MAXIMAL HÖLDER REGULARITY OF ELLIPTIC TRANSMISSION PROBLEMS IN UNBOUNDED DOMAINS

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Abstract. In the present paper, we study a class of abstract elliptic transmission problems family, posed in unbounded heterogeneous domains $]-\infty,\delta]$, with non-homogeneous Dirichlet boundary conditions at $-\infty$, when the right-hand side is a Hölder continuous function. Our approach is based on the generalized analytic semigroup theory and Sinestrari's results, to ensure existence, uniqueness, and maximal regularity of the strict solution.

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1. Introduction

Consider the following abstract transmission problem for a positive parameter $\delta \in (0,1]$:

$$\begin{cases} u''(x) + Au(x) = g^{\delta}(x), \quad x \in\,]-\infty,0[\,\cup\,]0,\delta[\,, \\ u(0_-) = u(0_+), \quad \mu_+ u'(0_+) = \mu_- u'(0_-) \\ \lim\limits_{x\to-\infty} u(x) = f_-, \quad u'(\delta) = f_+^{\delta}, \end{cases} \tag{P_δ}$$

where μ_- and μ_+ are two real positive numbers, f_+^{δ} and f_- are two given elements in a complex Banach space E. The operator A is a closed linear operator with domain $D(A)$ not necessarily dense in E, and which satisfies that: The resolvent set $\rho(A)$ satisfies $\rho(A) \supset [0,+\infty[$, and there exists a constant $C > 0$ such that

$$\forall \lambda \in [0,+\infty[\, : \left\| (A-\lambda I)^{-1} \right\|_{\mathcal{L}(E)} \leq \frac{C}{1+\lambda}. \tag{H}$$

This assumption allows us to define $\sqrt{-A}$ the square root of $(-A)$, for more details, see Balakrishnan [1] for dense domains and Martinez [13] for nondense domains. Furthermore, the operator $B := -\sqrt{-A}$ generates an analytic semigroup which is not strongly continuous at 0 when $\overline{D(A)} \neq E$ (see, for

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instance [4] and [7]). This semigroup satisfies that: for all $t > 0$ and $x \in E$, $e^{tB}x \in D\left(B^k\right)$ holds for every $k \in \mathbb{N}^*$. Moreover, it satisfies the negative exponential property, see [7] and [14]; for each $k \in \mathbb{N}$, there exist constants $M_k > 0$ and $a > 0$ such that

$$\left\|t^k B^k e^{tB}\right\|_{\mathcal{L}(E)} \leq M_k e^{-at}, \quad \forall t > 0.$$

Given the heterogeneity of the domain $]-\infty, \delta]$, and the permeability of the interface at 0, it is possible to allow a discontinuity in the second member $g^\delta(0^-) \neq g^\delta(0^+)$. More precisely, we assume that

$$g_- \in BUC^{2\alpha}\left(]-\infty, 0]; E\right), \quad g_+^\delta \in C^{2\alpha}\left([0, \delta]; E\right),$$

for $0 < 2\alpha < 1$, where g_- and g_+^δ denote the restrictions of g^δ to $]-\infty, 0]$ and $[0, \delta]$, respectively. Recall that $C^{2\alpha}\left([0, \delta]; E\right)$ is the Banach space of vector-valued function $f : [0, \delta] \to E$ which satisfy the Hölder condition

$$[f]_{2\alpha} := \sup_{\substack{x,y \in [0,\delta] \\ x \neq y}} \frac{\|f(x) - f(y)\|_E}{|x-y|^{2\alpha}} < \infty,$$

with exponent 2α, equipped with the norm $\|f\|_{2\alpha} = \|f\|_{C([0,\delta];E)} + [f]_{2\alpha}$. The space $BUC^{2\alpha}(]-\infty, 0]; E)$ is the space of bounded uniformly Hölder continuous function from $]-\infty, 0]$ into E. The present study will focus on the existence, uniqueness and maximal regularity for the strict solution of (P_δ); we seek a solution u equal to u_- on $]-\infty, 0]$ and u_+ on $[0, \delta]$, and fulfills

$$\begin{cases} u_- \in BUC^2\left(]-\infty, 0]; E\right) \cap BUC\left(]-\infty, 0]; D(A)\right) \\ u_+ \in C^2\left([0, \delta]; E\right) \cap C\left([0, \delta]; D(A)\right), \end{cases}$$

which satisfies the maximal regularity property

$$u_-'', Au_- \in BUC^{2\alpha}\left(]-\infty, 0]; E\right) \text{ and } u_+'', Au_+ \in C^{2\alpha}\left([0, \delta]; E\right).$$

Several researchers have contributed to this subject in different spaces, with diverse boundary conditions. For instance, within the continuous framework, F. Bouziani et al. [3] have studied a family of abstract transmission problems with variable operator coefficients, verifying the Labbas-Terreni hypothesis inspired by sum theory, by using the Dunford functional calculus, and interpolation spaces to prove the regularity of the strict solution.

In L^p spaces, Dore et al. [5] obtained maximal regularity results for an elliptic transmission problem, which is considered on juxtaposition of two bounded intervals, by using impedance and admittance operators, and H^∞ functional calculus. In [10], the author used the analytic semigroups theory and the Dore-Venni's theorem to study the same problem (P_δ), but in a UMD Banach space with a homogeneous boundary condition near infinity. In the same framework, R. Labbas et al. [9] have studied some transmission problems concerning conductivity in a biological cell by transforming it into an unbounded heterogeneous cylindrical body via natural changes of variables. The study utilized analytic semigroup theory to establish the existence, uniqueness, and maximal regularity of the classical solution.

Recently, in [2], the study has explored elliptic transmission problems on heterogeneous unbounded domains in UMD spaces. The authors examine the impact of two spectral parameters, one in the differential equation and the other in the interface permeability condition, to establish conditions ensuring maximal solution regularity.

In this study, within the framework of Hölder spaces, to establish the regularity of the strict solution, our approach is essentially based on semigroup theory, Sinestrari's results, and real Banach interpolation spaces $D_B(\alpha,\infty)$, for $0<\alpha<1$, which are well known in many concrete cases. When B generates an analytic semigroup, these spaces can be characterized as

$$D_B(\alpha,\infty)=\left\{x\in E:[x]_\alpha=\sup_{t>0}\left\|t^{1-\alpha}Be^{tB}x\right\|<\infty\right\},$$

and equipped with the norm $\|x\|_{D_B(\alpha,\infty)}:=\|x\|_{2\alpha}=\|x\|_E+[x]_\alpha$.

The following theorems present our main results.

Theorem 1. *Let A satisfy (H). For $\alpha\in\left]0,\frac{1}{2}\right[$, assume that*

$$g_+\in C^{2\alpha}([0,\delta];E)\text{ and }g_-\in BUC^{2\alpha}((-\infty,0];E),\text{ with }g_-(-\infty)=0.$$

Then, for any $f_-\in D(A)$, $f_+\in D\left(\sqrt{-A}\right)$, the problem (P_δ) has a unique strict solution if

$$g_+(0)-g_-(0),\ \sqrt{-A}f_+,\text{ and }Af_-\text{ belong to }\overline{D(A)}.$$

Theorem 2. *Let A satisfy (H). For $\alpha\in\left]0,\frac{1}{2}\right[$, assume that*

$$g_+\in C^{2\alpha}([0,\delta];E)\text{ and }g_-\in BUC^{2\alpha}((-\infty,0];E),\text{ with }g_-(-\infty)=0.$$

Then, for any $f_-\in D(A)$, $f_+\in D\left(\sqrt{-A}\right)$, the unique strict solution of (P_δ) satisfies the maximal regularity property:

$$u_+'',\ Au_+\in C^{2\alpha}([0,\delta];E),\text{ and }u_-'',\ Au_-\in BUC^{2\alpha}((-\infty,0];E)$$

if $g_+(0)-g_-(0)$, $\sqrt{-A}f_+$, and Af_- belong to $D_A(\alpha,\infty)$.

The organization of this paper is as follows:

Section 2 is devoted to several technical lemmas that will be useful in the analysis of the abstract problem. In Section 3, we use analytic semigroup theory to determine an explicit representation of the solution, after solving two auxiliary problems via the Krien transformation. Section 4 presents the proofs of the fundamental theorems, while Section 5 illustrates their practical applications.

