}(u, Gu, \bar{\xi}) = \mathbb{H}(\mathring{P}, \mathring{Q}, \bar{\xi}),$$

$$\mathbb{O}(u, Uu, \bar{\xi}) = \mathbb{O}(\mathring{P}, \mathring{Q}, \bar{\xi}) \text{ and } \mathbb{O}(u, Gu, \bar{\xi}) = \mathbb{O}(\mathring{P}, \mathring{Q}, \bar{\xi}).$$

Proof. Let $u_0 \in \mathring{P}_0, \mathring{P}_1$. Since $G(\mathring{P}_0) \subseteq U(\mathring{P}_0)$ and $G(\mathring{P}_1) \subseteq U(\mathring{P}_1)$ assurances the existence of an element $u_1 \in \mathring{P}_0, \mathring{P}_1$ then

$$Gu_0 = Uu_1.$$

Also, we have $G(\mathring{P}_0) \subseteq U(\mathring{P}_0)$, $G(\mathring{P}_1) \subseteq U(\mathring{P}_1)$ $\exists$ an element $u_2 \in \mathring{P}_0, \mathring{P}_1$ such that

$$Gu_1 = Uu_2.$$

Using the process of iteration there exist $u_n \subseteq \mathring{P}_0, \mathring{P}_1$ such that

$$Gu_n = Uu_{n-1} \; for \; all \; n.$$

Since $G(\mathring{P}_0) \subseteq \mathring{Q}_0$,$G(\mathring{P}_1) \subseteq \mathring{Q}_1$,$G(\mathring{P}_0) \subseteq U(\mathring{P}_0)$, $G(\mathring{P}_1) \subseteq U(\mathring{P}_1)$ $\exists$ an element $\varpi_n \in \mathring{P}_0, \mathring{P}_1$ such that

$$\mathbb{H}(\varpi_n, Gu_n, \bar{\xi}) = \mathbb{H}(\mathring{P}, \mathring{Q}, \bar{\xi}) \text{ and } \mathbb{O}(\varpi_n, Gu_n, \bar{\xi}) = \mathbb{O}(\mathring{P}, \mathring{Q}, \bar{\xi}) \quad \text{for all } n \in \mathbb{N}.$$

Certainly, it follows from the choice of u_n and $\{\varpi_n\}$ that

$$\mathbb{H}(\varpi_{n+1}, G(u_{n+1}), \bar{\xi}) = \mathbb{H}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{H}(\varpi_n, U(u_{n+1}), \bar{\xi}),$$

$$\mathbb{H}(\varpi_n, Gu_n, \bar{\xi}) = \mathbb{H}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{H}(\varpi_{n-1}, U(u_n), \bar{\xi}),$$

and

$$\mathbb{O}(\varpi_{n+1}, G(u_{n+1}), \bar{\xi}) = \mathbb{O}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{O}(\varpi_n, U(u_{n+1}), \bar{\xi}),$$

$$\mathbb{O}(\varpi_n, Gu_n, \bar{\xi}) = \mathbb{O}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{O}(\varpi_{n-1}, U(u_n), \bar{\xi}).$$

If,

$$\mathbb{H}(\varpi_n, Gu_n, \bar{\xi}) = \mathbb{H}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{H}(\varpi_{n-1}, U(u_n), \bar{\xi}), \tag{10}$$

$$\mathbb{O}(\varpi_n, Gu_n, \bar{\xi}) = \mathbb{O}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{O}(\varpi_{n-1}, U(u_n), \bar{\xi}). \tag{11}$$

From Equation (10-11), $\exists\, n \in \mathbb{N}$ such that $\varpi_n = \varpi_{n-1}$ then, ϖ_n is a CBP point of G,U. Conversely, if $\varpi_{n-1} \neq \varpi_n$, then from Equation (10-11), implies that

$$\mathbb{H}(\varpi_{n+1}, G(u_{n+1}), \bar{\xi}) = \mathbb{H}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{H}(\varpi_n, U(u_{n+1}), \bar{\xi}),$$

$$\mathbb{H}(\varpi_n, G(u_n), \bar{\xi}) = \mathbb{H}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{H}(\varpi_{n-1}, U(u_n), \bar{\xi}),$$

and

$$\mathbb{O}(\varpi_{n+1}, G(u_{n+1}), \bar{\xi}) = \mathbb{O}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{O}(\varpi_n, U(u_{n+1}), \bar{\xi}),$$

$$\mathbb{O}(\varpi_n, G(u_n), \bar{\xi}) = \mathbb{O}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{O}(\varpi_{n-1}, U(u_n), \bar{\xi}).$$