2. Technical Lemmas

In this study, we will consider several technical lemmas, taking into consideration that $B := -\sqrt{-A}$, and the fact that the function $x \to |x|^{2\alpha}$ is Hölder continuous with the exponent $0 < 2\alpha < 1$, so there exists a positive constant C such that

$$\left||x|^{2\alpha} - |y|^{2\alpha}\right| \leq C\,|x-y|^{2\alpha}.$$

Some techniques used in proving these technical lemmas are inspired by the works [15], [6], and [8].

Lemma 1. *Let A verify (H). Then for $\varphi \in D(B)$ we have*

1) $x \mapsto Be^{(\delta-x)B}\varphi$, $x \mapsto Be^{xB}\varphi \in C([0,\delta],E)$ if and only if $B\varphi \in \overline{D(A)}$.

2) $x \mapsto Be^{(\delta-x)B}\varphi$, $x \mapsto Be^{xB}\varphi \in C^{2\alpha}([0,\delta],E)$ if and only if $\varphi \in D_A(1/2+\alpha,+\infty)$.

Proof. We use Sinestrari's results [15, Theorem 3.1 and Remark, p. 39], and the fact that $\overline{D(B)} := \overline{D(A)}$ for the first point and the reiteration theorem, which establishes that

$$D_B(2\alpha,+\infty) := D_{B^2}(\alpha,+\infty) = D_A(\alpha,+\infty).$$

for the second point. □

Remark 1. *Du to [15, Proposition 1.5, p. 24], for each $\alpha \in \,]0,1[$, we have*

$$D(A) \subset D_A(\alpha,\infty) \subset \overline{D(A)}.$$

Lemma 2. *Let A verify (H). If $g_+^\delta \in C^{2\alpha}([0,\delta];E)$, then*

$$x \mapsto e^{(\delta-x)B}\left(g_+^\delta(x) - g_+^\delta(\delta)\right) \in C^{2\alpha}([0,\delta];E).$$

Proof. Let $g_+^\delta \in C^{2\alpha}([0,\delta];E)$. For all $0 \leq s < x \leq \delta$, we write

$$\begin{aligned} & e^{(\delta-s)B}\left(g_+^\delta(s) - g_+^\delta(\delta)\right) - e^{(\delta-x)B}\left(g_+^\delta(x) - g_+^\delta(\delta)\right) \\ : \; &= e^{(\delta-s)B}\left(g_+^\delta(s) - g_+^\delta(x)\right) + \left(e^{(\delta-s)B} - e^{(\delta-x)B}\right)\left(g_+^\delta(x) - g_+^\delta(\delta)\right). \end{aligned}$$

It is clear that $\left\|e^{(\delta-s)B}\left(g_+^\delta(s) - g_+^\delta(x)\right)\right\| \leq M_0\,(x-s)^{2\alpha}\left\|g_+^\delta\right\|_{2\alpha}$. For the second term, we use the relation $e^{(\delta-s)B} - e^{(\delta-x)B} = B\int_{\delta-s}^{\delta-x} e^{\tau B}d\tau$, from which we get

$$\begin{aligned} \left\|\left(e^{(\delta-s)B} - e^{(\delta-x)B}\right)\left(g_+^\delta(x) - g_+^\delta(\delta)\right)\right\| &\leq M_1\left\|g_+^\delta\right\|_{2\alpha}\int_{\delta-s}^{\delta-x}\frac{d\tau}{\tau}(\delta-x)^{2\alpha} \\ &\leq M_1\left\|g_+^\delta\right\|_{2\alpha}\int_{\delta-x}^{\delta-s}\frac{d\tau}{\tau^{1-2\alpha}} \leq M_1\,(x-s)^{2\alpha}\left\|g_+^\delta\right\|_{2\alpha}, \end{aligned}$$

therefore, $x \to e^{xB}\left(g_+^\delta(x) - g_+^\delta(\delta)\right)$ is Hölder continuous on $[0,\delta]$. □

Lemma 3. *Let A verify (H). We hold*

1. *If $\varphi \in \overline{D(A)}$, then $x \mapsto e^{-xB}\varphi - \varphi \in BUC\left(\left]-\infty, 0\right]; E\right)$,*

2. *If $\varphi \in D_A(\alpha, +\infty)$, then $x \mapsto e^{-xB}\varphi - \varphi \in BUC^{2\alpha}\left(\left]-\infty, 0\right]; E\right)$,*

3. *If $\varphi \in D(A)$, and $A\varphi \in \overline{D(A)}$, then $x \mapsto A\left(e^{-xB}\varphi - \varphi\right) \in BUC\left(\left]-\infty, 0\right]; E\right)$,*

4. *If $\varphi \in D(A)$ and $A\varphi \in D_A(\alpha, +\infty)$, then $x \mapsto A\left(e^{-xB}\varphi - \varphi\right) \in BUC^{2\alpha}\left(\left]-\infty, 0\right]; E\right)$.*

Proof. 1) For $x \in \left]-\infty, -1\right]$ and $h > 0$, we have

$$\begin{aligned}\left\|\left(e^{-xB}\varphi - \varphi\right) - \left(e^{-(x-h)B}\varphi - \varphi\right)\right\| &\leq \left\|B \int\limits_{x-h}^{x} e^{-sB}\varphi\, ds\right\| \leq M_1 \int\limits_{x-h}^{x} \frac{e^{-sa}ds}{s} \|\varphi\| \\ &\leq M_1 \int\limits_{x-h}^{x} ds\, \|\varphi\| \leq M_1 h \|\varphi\|,\end{aligned}$$

thus, the function $x \mapsto e^{-xB}\varphi - \varphi$ is Lipschitz on $\left]-\infty, -1\right]$. Consequently, it is uniformly continuous on $\left]-\infty, -1\right]$. That is, for every $\varepsilon > 0$, there exists $\eta_1 > 0$ such that for all $x, y \in \left]-\infty, -1\right]$, if $|x - y| \leq \eta_1$, then

$$\left\|e^{-xB}\varphi - e^{-yB}\varphi\right\|_E \leq \frac{\varepsilon}{2}.$$

On the compact interval $[-1, 0]$, and under the assumption that $\varphi \in \overline{D(B)} = \overline{D(A)}$, the continuity of the function $x \mapsto e^{-xB}\varphi - \varphi$ follows directly from Sinestrari's result. Consequently, it is uniformly continuous. That is, for every $\varepsilon > 0$, there exists $\eta_2 > 0$ such that for all $x, y \in [-1, 0]$, if $|x - y| \leq \eta_2$, then

$$\left\|e^{-xB}\varphi - e^{-yB}\varphi\right\|_E \leq \frac{\varepsilon}{2}.$$

Therefore, $x \mapsto e^{-xB}\varphi - \varphi$ is uniformly continuous on $\left]-\infty, 0\right]$. To see that, let $\varepsilon > 0$ be given, choosing $\eta = \inf(\eta_1, \eta_2)$ such that $|x - y| \leq \eta$, one obtains a uniform estimate for $\left\|e^{-xB}\varphi - e^{-yB}\varphi\right\|_E$ valid for all $x, y \in \left]-\infty, 0\right]$, for example, in the case where $x \leq -1 \leq y \leq 0$, we have $|x - y| = |(x - (-1)) - (y - (-1))|$, therefore

$$|x - (-1)| \leq |x - y| \leq \eta \leq \eta_1 \text{ and } |y - (-1)| \leq |x - y| \leq \eta \leq \eta_2,$$

then

$$\left\|e^{-xB}\varphi - e^{-yB}\varphi\right\|_E \leq \left\|e^{-yB}\varphi - e^{-B}\varphi\right\|_E + \left\|e^{-B}\varphi - e^{-xB}\varphi\right\|_E \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

However, it is clear that $\left\|e^{-xB}\varphi - \varphi\right\|_E \leq (M_0 + 1)\|\varphi\|_E$ for every $x \in \left]-\infty, 0\right]$. This shows that the function $x \mapsto e^{-xB}\varphi - \varphi$ is uniformly continuous and bounded on $\left]-\infty, 0\right]$ if $\varphi \in \overline{D(A)}$.

2) By the reiteration theorem, we identify $D_A(\alpha, +\infty) = D_B(2\alpha, +\infty)$, This implies that for every $\varphi \in D_A(\alpha, \infty)$, we have $\left\|t^{1-2\alpha}Be^{Bt}\varphi\right\| \leq \|\varphi\|_{2\alpha}$ for all $t > 0$.

Moreover, for every $x \in \left]-\infty, 0\right]$, we have

$$\left\| \left(e^{-xB}\varphi - \varphi\right) - \left(e^{-(x-h)B}\varphi - \varphi\right) \right\| \leq \int\limits_{x-h}^{x} \frac{1}{s^{1-2\alpha}} \left\| s^{1-2\alpha} B e^{-sB}\varphi \right\| ds \leq \|\varphi\|_{2\alpha} \int\limits_{x-h}^{x} \frac{ds}{s^{1-2\alpha}},$$

hence, we find

$$\left\| \left(e^{-xB}\varphi - \varphi\right) - \left(e^{-(x-h)B}\varphi - \varphi\right) \right\|_E \leq \frac{1}{2\alpha} h^{2\alpha} \|\varphi\|_{2\alpha}.$$

This implies that $x \mapsto e^{-xB}\varphi - \varphi \in BUC^{2\alpha}\left(\left]-\infty, 0\right]; E\right)$.

For points 3) and 4), we assume that $\varphi \in D(A)$. Since $-A = B^2$ so $e^{-xB}\varphi - \varphi \in D\left(B^2\right)$, and

$$B^2\left(e^{-xB}\varphi - \varphi\right) = e^{-xB}B^2\varphi - B^2\varphi = A\varphi - Ae^{-xB}\varphi.$$

It follows from point 1) that, if $A\varphi \in \overline{D(A)}$ then $x \to -Ae^{-xB}\varphi + A\varphi \in BUC\left(\left]-\infty, 0\right]; E\right)$. Moreover, by point 2), if $A\varphi \in D_A(\alpha, +\infty)$, it follows that

$$x \to -Ae^{-xB}\varphi + A\varphi \in BUC^{2\alpha}\left(\left]-\infty, 0\right]; E\right).$$

□

Lemma 4. *Let A verify (H). For each $g_+^\delta \in C^{2\alpha}\left([0, \delta]; E\right)$ with $0 < 2\alpha < 1$, we have*

1. $x \mapsto B \int\limits_0^x e^{(x-s)B}\left(g_+^\delta(s) - g_+^\delta(0)\right) ds \in C^{2\alpha}\left([0, \delta]; E\right)$,
2. $x \mapsto B \int\limits_x^\delta e^{(s-x)B}\left(g_+^\delta(s) - g_+^\delta(x)\right) ds \in C^{2\alpha}\left([0, \delta]; E\right)$.

Proof. The Hölder continuity of the first application is ensured by Sinestrari's result (see [15] for further details). For the second application, the continuity can be established by applying a similar argument to that presented in Sinestrari's proof. □

Lemma 5. *Let A verify (H). For $g_- \in BUC^{2\alpha}\left(\left]-\infty, 0\right]; E\right)$ with $0 < 2\alpha < 1$, we have*

1. $x \mapsto B \int\limits_{-\infty}^x e^{(x-s)B} g_-(s)\, ds \in BUC^{2\alpha}\left(\left]-\infty, 0\right]; E\right)$,
2. $x \mapsto B \int\limits_x^0 e^{(s-x)B}\left(g_-(s) - g_-(x)\right) ds \in BUC^{2\alpha}\left(\left]-\infty, 0\right]; E\right)$.

Proof. 1. The integral defined in the first point is well-defined. Indeed, for $x \in \left]-\infty, 0\right]$, we have

$$\left\| \int\limits_{-\infty}^x e^{(x-s)B} g_-(s)\, ds \right\|_E \leq M_0 \int\limits_{-\infty}^x e^{-a(x-s)} ds \max_{s \in \left]-\infty, 0\right]} \|g_-(s)\| \leq \frac{M_0}{a} \|g_-\|_{2\alpha}.$$

Since g_- is uniformly continuous on $\left]-\infty, 0\right]$, then for every $\varepsilon > 0$, there exists $\eta > 0$ such that for all $x, y \in]-\infty, 0]$, if $|y - x| \leq \eta$, then

$$\|g_-(y) - g_-(x)\|_E \leq \varepsilon.$$

For $x \in]-\infty, 0]$, we write

$$\begin{aligned} B\int_{-\infty}^{x} e^{(x-s)B} g_-(s)\,ds &= \int_{-\infty}^{x-1} Be^{(x-s)B}\left(g_-(s)-g_-(x)\right)\,ds + \int_{x-1}^{x} Be^{(x-s)B}\left(g_-(s)-g_-(x)\right)\,ds \\ &\quad + \int_{-\infty}^{x} Be^{(x-s)B} g_-(x)\,ds. \end{aligned}$$

For the first one, since $s \leq x-1$, we have $(x-s)^{2\alpha-1} \leq 1$, and thus we get

$$\left\| \int_{-\infty}^{x-1} Be^{(x-s)B}\left(g_-(s)-g_-(x)\right)\,ds \right\|_E \leq M_1 \int_{-\infty}^{x-1} e^{-a(x-s)} ds \left\| g_- \right\|_{2\alpha} \leq \frac{M_1}{a} \left\| g_- \right\|_{2\alpha}.$$

For the second one, we have

$$\left\| \int_{x-1}^{x} Be^{(x-s)B}\left(g_-(s)-g_-(x)\right)\,ds \right\|_E \leq M_1 \int_{x-1}^{x} (x-s)^{2\alpha-1}\,ds \left\| g_- \right\|_{2\alpha} \leq \frac{M_1}{2\alpha} \left\| g_- \right\|_{2\alpha}.$$

For the last one, it is clear that

$$\int_{-\infty}^{x} Be^{(x-s)B} g_-(x)\,ds = \lim_{r\to+\infty} B\int_{-r}^{x} e^{(x-s)B} g_-(x)\,ds = \lim_{r\to+\infty}\left(e^{(x+r)B} g_-(x) - g_-(x)\right),$$

However $\left\| e^{(x+r)B} g_-(x) \right\|_E \leq M_0 e^{-a(x+r)} \left\| g_- \right\|_{2\alpha}$, as r tends to infinity, this latter tends to 0, so

$$\int_{-\infty}^{x} Be^{(x-s)B} g_-(x) ds = -g_-(x).$$

Then $\left\| \int_{-\infty}^{x} Be^{(x-s)B} g_-(x)\,ds \right\|_E = \left\| -g_-(x) \right\|_E \leq \left\| g_- \right\|_{2\alpha}$. In summary,

$$\left\| B\int_{-\infty}^{x} e^{(x-s)B} g_-(s)\,ds \right\| \leq \left(\frac{M_1}{a} + \frac{M_1}{2\alpha} + 1 \right) \left\| g_- \right\|_{2\alpha},$$

this shows that $x \mapsto B\int_{-\infty}^{x} e^{(x-s)B} g_-(s)\,ds$ is bounded. It remains to prove that it is uniformly Hölder continuous. Let $x, y \in]-\infty, 0]$ with $x < y$, and denote $\zeta = y - x > 0$. Then, we can write

$$B\int_{-\infty}^{y} e^{(y-s)B} g_-(s)\,ds - B\int_{-\infty}^{x} e^{(x-s)B} g_-(s)\,ds = \sum_{i=1}^{5} I_i(x,y)$$

where $I_1(x,y) = B\int_{-\infty}^{x-\zeta}\left(e^{(y-s)B} - e^{(x-s)B}\right)\left(g_-(s) - g_-(x)\right)ds$,

$$I_2(x,y) = B\int_{x-\zeta}^{x}\left(e^{(y-s)B} - e^{(x-s)B}\right)\left(g_-(s) - g_-(x)\right)ds,\ I_3(x,y) = B\int_{x}^{y} e^{(y-s)B}\left(g_-(s) - g_-(y)\right)ds,$$

$$I_4(x,y) = B\int_{-\infty}^{x}\left(e^{(y-s)B} - e^{(x-s)B}\right)g_-(x)\,ds \text{ and } I_5(x,y) = B\int_{x}^{y} e^{(y-s)B} g_-(y)\,ds.$$