Thus, from Equation (1-2) we have,

$$J(\mathbb{H}(\varpi_{n+1}, \varpi_n, \bar{\xi})) \geq S(\mathbb{H}(\varpi_n, \varpi_{n-1}, \bar{\xi})), \tag{12}$$

$$J(\mathbb{O}(\varpi_{n+1}, \varpi_n, \bar{\xi})) \leq S(\mathbb{O}(\varpi_n, \varpi_{n-1}, \bar{\xi})), \tag{13}$$

for all $\varpi_{n-1}, \varpi_n, \varpi_{n+1}, u_{n+1}, u_n \in \mathring{P}$.
Let $q_n = \mathbb{H}(\varpi_{n+1}, \varpi_n, \bar{\xi})$, $q'_n = \mathbb{O}(\varpi_{n+1}, \varpi_n, \bar{\xi})$ we have

$$J(q_n) \geq S(q_{n-1}) > J(q_{n-1}),$$

$$J(q'_n) \leq S(q'_{n-1}) < J(q'_{n-1}).$$

From Equation (12-13), Also J is increasing function, we have

$$q'_n < q'_{n-1} \text{ and } q_n > q_{n-1} \forall n \in \mathbb{N}.$$

To prove that $\{q_n\}$and $\{q'_n\}$ is strictly non-decreasing and strictly non increasing then, its converges to $q \geq 0$ and $q' \leq 1$ respectively. To prove that $q = 0$ and $q' = 1$. In contrary suppose that $q > 0$ and $q' < 1$ and from Equation (12-13), we have,

$$J(\epsilon+) = \lim_{n\to\infty} J(q'_n) \leq \lim_{n\to\infty} S(q'_{n-1}) \leq \lim_{n\to q+} \sup S(t),$$

and

$$\lim_{n\to q+} \inf S(t) \leq \lim_{n\to\infty} S(q_{n-1}) \leq \lim_{n\to\infty} J(q_n) = J(\epsilon+).$$

which is a contradiction to the condition (III). Therefore $q = 1$, $q' = 0$ and

$$\lim_{n\to\infty} \mathbb{H}(\varpi_n, \varpi_{n+1}, \bar{\xi}) = 1 \; and \; \lim_{n\to\infty} \mathbb{O}(\varpi_n, \varpi_{n+1}, \bar{\xi}) = 0.$$

By Lemma 3.2 and condition (III), we prove that $\{\varpi_n\}$ is a Cauchy sequence. Since $(Z, \mathbb{H}, \mathbb{O}, *, \diamond)$ is a complete IFMS, $\mathring{P} \subseteq Z$. Since $G(\mathring{P}_0) \subseteq \mathring{Q}_0$, $G(\mathring{P}_1) \subseteq \mathring{Q}_1$, $\exists$ an element ϖ^* such that

$$\lim_{n\to\infty} \mathbb{H}(\varpi_n, \varpi_*) = 0 \; and \; \lim_{n\to\infty} \mathbb{O}(\varpi_n, \varpi_*) = 1.$$

Moreover,

$$\mathbb{H}(\varpi^*, G(u_n), \bar{\xi}) \geq \mathbb{H}(\varpi^*, \varpi_n, \bar{\xi}), \mathbb{H}(\varpi_n, G(u_n), \bar{\xi}),$$

$$\mathbb{H}(\varpi^*, U(u_n), \bar{\xi}) \geq \mathbb{H}(\varpi^*, \varpi_n, \bar{\xi}).\mathbb{H}(\varpi_n, U(u_n), \bar{\xi}),$$

and

$$\mathbb{O}(\varpi^*, G(u_n), \bar{\xi}) \leq \mathbb{O}(\varpi^*, \varpi_n, \bar{\xi}), \mathbb{O}(\varpi_n, G(u_n), \bar{\xi}),$$

$$\mathbb{O}(\varpi^*, U(u_n), \bar{\xi}) \leq \mathbb{H}(\varpi^*, \varpi_n, \bar{\xi}).\mathbb{O}(\varpi_n, U(u_n), \bar{\xi}).$$