For the first one, we have

$$\|I_1(x,y)\|_E \leq \int_{-\infty}^{x-\zeta}\left\|Be^{(y-s)B} - Be^{(x-s)B}\right\|\|g_-(s) - g_-(x)\|_E\,ds,$$

and

$$\left\|Be^{(y-s)B} - Be^{(x-s)B}\right\| = \left\|\int_{x-s}^{y-s} B^2 e^{tB}dt\right\| \leq M_2\int_{x-s}^{y-s}\frac{dt}{t^2} \leq \frac{M_2(y-x)}{(x-s)(y-s)},$$

since $y-s>0$ and $x-s>0$ for all $s \leq x-\zeta$, it follows that

$$\begin{aligned}\|I_1(x,y)\|_E &\leq M_2\|g_-\|_{2\alpha}\int_{-\infty}^{x-\zeta}\frac{y-x}{(y-s)(x-s)}(x-s)^{2\alpha}ds\\ &\leq M_2(y-x)\|g_-\|_{2\alpha}\int_{-\infty}^{x-\zeta}(x-s)^{2\alpha-2}ds \leq \frac{M_2}{1-2\alpha}\|g_-\|_{2\alpha}(y-x)^{2\alpha}.\end{aligned}$$

For the second term, we have

$$\|I_2(x,y)\|_E \leq \int_{x-\zeta}^{x}\left(\frac{M_1}{y-s} + \frac{M_1}{x-s}\right)(x-s)^{2\alpha}ds\,\|g_-\|_{2\alpha} \leq \frac{M_1}{\alpha}\|g_-\|_{2\alpha}(y-x)^{2\alpha}.$$

As for the third one, we obtain

$$\|I_3(x,y)\| \leq M_1\int_{x}^{y}(y-s)^{2\alpha-1}ds\,\|g_-\|_{2\alpha} \leq \frac{M_1}{2\alpha}\|g_-\|_{2\alpha}(y-x)^{2\alpha}.$$

Now, we examine the sum of the fourth and fifth terms. First, we can see that

$$\begin{aligned}I_4(x,y) &= \lim_{r\to+\infty} B\int_{-r}^{x}\left(e^{(y-s)B} - e^{(x-s)B}\right)g_-(x)\,ds\\ &= \left(I - e^{(y-x)B}\right)g_-(x) + \lim_{r\to+\infty}\left(e^{(y+r)B} - e^{(x+r)B}\right)g_-(x),\end{aligned}$$

moreover

$$\left\|\left(e^{(y+r)B} - e^{(x+r)B}\right)g_-(x)\right\|_E \leq M_0\left(e^{-a(y+r)} - e^{-a(x+r)}\right)\|g_-\|_{2\alpha},$$

this latter tends to 0, as r tends to infinity, so $I_4(x,y) = \left(I - e^{(y-x)B}\right) g_-(x)$. While the fifth term can be explicitly computed and yields $I_5(x,y) = \left(e^{(y-x)B} - I\right) g_-(y)$, this implies that

$$\|I_4(x,y) + I_5(x,y)\| \leq \left\|\left(e^{(y-x)B} - I\right)\right\| \|g_-(y) - g_-(x)\| \leq (M_0 + 1)\|g_-\|_{2\alpha}(y-x)^{2\alpha}.$$

In summary, we obtain

$$\left\| B\int_{-\infty}^{y} e^{(y-s)B} g_-(s)\,ds - B\int_{-\infty}^{x} e^{(x-s)B} g_-(s)\,ds \right\| \leq C\|g_-\|_{2\alpha}(y-x)^{2\alpha},$$

where $C = \dfrac{M_2}{1-2\alpha} + \dfrac{3M_1}{2\alpha} + M_0 + 1$, this shows the desired result.

2. We now prove that $x \mapsto B\int_x^0 e^{(s-x)B}(g_-(s) - g_-(x))\,ds$ is uniformly Hölder continuous on $]-\infty, 0]$. Indeed, for $x \leq y \leq 0$, we can write

$$B\int_x^0 e^{(s-x)B}(g_-(s) - g_-(x))\,ds - B\int_y^0 e^{(s-y)B}(g_-(s) - g_-(y))\,ds = \sum_{i=1}^{3} I_i(x,y),$$

where

$$I_1(x,y) = B\int_x^y e^{(s-x)B}(g_-(s) - g_-(x))\,ds,\ I_2(x,y) = B\int_y^0 \left(e^{(s-x)B} - e^{(s-y)B}\right)(g_-(y) - g_-(x))\,ds,$$

and $I_3(x,y) = B\int_y^0 e^{(s-x)B}(g_-(s) - g_-(y))\,ds$. For the first one, we observe that

$$\|I_1(x,y)\| \leq M_1\int_x^y \frac{1}{s-x}(s-x)^{2\alpha}\,ds\,\|g_-\|_{2\alpha} \leq \frac{M_1}{2\alpha}(y-x)^{2\alpha}\|g_-\|_{2\alpha}.$$

For the second one, we obtain

$$|I_2(x,y)| \leq M_2.\|g_-\|_{2\alpha}\int_y^0\int_{s-y}^{s-x}\frac{d\tau}{\tau^2}(s-y)^{2\alpha}\,ds \leq M_2(y-x)\|g_-\|_{2\alpha}\int_y^0 \frac{(s-y)^{2\alpha-1}}{(s-x)}ds$$

$$\leq M_2(y-x)\|g_-\|_{2\alpha}\int_y^0 \frac{(s-y)^{2\alpha-1}}{(s-y+y-x)}ds,$$

using this change $s - y = t(y-x)$, it follows that

$$|I_2(x,y)| \leq M_2(y-x)^{2\alpha}\|g_-\|_{2\alpha}\int_0^{\infty}\frac{t^{2\alpha-1}}{t+1}dt,$$

this improper integral converges. For the third integral, we have

$$\|I_3(x,y)\| \leq M_1\left|\int_y^0 (s-x)^{2\alpha-1}ds\right|\|g_-\|_{2\alpha} \leq \frac{M_1}{2\alpha}(y-x)^{2\alpha}\|g_-\|_{2\alpha}.$$

Thus, the function $x \mapsto B \int\limits_{x}^{0} e^{(s-x)B} \left(g_{-}(s) - g_{-}(x)\right) ds$ is Hölder continuous on $]-\infty, 0]$ with a Hölder constant $M_1 \left(1 + \dfrac{1}{2\alpha}\right) + M_2. \int\limits_{0}^{\infty} \dfrac{t^{2\alpha-1}}{t+1} dt.$ □

Lemma 6. *Let A verify (H).*
If $g_{-} \in BUC^{2\alpha}\left(]-\infty, 0] ; E\right)$, then $x \mapsto e^{-xB}\left(g_{-}(x) - g_{-}(0)\right) \in BUC^{2\alpha}\left(]-\infty, 0] ; E\right)$.

Proof. It is clear that, for every $x \in]-\infty, 0]$, we have

$$\left\| e^{-xB}\left(g_{-}(x) - g_{-}(0)\right)\right\|_{E} \leq 2M_0 \left\|g_{-}\right\|_{2\alpha}.$$

then the function $x \to e^{-xB}\left(g_{-}(x) - g_{-}(0)\right)$ is bounded. Moreover, for $h > 0$, we write

$$\begin{aligned}
& e^{-xB}\left(g_{-}(x) - g_{-}(0)\right) - e^{-(x-h)B}\left(g_{-}(x-h) - g_{-}(0)\right) \\
& = \left(e^{-xB} - e^{-(x-h)B}\right)\left(g_{-}(x) - g_{-}(0)\right) + e^{-(x-h)B}\left(g_{-}(x) - g_{-}(x-h)\right) \\
& = B \int\limits_{x-h}^{x} e^{-sB}\left(g_{-}(x) - g_{-}(0)\right) ds + e^{-(x-h)B}\left(g_{-}(x) - g_{-}(x-h)\right).
\end{aligned}$$

where

$$\begin{aligned}
\left\| B \int\limits_{x-h}^{x} e^{-sB}\left(g_{-}(x) - g_{-}(0)\right) ds \right\| & \leq M_1 \int\limits_{x-h}^{x} \frac{ds}{-s}(-x)^{2\alpha}\left\|g_{-}\right\|_{2\alpha} \\
& \leq M_1 \int\limits_{x-h}^{x} \frac{ds}{(-s)^{1-2\alpha}}\left\|g_{-}\right\|_{2\alpha} \leq M_1 h^{2\alpha}\left\|g_{-}\right\|_{2\alpha},
\end{aligned}$$

and

$$\left\| e^{-(x-h)B}\left(g_{+}^{\delta}(x) - g_{+}^{\delta}(x-h)\right)\right\| \leq M_0.h^{2\alpha}.M\left\|g_{-}\right\|_{2\alpha}.$$

Since the Hölder constant is independent of the points x and $x+h$, the function

$$x \mapsto e^{-xB}\left(g_{-}(x) - g_{-}(0)\right)$$

is uniformly Hölder continuous and bounded on $]-\infty, 0]$. □

Lemma 7. *Let A verify (H). For each $g_{+}^{\delta} \in C^{2\alpha}\left([0, \delta] ; E\right)$ with $0 < 2\alpha < 1$, we have*

$$B \int\limits_{0}^{\delta} e^{sB}\left(g_{+}^{\delta}(s) - g_{+}^{\delta}(0)\right) ds,\ B \int\limits_{0}^{\delta} e^{(\delta-s)B}\left(g_{+}^{\delta}(s) - g_{+}^{\delta}(\delta)\right) ds \in D_B\left(2\alpha, +\infty\right).$$

Proof. To establish these results, we set

$$\chi = B \int\limits_{0}^{\delta} e^{sB}\left(g_{+}^{\delta}(s) - g_{+}^{\delta}(0)\right) ds \text{ or } \chi = B \int\limits_{0}^{\delta} e^{(\delta-s)B}\left(g_{+}^{\delta}(s) - g_{+}^{\delta}(\delta)\right) ds,$$

and recall that $\|\chi\|_{D_B(2\alpha,+\infty)} := \|\chi\|_{2\alpha} = \|\chi\| + [\chi]_{2\alpha}$ where $[\chi]_{2\alpha} = \sup_{t>0} \left\|t^{1-2\alpha} B e^{Bt}\chi\right\| < \infty$. Indeed, for the first case, we have

$$\left\|\int_0^\delta t^{1-2\alpha} B^2 e^{(t+s)B}\left(g_+^\delta(s) - g_+^\delta(0)\right) ds\right\| \leq M_2 \left\|\int_0^\delta \frac{t^{1-2\alpha} s^{2\alpha}}{(t+s)^2} ds\right\| \left\|g_+^\delta\right\|_{2\alpha}.$$

Using the change of variable $s = t\tau$, we obtain

$$\left\|\int_0^\delta t^{1-2\alpha} B^2 e^{(t+s)B}\left(g_+^\delta(s) - g_+^\delta(0)\right) ds\right\| \leq M_2 \left\|g_+^\delta\right\|_{2\alpha} \int_0^\infty \frac{\tau^{2\alpha}}{(1+\tau)^2} d\tau < \infty.$$

Similarly, for the second case, we obtain

$$\begin{aligned}\left\|\int_0^\delta t^{1-2\alpha} B^2 e^{(t+\delta-s)B}\left(g_+^\delta(s) - g_+^\delta(\delta)\right) ds\right\| &\leq M_2 \left\|\int_0^\delta \frac{t^{1-2\alpha} (\delta-s)^{2\alpha}}{(t+\delta-s)^2} ds\right\| \left\|g_+^\delta\right\|_{2\alpha} \\ &\leq M_2 \left\|g_+^\delta\right\|_{2\alpha} \int_0^{+\infty} \frac{\tau^{2\alpha}}{(1+\tau)^2} d\tau.\end{aligned}$$

Since the integral $\int_0^{+\infty} \frac{\tau^{2\alpha}}{(1+\tau)^2} d\tau$ converges, thus $B \int_0^\delta e^{(\delta-s)B}\left(g_+^\delta(s) - g_+^\delta(\delta)\right) ds \in D_B(2\alpha, +\infty)$. □

Lemma 8. *Let A verify (H). For $g_- \in BUC^{2\alpha}(]-\infty, 0]; E)$ with $0 < 2\alpha < 1$, we have*

$$B \int_{-\infty}^0 e^{-sB}(g_-(s) - g_-(0))\, ds \in D_B(2\alpha, \infty)$$

Proof. To begin, we split

$$B \int_{-\infty}^0 e^{-sB}(g_-(s) - g_-(0))\, ds = B \int_{-\infty}^{-1} e^{-sB}(g_-(s) - g_-(s))\, ds + B \int_{-1}^0 e^{-sB}(g_-(s) - g_-(0))\, ds.$$

For the first, we have

$$B \int_{-\infty}^{-1} e^{-sB}(g_-(s) - g_-(0))\, ds \in D(B^k) \subset D_B(2\alpha, \infty)$$

for $k \in \mathbb{N} \setminus \{0\}$, since

$$\begin{aligned}\left\|B^{k+1} \int_{-\infty}^{-1} e^{-sB}(g_-(s) - g_-(0))\, ds\right\| &\leq M_{k+1} \int_{-\infty}^{-1} \frac{(-s)^{2\alpha}}{(-s)^{k+1}} ds \, \|g_-\|_{2\alpha} \\ &\leq M_{k+1} \int_1^\infty \frac{ds}{s^{k+1-2\alpha}} \|g_-\|_{2\alpha} < \infty\end{aligned}$$

For the second integral, we have

$$\begin{aligned}\left[B\int_{-1}^{0}e^{-sB}\left(g_{-}(s)-g_{-}(0)\right)ds\right]_{2\alpha} &= \left\|\int_{-1}^{0}t^{1-2\alpha}B^{2}e^{(t-s)B}\left(g_{-}(s)-g_{-}(0)\right)ds\right\| \\ &\leq M_{2}\left\|\int_{-1}^{0}\frac{t^{1-2\alpha}\left(-s\right)^{2\alpha}}{\left(t-s\right)^{2}}ds\right\|\left\|g_{-}\right\|_{2\alpha},\end{aligned}$$

Letting $-s=t\tau$, it follows that

$$\int_{-1}^{0}\frac{t^{1-2\alpha}\left(-s\right)^{2\alpha}}{\left(t-s\right)^{2}}ds=-\int_{\frac{1}{t}}^{0}\frac{t^{1-2\alpha}\left(t\tau\right)^{2\alpha}}{\left(t+t\tau\right)^{2}}td\tau=\int_{0}^{\frac{1}{t}}\frac{\tau^{2\alpha}}{\left(1+\tau\right)^{2}}d\tau<\infty.$$

Then $B\int_{-\infty}^{0}e^{-sB}\left(g_{-}(s)-g_{-}(0)\right)ds\in D_{B}\left(2\alpha,\infty\right)$. □

3. Explicit Solution Representation

3.1. **Auxiliary Problems.** Our approach consists of solving the following auxiliary problems by introducing two auxiliary elements ψ and φ, which belong to E. The first problem is posed on $(0,\delta)$

$$\begin{cases} u_{+}''(x)+Au_{+}(x)=g_{+}^{\delta}(x), \\ u_{+}'(\delta)=f_{+}^{\delta},\;\; u_{+}(0)=\psi, \end{cases} \tag{P_{+}}$$

and the second problem is posed on the negative half-line

$$\begin{cases} u_{-}''(x)+Au_{-}(x)=g_{-}(x), \\ u_{-}'(0)=\varphi,\;\; u_{-}(-\infty)=f_{-}. \end{cases} \tag{P_{-}}$$

Using Krein's method [16], we obtain for $x\in\,]0,\delta[$

$$u_{+}(x)=e^{xB}\zeta_{0}+e^{(\delta-x)B}\zeta_{1}+\frac{1}{2}B^{-1}\int_{0}^{x}e^{(x-s)B}g_{+}^{\delta}(s)ds+\frac{1}{2}B^{-1}\int_{x}^{\delta}e^{(s-x)B}g_{+}^{\delta}(s)ds, \tag{1}$$

where ζ_0 and ζ_1 are arbitrary constants in E. By using the boundary condition and the fact that the operator $(I+e^{2\delta B})$ is invertible (see Lunardi [12, Proposition 2.3.6, p. 60]), we obtain

$$\zeta_{0}=(I+e^{2\delta B})^{-1}\left[\psi+B^{-1}e^{\delta B}f_{+}^{\delta}-\frac{1}{2}B^{-1}\int_{0}^{\delta}\left(I+e^{2(\delta-s)B}\right)e^{sB}g_{+}^{\delta}(s)ds,\right] \tag{2}$$

and

$$\zeta_{1}=(I+e^{2\delta B})^{-1}\left[e^{\delta B}\psi-B^{-1}f_{+}^{\delta}+\frac{1}{2}B^{-1}\int_{0}^{\delta}\left(I-e^{2sB}\right)e^{(\delta-s)B}g_{+}^{\delta}(s)ds.\right] \tag{3}$$