Therefore, $\mathbb{H}(\varpi^*, U(u_n), \bar{\xi}) \to \mathbb{H}(\varpi^*, \mathring{Q}, \bar{\xi})$, and $\mathbb{O}(\varpi^*, U(u_n), \bar{\xi}) \to \mathbb{O}(\varpi^*, \mathring{Q}, \bar{\xi})$ and also

$$\mathbb{H}(\varpi^*, G(u_n), \bar{\xi}) \to \mathbb{H}(\varpi^*, \mathring{Q}, \bar{\xi}),$$

and

$$\mathbb{O}(\varpi^*, G(u_n), \bar{\xi}) \to \mathbb{O}(\varpi^*, \mathring{Q}, \bar{\xi}) \; as \; n \to \infty.$$

Given G and U commute proximally, $U\varpi^*$ and $G\varpi^*$ are identical. Also, $\mathring{Q}$ is AC w.r.t $\mathring{P}$, $\exists$ a sub-sequence $U(u_{n_{\bar{\xi}}})$ and $G(u_{n_{\bar{\xi}}})$ of $U(u_n)$, $G(u_n)$ respectively such that

$$U(u_{n_{\bar{\xi}}}) \to \hat{e}_* \in \mathring{Q} \; and \; G(u_{n_{\bar{\xi}}}) \to \hat{e}_* \in \mathring{Q} \; as \; \bar{\xi} \to \infty.$$

Furthermore, by allowing $\bar{\xi} \to \infty$ in the following equation, we have,

$$\mathbb{H}(\hat{e}_*, G(u_{n_{\bar{\xi}}}), \bar{\xi}) = \mathbb{H}(\mathring{P}, \mathring{Q}, \bar{\xi}) \; and \; \mathbb{H}(\hat{e}_*, Uu_{n_{\bar{\xi}}}, \bar{\xi}) = \mathbb{H}(\mathring{P}, \mathring{Q}, \bar{\xi}), \tag{14}$$

and

$$\mathbb{O}(\hat{e}_*, G(u_{n_{\bar{\xi}}}), \bar{\xi}) = \mathbb{O}(\mathring{P}, \mathring{Q}, \bar{\xi}) \; and \; \mathbb{O}(\hat{e}_*, Uu_{n_{\bar{\xi}}}, \bar{\xi}) = \mathbb{O}(\mathring{P}, \mathring{Q}, \bar{\xi}). \tag{15}$$

Also,

$$\mathbb{H}(\hat{e}_*, \varpi^*, \bar{\xi}) = \mathbb{H}(\mathring{P}, \mathring{Q}, \bar{\xi}) \; and \; \mathbb{O}(\hat{e}_*, \varpi^*, \bar{\xi}) = \mathbb{O}(\mathring{P}, \mathring{Q}, \bar{\xi}).$$

Since, $\varpi^* \in \mathring{P}_0, \mathring{P}_1$, so $G(\varpi^*) \in G(\mathring{P}_0) \subseteq \mathring{Q}_0$ and $\exists x \in \mathring{P}_0$ and $G(\varpi^*) \in G(\mathring{P}_1) \subseteq \mathring{Q}_1$ and $\exists x' \in \mathring{P}_1$.