For the second auxiliary problem, the solution is obtained using the following representation found in [10], under the condition that $u_-(-\infty) = g_-(-\infty) = 0$, which is defined by

$$u_-(x) = e^{-xB}\zeta_2 + \frac{1}{2}B^{-1}\int_{-\infty}^{x} e^{(x-s)B}g_-(s)ds + \frac{1}{2}B^{-1}\int_{x}^{0} e^{(s-x)B}g_-(s)ds,$$

where $\zeta_2 = -B^{-1}\varphi + \frac{1}{2}B^{-1}\int_{-\infty}^{0} e^{-sB}g_-(s)ds$. To address the situation where $u_-(-\infty) = f_- \neq 0$, we define the function $v_-(x) = u_-(x) - f_-$. Consequently, the new function v_- fulfills the following problem for $f_- \in D(A)$:

$$\begin{cases} v_-''(x) + Av_-(x) = -Af_- + g_-(x), \\ v_-'(0) = \varphi, \ \lim\limits_{x\to-\infty} v_-(x) = 0. \end{cases} \qquad (\widetilde{P})$$

Consequently, we adapt the representation to the new function v_- and obtain

$$u_-(x) = f_- + \frac{1}{2}B\int_{-\infty}^{x} e^{-\tau B}f_- d\tau + e^{-xB}\zeta_2 + \frac{1}{2}B^{-1}\int_{x}^{0} e^{(s-x)B}g_-(s)ds + \frac{1}{2}B^{-1}\int_{-\infty}^{x} e^{(x-s)B}g_-(s)ds$$

where

$$e^{-xB}\zeta_2 = -B^{-1}e^{-xB}\varphi + \frac{1}{2}\int_{-\infty}^{0} e^{-(s+x)B}Bf_- ds + \frac{1}{2}B^{-1}\int_{-\infty}^{0} e^{-(x+s)B}g_-(s)ds.$$

On the other hand, using the following relations

$$B\int_{-\infty}^{x} e^{-sB}f_- ds = -e^{-xB}f_- \text{ and } B\int_{-\infty}^{0} e^{-sB}f_- ds = -f_- \text{ if } f_- \in \overline{D(B)},$$

simplifies the expressions of u_- and ξ_2, leading to

$$u_-(x) = f_- - e^{-xB}f_- + e^{-xB}\xi_2 + \frac{1}{2}B^{-1}\int_{x}^{0} e^{(s-x)B}g_-(s)ds + \frac{1}{2}B^{-1}\int_{-\infty}^{x} e^{(x-s)B}g_-(s)ds, \qquad (4)$$

and

$$\xi_2 = -B^{-1}\varphi + \frac{1}{2}B^{-1}\int_{-\infty}^{0} e^{-sB}g_-(s)ds \qquad (5)$$

Remark 2. *The condition at infinity is satisfied because*

$$\begin{aligned} \|u_-(x) - f_-\| \ &\leq \ \left\|e^{-xB}(\zeta_2 - f_-)\right\| + \frac{1}{2}\left\|B^{-1}\int_{-\infty}^{x} e^{(x-s)B}g_-(s)\,ds\right\| \\ &+ \frac{1}{2}\left\|B^{-1}\int_{x}^{0} e^{(s-x)B}g_-(s)\,ds\right\|. \end{aligned}$$

However, there exists $a > 0$ and $M_0 > 0$ such that

$$\left\|e^{-xB}(\zeta_2 - f)\right\| \leq M_0 e^{ax}\|\zeta_2 - f_-\|,$$

and

$$\begin{aligned}\left\| B^{-1}\int_{-\infty}^{x} e^{(x-s)B} g_-(s)ds \right\| &\leq M_0 \left\| B^{-1} \right\| \left(\int_{-\infty}^{x} e^{-a(x-s)} ds \right) \sup_{s\leq x} \| g_-(s) \| \\ &\leq \frac{M_0}{a} \left\| B^{-1} \right\| \sup_{s\leq x} \| g_-(s) \| ,\end{aligned}$$

therefore, as x tends to $-\infty$, it follows that $\lim_{x\to-\infty} \left\| e^{-xB} (\zeta_2 - f) \right\| = 0$ *and* $\lim_{x\to-\infty} \sup_{s\leq x} \| g_-(s) \| = 0$ *(This follows because the function g_- is uniformly Hölder continuous and bounded on* $]-\infty, 0]$ *, and its limit goes to zero as x tends to $-\infty$) therefore*

$$\lim_{x\to-\infty} \left\| B^{-1}\int_{-\infty}^{x} e^{(x-s)B} g_-(s)ds \right\| = 0.$$

For the third term, we decompose it as follows

$$\frac{1}{2}B^{-1}\int_{x}^{\frac{x}{2}} e^{(s-x)B} g_-(s)ds + \frac{1}{2}B^{-1}\int_{\frac{x}{2}}^{0} e^{(s-x)B} g_-(s)ds := I_1(x) + I_2(x)$$

where

$$\| I_1(x) \| \leq \frac{M_0}{2} \left\| B^{-1} \right\| \left(\int_{x}^{\frac{x}{2}} e^{-a(s-x)} ds \right) \sup_{s\in\left[x,\frac{x}{2}\right]} \| g_-(s) \| \leq \frac{M_0}{2a} \left\| B^{-1} \right\| \left(e^{a\frac{x}{2}} - 1 \right) \sup_{s\in\left[x,\frac{x}{2}\right]} \| g_-(s) \| ,$$

and

$$\begin{aligned}\| I_2(x) \| &\leq \left\| \frac{1}{2}B^{-1}\int_{\frac{x}{2}}^{0} e^{(s-x)B} g_-(s)ds \right\| \leq \frac{M_0}{2} \left\| B^{-1} \right\| \| g_- \|_{2\alpha} \int_{\frac{x}{2}}^{0} e^{-a(s-x)} ds \\ &\leq C \| g_- \|_{2\alpha} \left[e^{ax} - e^{a\frac{x}{2}} \right].\end{aligned}$$

Since $g_-(-\infty) = 0$, the limit of $\sup_{s\in\left[x,\frac{x}{2}\right]} \| g_-(s) \|$ *tends to zero when $x \to -\infty$. This implies that* $\lim_{x\to-\infty} \| I_1(x) \| = 0$, *and the term* $\left(e^{ax} - e^{a\frac{x}{2}} \right)$ *tends to 0 as $x \to -\infty$, so* $\lim_{x\to-\infty} \| I_2(x) \| = 0$. *As a result* $\lim_{x\to-\infty} \| u_-(x) - f_- \| = 0$.

3.2. **Solution Formula.** Using the expressions of u_- and u_+, given in $(4-5)$ and $(1-2-3)$ along with the following transmission conditions

$$u_-(0) = \psi \text{ and } \varphi = \mu u_+'(0) \text{ where } \mu = \frac{\mu_+}{\mu_-},$$

we deduce the following system

$$\begin{cases} B^{-1}\varphi + \psi = f_- + \int\limits_{-\infty}^{0} e^{-sB} B f_- ds + B^{-1} \int\limits_{-\infty}^{0} e^{-sB} g_-(s) ds \\ \varphi - \mu B (I + e^{2\delta B})^{-1} (I - e^{2\delta B}) \psi = b_1 \end{cases} \tag{S}$$

where

$$b_1 = 2\mu(I+e^{2\delta B})^{-1}e^{\delta B}f_+^\delta - \mu(I+e^{2\delta B})^{-1}\int_0^\delta \left(e^{sB}+e^{(2\delta-s)B}\right)g_+^\delta(s)\,ds.$$

The determinant of this system is of the form

$$(I+e^{2\delta B})^{-1}\left(I+e^{2\delta B}+\mu(I-e^{2\delta B})\right) := (I+e^{2\delta B})^{-1}D_\mu$$

which is invertible according to [10, Lemma 7.]. After solving system (S), we find

$$\psi = -2\mu D_\mu^{-1}B^{-1}e^{\delta B}f_+^\delta + D_\mu^{-1}(I+e^{2\delta B})f_- + D_\mu^{-1}(I+e^{2\delta B})\int_{-\infty}^0 e^{-sB}Bf_-ds$$
$$+\mu D_\mu^{-1}B^{-1}\int_0^\delta \left(e^{sB}+e^{(2\delta-s)B}\right)g_+^\delta(s)ds + D_\mu^{-1}\left(I+e^{2\delta B}\right)B^{-1}\int_{-\infty}^0 e^{-sB}g_-(s)ds.$$

and

$$\varphi = 2\mu D_\mu^{-1}e^{\delta B}f_+^\delta + \mu D_\mu^{-1}B(I-e^{2\delta B})f_- + \mu D_\mu^{-1}B(I-e^{2\delta B})\int_{-\infty}^0 e^{-sB}Bf_-ds$$
$$-\mu D_\mu^{-1}\int_0^\delta \left(e^{sB}+e^{(2\delta-s)B}\right)g_+^\delta(s)ds + \mu D_\mu^{-1}\left(I-e^{2\delta B}\right)\int_{-\infty}^0 e^{-sB}g_-(s)ds.$$