Similarly, $\varpi^* \in \mathring{P}_0$, so $U(\varpi^*) \in U(\mathring{P}_0) \subseteq \mathring{Q}_0$ and $\exists x \in \mathring{P}_0$, $\varpi^* \in \mathring{P}_1$, so $U(\varpi^*) \in U(\mathring{P}_1) \subseteq \mathring{Q}_1$ and $\exists x' \in \mathring{P}_1$ such that

$$\begin{aligned} \mathbb{H}(\varpi^*, G(\varpi^*), \bar{\xi}) = \mathbb{H}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{H}(\varpi^*, U(\varpi^*, \bar{\xi}), \\ \mathbb{H}(x, G(\varpi^*), \bar{\xi}) = \mathbb{H}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{H}(x, U(\varpi^*, \bar{\xi}), \end{aligned} \tag{16}$$

and

$$\begin{aligned} \mathbb{O}(\varpi^*, G(\varpi^*), \bar{\xi}) = \mathbb{O}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{O}(\varpi^*, U(\varpi^*, \bar{\xi}), \\ \mathbb{O}(x', G(\varpi^*), \bar{\xi}) = \mathbb{O}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{O}(x', U(\varpi^*, \bar{\xi}). \end{aligned} \tag{17}$$

Now, by Equation (14-17) and (1-2), we get,

$$J(\mathbb{H}(\varpi^*, x, \bar{\xi})) \geq S(\mathbb{H}(\varpi^*, x, \bar{\xi})) < J(\mathbb{H}(\varpi^*, x, \bar{\xi})),$$

$$J(\mathbb{O}(\varpi^*, x', \bar{\xi})) \leq S(\mathbb{O}(\varpi^*, x', \bar{\xi})) > J(\mathbb{O}(\varpi^*, x', \bar{\xi})).$$

Since J is non decreasing function, we get,

$$\mathbb{H}(\varpi^*, x, \alpha\bar{\xi}) \geq \mathbb{H}(\varpi^*, x, \bar{\xi}) < \mathbb{H}(\varpi^*, x, \bar{\xi}),$$

$$\mathbb{O}(\varpi^*, x', \alpha\bar{\xi}) \leq \mathbb{O}(\varpi^*, x', \bar{\xi}) > \mathbb{O}(\varpi^*, x', \bar{\xi}).$$

This implies ϖ^* and x,x' are identical. Finally, by Equation (10-11), we have

$$\mathbb{H}(\varpi^*, U(\varpi^*), \bar{\xi}) = \mathbb{H}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{H}(\varpi^*, G(\varpi^*), \bar{\xi}),$$

$$\mathbb{O}(\varpi^*, U(\varpi^*), \bar{\xi}) = \mathbb{O}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{O}(\varpi^*, G(\varpi^*), \bar{\xi}).$$

This shows that ϖ^* is a CBP point of mappings U and G.

□

Theorem 3.2. *Suppose $\mathring{P}, \mathring{Q} \subseteq (Z, \mathbb{H}, \mathbb{O}, *, \diamond)$ in a complete IFMS such that $\mathring{Q}$ is AC w.r.t $\mathring{P}$. Also, assume that $\lim_{\bar{\xi}\to\infty} \mathbb{H}(\varpi_1, \varpi_2, \bar{\xi}) = 1$, $\lim_{\bar{\xi}\to\infty} \mathbb{O}(\varpi_1, \varpi_2, \bar{\xi}) = 0$ and $\mathring{P}_1, \mathring{Q}_1, \mathring{P}_0, \mathring{Q}_0 \neq \emptyset$. Let $G, U : \mathring{P} \to \mathring{Q}$ satisfy the following circumstance:*

(I) *U dominates G and is $IF_{(J,S)} - proximal$;*
(II) *G and U are continuous and compact proximal;*
(III) *$G(\mathring{P}_0) \subseteq \mathring{Q}_0$, $G(\mathring{P}_0) \subseteq U(\mathring{P}_0)$ and $G(\mathring{P}_1) \subseteq \mathring{Q}_1$, $G(\mathring{P}_1) \subseteq U(\mathring{P}_1)$;*
(IV) *$\{S(\bar{\xi}_n)\}$ and $\{J(\bar{\xi}_n)\}$ are convergent sequences such that $\lim_{n\to\infty} J(\bar{\xi}_n) = \lim_{n\to\infty} S(\bar{\xi}_n)$, then $\lim_{n\to\infty} \bar{\xi}_n = 1$. Also J is non-decreasing.*

Then U and G have a unique $u \in \mathring{P}$ such that

$$\mathbb{H}(u, Uu, \bar{\xi}) = \mathbb{H}(\mathring{P}, \mathring{Q}, \bar{\xi}), \text{ and } \mathbb{H}(u, Gu, \bar{\xi}) = \mathbb{H}(\mathring{P}, \mathring{Q}, \bar{\xi}),$$

$$\mathbb{O}(u, Uu, \bar{\xi}) = \mathbb{O}(\mathring{P}, \mathring{Q}, \bar{\xi}), \text{ and } \mathbb{O}(u, Gu, \bar{\xi}) = \mathbb{O}(\mathring{P}, \mathring{Q}, \bar{\xi}).$$

Proof. As we proceed with Theorem 3.1's proof, we obtain

$$J(q_n) \geq S(q_{n-1}) < J(q_{n-1}), \ and\ J(q'_n) \leq S(q'_{n-1}) > J(q'_{n-1}). \tag{18}$$

By using Equation (18), we concludes that $\{J(q_n)\}$ is a non-decreasing sequence either, $\{J(q_n)\}$ is bounded above, or not then

$$\inf_{n>\epsilon} J(q_n) > -\infty.$$

From Lemma 3.2 then $q_n \to 1$. Next, should $\{J(q_n)\}$ be bounded, it implies that the sequence converges.

Similarly $\{J(q'_n)\}$ is non-increasing sequence either, $\{J(q'_n)\}$ is bounded below, or not then

$$\sup_{n>\epsilon} J(q'_n) < -\infty \quad \text{for every } \epsilon > 0, n \in \mathbb{N}.$$

By using Lemma 3.2 then $q'_n \to 0$. On the other side, if $\{J(q'_n)\}$ is bounded below then it is a convergent sequence.

By using Equation (18), the sequence $\{S(q_n)\}$,$\{S(q'_n)\}$ also converges. Also, from condition (III), we get

$$\lim_{n\to\infty} q_n = 1, or \lim_{n\to\infty} J(\varpi_n, u_{n+1}, \bar{\xi}) = 1,$$

$$\lim_{n\to\infty} q'_n = 0, or \lim_{n\to\infty} J(\varpi_n, u_{n+1}, \bar{\xi}) = o,$$

for any sequence $\{\varpi_n\}$. Now, follow the proof of Theorem 3.1, we get,

$$\mathbb{H}(\varpi^*, U(\varpi^*), \bar{\xi}) = \mathbb{H}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{H}(\varpi^*, G(\varpi^*), \bar{\xi}),$$

and

$$\mathbb{O}(\varpi^*, U(\varpi^*), \bar{\xi}) = \mathbb{O}(\mathring{P}, \mathring{Q}, \bar{\xi}) = \mathbb{O}(\varpi^*, G(\varpi^*), \bar{\xi}).$$

This prove that ϖ^* is a CBP point of the mappings U and G. □

Now, first of all we have to construct some conditions:

(K1) The Kernel $\bar{\xi}_1 : I_R \times I_R \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions, and $|\bar{\xi}_1(\overline{p}, s, \varpi((s))| \leq w(\overline{p}, s) + e(\overline{p}, s)$, where $w, e \in \mathcal{L}^2(I_R \times I_R)$, $e(\overline{p}, s) > 0$.

(K2) The function $f : I_R \to [1, \infty)$ is continuous and bounded.

(K3) The $\sup_{\overline{p}\in I_R} \int_{I_R} |\bar{\xi}_1(\overline{p}, s)|\, ds \leq C$, where C a positive constant.

(K4) Let $\varpi_0 \in \mathring{P}$. Since $G(\mathring{P}) \subseteq U(\mathring{P})$ assurances the existence of $\varpi_1 \in \mathring{P}$ such that $G\varpi_0 = U\varpi_1$. Also, we have $G(\mathring{P}) \subseteq U(\mathring{P})$.

(K5) The function $q : I_R \times I_R \to \mathbb{R}$ such that

$$\alpha(\overline{p}) = \int_{I_R} q^2(\overline{p}, s)\, ds \leq \frac{1}{v\mathring{P}^2}$$

is non negative, measurable and integrable over I_R.

$$e^{\frac{-|\bar{\xi}_1(\overline{p},s,\varpi(s))-\bar{\xi}_1(\overline{p},s,f(s))|}{\bar{\xi}}} \geq e^{\frac{-q(\overline{p},s)}{\bar{\xi}}} e^{\frac{-|\varpi(s)-f(s)|}{\bar{\xi}}}$$

for all $\overline{p}, s \in I_R$ and $\varpi, f \in \mathring{P}$.