Substituting these expressions into ζ_0, ζ_1 and ξ_2, and simplifying, we obtain

$$\zeta_0 = (1-\mu)\,D_\mu^{-1}B^{-1}e^{\delta B}f_+^\delta - \frac{1-\mu}{2}D_\mu^{-1}B^{-1}\int_0^\delta e^{sB}g_+^\delta(s)ds$$
$$-\frac{1-\mu}{2}D_\mu^{-1}B^{-1}\int_0^\delta e^{(2\delta-s)B}g_+^\delta(s)ds + D_\mu^{-1}B^{-1}\int_{-\infty}^0 e^{-sB}g_-(s)ds,$$

$$\zeta_1 = -\,(1+\mu)\,B^{-1}D_\mu^{-1}f_+^\delta - \frac{(1-\mu)}{2}e^{\delta B}D_\mu^{-1}B^{-1}\int_0^\delta e^{sB}g_+^\delta(s)ds$$
$$+\frac{1+\mu}{2}D_\mu^{-1}B^{-1}\int_0^\delta e^{(\delta-s)B}g_+^\delta(s)ds + e^{\delta B}D_\mu^{-1}B^{-1}\int_{-\infty}^0 e^{-sB}g_-(s)ds,$$

and

$$\xi_2 = -2\mu B^{-1}D_\mu^{-1}e^{\delta B}f_+^\delta + \mu B^{-1}D_\mu^{-1}\int_0^\delta \left(e^{sB}+e^{(2\delta-s)B}\right)g_+^\delta(s)ds$$
$$+\frac{1}{2}D_\mu^{-1}\left[I+e^{2\delta B}-\mu(I-e^{2\delta B})\right]B^{-1}\int_{-\infty}^0 e^{-sB}g_-(s)ds.$$

4. Proof of Theorems

To establish our theorems through the technical lemmas, we represent the solution as follows

$$u_-(x) = \left(I - e^{-xB}\right) f_- - \frac{1}{2}B^{-2}e^{-xB}g_-(x) + \frac{1}{2}B^{-2}e^{-xB}\left(g_-(x) - g_-(0)\right)$$
$$+ e^{-xB}\xi_2^* + \frac{1}{2}B^{-1}\int_{-\infty}^{x} e^{(x-s)B}g_-(s)ds + \frac{1}{2}B^{-1}\int_{x}^{0} e^{(s-x)B}\left(g_-(s) - g_-(x)\right)ds,$$

for $x \in \left]-\infty, 0\right]$ and

$$u_+(x) = e^{xB}\zeta_0^* + e^{(\delta-x)B}\zeta_1^* + \frac{1}{2}B^{-2}e^{(\delta-x)B}\left(g_+^\delta(x) - g_+^\delta(\delta)\right) - \frac{1}{2}B^{-2}\left(g_+^\delta(x) - g_+^\delta(0)\right)$$
$$+ \frac{1}{2}B^{-1}\int_{0}^{x} e^{(x-s)B}\left(g_+^\delta(s) - g_+^\delta(0)\right)ds + \frac{1}{2}B^{-1}\int_{x}^{\delta} e^{(s-x)B}\left(g_+^\delta(s) - g_+^\delta(x)\right)ds,$$

for $x \in [0, \delta]$, where

$$\zeta_0^* = \zeta_0 + \frac{1}{2}B^{-2}g_+^\delta(0),\ \zeta_1^* = \zeta_1 + \frac{1}{2}B^{-2}g_+^\delta(\delta) \text{ and } \zeta_2^* = \xi_2 + \frac{1}{2}B^{-2}g_-(0).$$

Hence, after simplification, we find:

$$\zeta_0^* = (1-\mu)\,D_\mu^{-1}B^{-1}e^{\delta B}f_+^\delta - \frac{1-\mu}{2}D_\mu^{-1}B^{-2}\left(e^{\delta B} - I\right)e^{\delta B}\left(g_+^\delta(\delta) - g_+^\delta(0)\right)$$
$$+ D_\mu^{-1}B^{-2}\left(g_+^\delta(0) - g_-(0)\right) + D_\mu^{-1}B^{-1}\int_{-\infty}^{0} e^{-sB}\left(g_-(s) - g_-(0)\right)ds$$
$$- \frac{1-\mu}{2}D_\mu^{-1}B^{-1}\int_{0}^{\delta}\left(e^{sB}\left(g_+^\delta(s) - g_+^\delta(0)\right) + e^{(2\delta-s)B}\left(g_+^\delta(s) - g_+^\delta(\delta)\right)\right)ds,$$

$$\zeta_1^* = -\,(1+\mu)\,B^{-1}D_\mu^{-1}f_+^\delta + e^{\delta B}D_\mu^{-1}B^{-1}\int_{-\infty}^{0} e^{-sB}\left(g_-(s) - g_-(0)\right)ds$$
$$+ \frac{1+\mu}{2}D_\mu^{-1}B^{-1}\int_{0}^{\delta} e^{(\delta-s)B}\left(g_+^\delta(s) - g_+^\delta(\delta)\right)ds - \frac{1-\mu}{2}e^{\delta B}D_\mu^{-1}B^{-1}\int_{0}^{\delta} e^{sB}\left(g_+^\delta(s) - g_+^\delta(0)\right)ds$$
$$+ e^{\delta B}B^{-2}D_\mu^{-1}\left(g_+^\delta(0) - g_-(0)\right) + \frac{1}{2}B^{-2}D_\mu^{-1}\left((1+\mu) + (I-\mu)\,e^{\delta B}\right)e^{\delta B}\left(g_+^\delta(\delta) - g_+^\delta(0)\right),$$

and

$$\xi_2^* = -2\mu B^{-1}D_\mu^{-1}e^{\delta B}f_+^\delta + \frac{1}{2}D_\mu^{-1}\left(I + e^{2\delta B} - \mu(I - e^{2\delta B})\right)B^{-1}\int_{-\infty}^{0} e^{-sB}\left(g_-(s) - g_-(0)\right)ds$$
$$+ \mu D_\mu^{-1}B^{-1}\int_{0}^{\delta} e^{(2\delta-s)B}\left(g_+^\delta(s) - g_+^\delta(\delta)\right)ds + \mu D_\mu^{-1}B^{-1}\int_{0}^{\delta} e^{sB}\left(g_+^\delta(s) - g_+^\delta(0)\right)ds$$
$$+ \mu B^{-2}D_\mu^{-1}\left(e^{2\delta B} - I\right)\left(g_+^\delta(0) - g_-(0)\right) + \mu D_\mu^{-1}e^{\delta B}B^{-2}\left(I - e^{\delta B}\right)\left(g_+^\delta(\delta) - g_+^\delta(0)\right).$$

By applying Lemmas 2, 4, 6 and 8, we conclude that:

- $x \mapsto e^{-xB}\left(g_{-}(x)-g_{-}(0)\right)+B\int_{x}^{0} e^{(s-x)B}\left(g_{-}(s)-g_{-}(x)\right)ds+B\int_{-\infty}^{x} e^{(x-s)B}g_{-}(s)ds$ belongs to $BUC^{2\alpha}\left(]-\infty,0];E\right)$.
- $x \mapsto B\int_{0}^{x} e^{(x-s)B}\left(g_{+}^{\delta}(s)-g_{+}^{\delta}(0)\right)ds+B\int_{x}^{\delta} e^{(s-x)B}\left(g_{+}^{\delta}(s)-g_{+}^{\delta}(x)\right)ds$ and $x \mapsto \left(e^{(\delta-x)B}-I\right)\left(g_{+}^{\delta}(x)-g_{+}^{\delta}(\delta)\right)$ belong to $C^{2\alpha}\left([0,\delta];E\right)$.