Theorem 3.3. *Let the mapping f and $\bar{\xi}_1$ satisfying the conditions (K1)–(K5), then, the integral Equation*

$$\varpi(\overline{p}) = f(\overline{p}) + \int_{I_R} \bar{\xi}_1(\overline{p}, s, \varpi(\overline{p})(s))\, ds. \tag{19}$$

has a singular solution.

Proof. Let us consider the mappings $G, U : \mathring{P} \to \mathring{Q}$,

$$(\mathbb{H}\varpi)(\overline{p}) = f(\overline{p}) + \int_{I_R} \bar{\xi}_1(\overline{p}, s, \varpi(s))\, ds, \tag{20}$$

$$(\mathbb{O}\varpi)(\overline{p}) = f(\overline{p}) + \int_{I_R} \bar{\xi}_1(\overline{p}, s, \varpi(s))\, ds. \tag{21}$$

Suppose $\varpi \in \mathring{P}$ and for every $\overline{p} \in I_R$,

$$(\mathbb{H}\varpi)(\overline{p}) = f(\overline{p}) + \int_{I_R} \bar{\xi}_1(\overline{p}, s, \varpi(s))\, ds \geq 1,$$

$$(\mathbb{O}\varpi)(\overline{p}) = f(\overline{p}) + \int_{I_R} \bar{\xi}_1(\overline{p}, s, \varpi(s))\, ds \leq 1.$$

Conditions (K1)–(K2) indicates that $\mathbb{H}$,$\mathbb{O}$ is a continuous and compact mapping from $\mathring{P}$ to $\mathring{Q}$. Also, check the contraction of Equation (12-13) of Theorem 3.1. For this by using (K4 -K5) and the Holder inequality, we get,

$$\begin{aligned}
e^{\frac{-|(\mathbb{H}\varpi)(\overline{p})-(\mathbb{H}f)(\overline{p})|^2}{k}} &= e^{\frac{-\left|\int_{I_R}\bar{\xi}_1(\overline{p},s,\varpi(s))\,ds-\int_{I_R}\bar{\xi}_1(\overline{p},s,f(s))\,ds\right|^2}{k}} \\
&\geq e^{\frac{[-\left|\int_{I_R}\bar{\xi}_1(\overline{p},s,\varpi(s))-\bar{\xi}_1(\overline{p},s,f(s))\,ds\right|]^2}{\bar{\xi}}} \\
&\geq e^{\frac{-\left(\int_{I_R} q(\overline{p},s)|\varpi(s)-f(s)|\,ds\right)^2}{\bar{\xi}}} \\
&\geq e^{\frac{-\int_{I_R} q^2(\overline{p},s)}{\bar{\xi}}ds} . e^{\frac{-\int_{I_R}|\varpi(s)-f(s)|^2}{\bar{\xi}}ds} \\
&\geq e^{\frac{-\alpha(\overline{p})}{\bar{\xi}}} . e^{\frac{-\int_{I_R}|\varpi(s)-f(s)|^2}{\bar{\xi}}ds},
\end{aligned}$$

and

$$1-e^{\frac{-|(\mathbb{O}\varpi)(\overline{p})-(\mathbb{O}f)(\overline{p})|^2}{\bar{\xi}}} = 1-e^{\frac{-\left|\int_{I_R}\bar{\xi}_1(\overline{p},s,\varpi(s))\,ds-\int_{I_R}\bar{\xi}_1(\overline{p},s,f(s))\,ds\right|^2}{\bar{\xi}}}$$
$$\leq 1-e^{\frac{[-\left|\int_{I_R}\bar{\xi}_1(\overline{p},s,\varpi(s))-\bar{\xi}_1(\overline{p},s,f(s))\,ds\right|]^2}{\bar{\xi}}}$$
$$\leq 1-e^{\frac{-\left(\int_{I_R}q(\overline{p},s)|\varpi(s)-f(s)|\,ds\right)^2}{\bar{\xi}}}$$
$$\leq 1-\left(e^{\frac{-\int_{I_R}q^2(\overline{p},s)}{\bar{\xi}}\,ds}.e^{\frac{-\int_{I_R}|\varpi(s)-f(s)|^2}{\bar{\xi}}\,ds}\right).$$

By integrating above equation w.r.t $\overline{p}$,

$$e^{\frac{-|(\mathbb{H}\varpi)(\overline{p})-(\mathbb{H}f)(\overline{p})|^2 d\bar{h}}{\bar{\xi}}} \geq e^{\frac{-\int_{I_R}\left(\alpha(\overline{p}).\int_{I_R}|\varpi(s)-f(s)|^2 ds\right)d\bar{h}}{\bar{\xi}}}$$
$$\geq e^{\frac{-\int_{I_R}\left(\alpha(\overline{p})e^{v}\int_{I_R}\alpha(s)ds.e^{-v}\int_{I_R}\alpha(s)ds\int_{I_R}|\varpi(s)-f(s)|^2 ds\right)d\bar{h}}{\bar{\xi}}}$$
$$\geq e^{\frac{-||\varpi-f||_2^2\int_{I_R}\left(\alpha(\overline{p})e^{v}\int_{I_R}\alpha(s)ds\right)d\bar{h}}{\bar{\xi}}}$$
$$\geq e^{\frac{-||\varpi-f||_2^2 e^{v}\int_{I_R}\alpha(s)dsds}{v\mathring{P}^2\bar{\xi},}}$$

and

$$1-e^{\frac{-|(\mathbb{O}\varpi)(\overline{p})-(\mathbb{O}f)(\overline{p})|^2 d\bar{h}}{\bar{\xi}}} \leq 1-e^{\frac{-\int_{I_R}\left(\alpha(\overline{p}).\int_{I_R}|\varpi(s)-f(s)|^2 ds\right)d\bar{h}}{\bar{\xi}}}$$
$$\leq 1-e^{\frac{-\int_{I_R}\left(\alpha(\overline{p})e^{v}\int_{I_R}\alpha(s)ds.e^{-v}\int_{I_R}\alpha(s)ds\int_{I_R}|\varpi(s)-f(s)|^2 ds\right)d\bar{h}}{\bar{\xi}}}$$
$$\leq 1-e^{\frac{-||\varpi-f||_2^2\int_{I_R}\left(\alpha(\overline{p})e^{v}\int_{I_R}\alpha(s)ds\right)d\bar{h}}{\bar{\xi}}}$$
$$\leq 1-e^{\frac{-||\varpi-f||_2^2 e^{v}\int_{I_R}\alpha(s)dsds}{v\mathring{P}^2\bar{\xi}.}}$$

Thus, we have

$$e^{-\frac{\mathring{P}^2 e^{-v\int_{I_R}\alpha(s)\,ds}|(\mathbb{H}\varpi)\overline{p}-(\mathbb{H}f)\overline{p}|^2}{v\bar{\xi}}} \geq e^{\frac{-||\varpi-f||_2^2}{\bar{\xi}}},$$
$$1-e^{-\frac{\mathring{P}^2 e^{-v\int_{I_R}\alpha(s)\,ds}|(\mathbb{O}\varpi)\overline{p}-(\mathbb{O}f)\overline{p}|^2}{v\bar{\xi}}} \leq 1-e^{\frac{-||\varpi-f||_2^2}{\bar{\xi}}}.$$

This implies that,

$$e^{-\frac{\mathring{P}^2|(\mathbb{H}\varpi)-(\mathbb{H}f)|^2}{v\bar{\xi}}} \geq e^{\frac{-||\varpi-f||_2^2}{\bar{\xi}}},$$

and

$$1-e^{-\frac{\mathring{P}^2|(\mathbb{H}\varpi)-(\mathbb{H}f)|^2}{v\bar{\xi}}} \leq 1-e^{\frac{-||\varpi-f||_2^2}{\bar{\xi}}},$$

i.e,

$$\mathcal{L}(\varpi,f)d_v(\mathbb{H}\varpi,(\mathbb{H}f),\bar{\xi}) \geq d_v(\varpi,f),$$

and

$$\mathcal{L}(\varpi, f) d_v(\mathbb{O}\varpi, (\mathbb{O}f), \bar{\xi}) \leq d_v(\varpi, f).$$

Consider $J(s) = \frac{1}{\ln(s)}$ and $S(s) = \frac{1}{\ln(s^2)}$ then,

$$J(\mathcal{L}(\varpi, f) d_v(\mathbb{H}\varpi, (Jf), \bar{\xi})) \geq S(F_1(\overline{p}, f)),$$

and

$$J(\mathcal{L}(\varpi, f) d_v(\mathbb{O}\varpi, (Jf), \bar{\xi})) \leq S(F_1(\overline{p}, f)).$$

Hence J and S follow Theorem 3.1. Therefore the integral Equation (19) has a unique solution. □

4. Application

The integral Equation (19) and its subsequent analysis can be discussed in terms of its applications in solving both Volterra and Fredholm integral equations. The provided framework and theorem offer a rigorous mathematical foundation for ensuring the existence and uniqueness of solutions to the integral equation:

$$\varpi(\overline{p}) = f(\overline{p}) + \int_{I_R} \bar{\xi}_1(\overline{p}, s, \varpi(\overline{p})(s))\, ds, \tag{22}$$

where $I_R = (a, x)$, and a is fixed. This equation, depending on the region of integration, can represent either a Volterra or Fredholm integral equation. We defined the norm on the space $\mathcal{L}^2(I_R)$ and demonstrated that it forms a Banach space.

The norm is defined by $\|.\| : \mathcal{L}^2(I_R) \to [0, \infty)$

$$\|\varpi\| = \sqrt{\int_{I_R} |\varpi(s)|^2\, ds}, \quad \text{for all } \varpi \in \mathcal{L}^2(I_R), \tag{23}$$

and equivalent norm is defined by:

$$\|\varpi\| = \sqrt{\sup\left(\int_{I_R} e^{-v\int_{I_R}\alpha(s)\, ds} \int_{I_R} |\varpi(s)|^2\, ds\right)}, \tag{24}$$

where

$$\mathcal{L}^2(I_R) = \left\{\varpi \,\middle|\, \int_{I_R} |\varpi(s)|^2\, ds < \infty\right\} \text{and } v > 1.$$

Then, $(\mathcal{L}^2(I_R), \|.\|_{2,v})$ is a Banach space.

Let $Z = \{\varpi \in \mathcal{L}^2(I_R) : \varpi(s) > 0 \text{ for all } s\}$. The functions $\mathbb{H}$ and $\mathbb{O}$ is defined as

$$\mathbb{H}(\varpi, f, \bar{\xi}) = e^{\frac{||\varpi - f||}{\bar{\xi}}} \; and \; \mathbb{O}(\varpi, f, \bar{\xi}) = 1 - e^{\frac{||\varpi - f||}{\bar{\xi}}} \; for \; all \; \varpi, f \in Z.$$

Then $(Z, \mathbb{H}, \mathbb{O}, *, \diamond)$ is a complete IFMS.

We also introduced functions $\mathbb{H}$ and $\mathbb{O}$ to construct an Intuitionistic Fuzzy Metric Space (IFMS), and provided a set of conditions (K1-K5) that ensure the existence of a singular solution to the integral equation.

Through the application of these conditions and utilizing properties such as Carathéodory conditions, continuity, boundedness, and integrability, we proved that the integral equation (19) has a unique solution. This was done by showing that the mappings $\mathbb{H}$ and $\mathbb{O}$ are continuous and compact, and by applying specific inequalities and integrability conditions.

Therefore, we have successfully established that under the given conditions, the integral equation has a singular solution, contributing to the theory of integral equations and their applications in various mathematical and applied contexts.

Abbreviations:

FP–fixed point

MS– metric space

FMS–fuzzy metric space

IFMS–intuitionistic fuzzy metric space

CBP–common best proximity

BPp–best proximity point

$IF_{(S,J)}$ proximal–intuitionistic fuzzy (S, J) proximal

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