Moreover, by applying Lemmas 6, 7 and 8, the applications

- $x \mapsto e^{-xB}B\int_{0}^{\delta} e^{sB}\left(g_{+}^{\delta}(s)-g_{+}^{\delta}(0)\right)ds$, $x \mapsto e^{-xB}B\int_{0}^{\delta} e^{(\delta-s)B}\left(g_{+}^{\delta}(s)-g_{+}^{\delta}(\delta)\right)ds$,
- $x \mapsto e^{-xB}B\int_{-\infty}^{0} e^{-sB}\left(g_{-}(s)-g_{-}(0)\right)ds$

are bounded and uniformly Hölder continuous with exponent 2α on $]-\infty,0]$.

Similarly, we can deduce from Lemma 2 that the following functions

$$x \mapsto e^{xB}B\int_{0}^{\delta} e^{sB}\left(g_{+}^{\delta}(s)-g_{+}^{\delta}(0)\right)ds,\;\; x \mapsto e^{xB}B\int_{0}^{\delta} e^{(\delta-s)B}\left(g_{+}^{\delta}(s)-g_{+}^{\delta}(\delta)\right)ds,$$

$$x \mapsto e^{xB}B\int_{-\infty}^{0} e^{-sB}\left(g_{-}(s)-g_{-}(0)\right)ds,\;\; x \mapsto e^{(\delta-x)B}B\int_{0}^{\delta} e^{sB}\left(g_{+}^{\delta}(s)-g_{+}^{\delta}(0)\right)ds,$$

$$x \mapsto e^{(\delta-x)B}B\int_{0}^{\delta} e^{(\delta-s)B}\left(g_{+}^{\delta}(s)-g_{+}^{\delta}(\delta)\right)ds,\;\; x \mapsto e^{(\delta-x)B}B\int_{-\infty}^{0} e^{-sB}\left(g_{-}(s)-g_{-}(0)\right)ds$$

belong to $C^{2\alpha}\left([0,\delta];E\right)$.

Moreover, for $\delta>0$, we know that $e^{\delta B}\chi,\; Be^{\delta B}\chi \in D\left(B^{k}\right) \subset D_{B}\left(2\alpha,\infty\right)$ for all $\chi \in E$, hence the following functions

$$x \mapsto e^{xB}Be^{\delta B}f_{+}^{\delta},\; x \mapsto e^{(\delta-x)B}e^{\delta B}\left(g_{+}^{\delta}(0)-g_{-}(0)\right),\; x \mapsto e^{xB}e^{\delta B}\left(g_{+}^{\delta}(\delta)-g_{+}^{\delta}(0)\right)$$

and $x \mapsto e^{(\delta-x)B}e^{\delta B}\left(g_{+}^{\delta}(\delta)-g_{+}^{\delta}(0)\right)$ belong to $C^{2\alpha}\left([0,\delta];E\right)$.

Similarly, the following functions

$$x \mapsto e^{\delta B}\left(I-e^{\delta B}\right)\left(g_{+}^{\delta}(\delta)-g_{+}^{\delta}(0)\right),\; x \mapsto Be^{-xB}e^{\delta B}f_{+}^{\delta}$$

and $x \mapsto e^{-xB}e^{2\delta B}\left(g_{+}^{\delta}(0)-g_{-}(0)\right)$ belong to $BUC^{2\alpha}\left(]-\infty,0];E\right)$.

Consequently, and by using this notation $g \simeq_{2\alpha} h$ if and only if $g-h \in C^{2\alpha}$, we obtain

$$Au_{-}(x) \simeq_{2\alpha} A\left(f_{-}-e^{-xB}f_{-}\right)+\mu D_{\mu}^{-1}e^{-xB}\left(g_{+}^{\delta}(0)-g_{-}(0)\right),$$

$$Au_{+}(x) \simeq_{2\alpha} \left(1+\mu\right)D_{\mu}^{-1}Be^{(\delta-x)B}f_{+}^{\delta}-e^{xB}D_{\mu}^{-1}\left(g_{+}^{\delta}(0)-g_{-}(0)\right).$$

On the other hand, for $f_{-} \in D\left(A\right)$, according to Lemma 3, it follows that the functions

$$x \mapsto A\left(f_{-}-e^{-xB}f_{-}\right),\; x \mapsto e^{-xB}\left(g_{+}^{\delta}(0)-g_{-}(0)\right)$$

are uniformly bounded and continuous on $]-\infty,0]$ if $Af_-,\ g_+^\delta(0)-g_-(0)\in\overline{D(A)}$, hence $Au_-\in BUC\left(]-\infty,0];E\right)$, and from the equation $Au_-+u_-''=g_-$, we deduce that

$$u_-\in BUC^2\left(]-\infty,0];E\right)\cap BUC\left(]-\infty,0];D(A)\right),$$

while

$$x\mapsto A\left(f_--e^{-xB}f_-\right),\ x\mapsto e^{-xB}\left(g_+^\delta(0)-g_-(0)\right)\in BUC^{2\alpha}\left(]-\infty,0];E\right)$$

if $Af_-,\ g_+^\delta(0)-g_-(0)\in D_A(\alpha,\infty)$. Thus Au_-,u_-'' belong to $BUC^{2\alpha}\left(]-\infty,0];E\right)$.

Similarly, by applying Lemma 1 for $f_+^\delta\in D(B)$, we obtain $u_+\in C^2\left([0,\delta];E\right)\cap C\left([0,\delta];D(A)\right)$ if $Bf_+^\delta\in\overline{D(A)}$, and $Au_+,u_+''\in C^{2\alpha}\left([0,\delta];E\right)$ if $Bf_+^\delta\in D_A(\alpha,\infty)$.

5. Applications

Example 1. *Let $E=C([0,1])$. We define the operator A on this space by*

$$\begin{cases} D(A)=\left\{\psi\in C^2([0,1]):\psi(0)=\psi(1)=0\right\}:=C_0^1([0,1])\cap C^2([0,1]) \\ A\psi=\psi'', \end{cases}$$

This operator satisfies Hypothesis (H) through a direct computation of its resolvent. Furthermore, we have

$$\overline{D(A)}=\{\psi\in C([0,1]):\psi(0)=\psi(1)=0\}:=C_0([0,1])$$

which indicates that the domain $\overline{D(A)}$ is not dense in $E=C([0,1])$, and the interpolation space $D_A(\alpha,+\infty)$ is characterized by

$$D_A(\alpha,+\infty)=\left\{\psi\in C^{2\alpha}([0,1]):\psi(0)=\psi(1)=0\right\}:=C_0^{2\alpha}([0,1]).$$

In the continuous framework $E=C([0,1])$, it is difficult to determine the square root of $-A$ precisely. Therefore, we consider $f_+^\delta=0$.

In the following proposition, we present the main regularity result concerning this problem

$$\begin{cases} (e.q)\begin{cases} \Delta u_+(x,y)=g_+^\delta(x,\theta) & in\ ]0,\delta[\times]0,1[\\ \Delta u_-(x,y)=g_-(x,\theta) & in\ ]-\infty,0[\times]0,1[\end{cases} \\ (t.c)\left\{u_-(0,y)=u_+(0,y),\ \mu_-\dfrac{\partial u_-}{\partial x}(0,y)=\mu_+\dfrac{\partial u_+}{\partial x}(0,y),\ for\ y\in]0,1[\right. \\ (b.c)\begin{cases} \dfrac{\partial u_+}{\partial x}(\delta,y)=f_+^\delta(y) & for\ \ y\in]0,1[\\ u(x,0)=u(x,1)=0 & for\ \ x\in]-\infty,\delta[. \end{cases} \end{cases} \tag{6}$$

as immediate consequences of Theorems 1 and 2.

Proposition 1. *Let $g_+^\delta\in C^{2\alpha}([0,1],C([0,1]))$ and*

$$g_-\in BUC^{2\alpha}\left(]-\infty,0];C([0,1])\right)\ with\ g_-(-\infty,y)=0.$$

For $f_- \in C_0^1([0,1]) \cap C^2([0,1])$, *we have*

1) *The problem* (6) *has a unique strict solution i.e.,*

$$u_- \in BUC^2(]-\infty,0]; C([0,1])) \cap BUC\left(]-\infty,0]; C_0^1([0,1]) \cap C^2([0,1])\right),$$

$$u_+ \in C^2(0,\delta; C([0,1])) \cap C\left(0,\delta; C_0^1([0,1]) \cap C^2([0,1])\right),$$

if $\frac{\partial^2}{\partial y^2} f_-, g_+^\delta(0,.) - g_-(0,.) \in C_0([0,1])$.

2) *The strict solution satisfies this property of maximal regularity*

$$\frac{\partial^2 u_+}{\partial x^2}, \frac{\partial^2 u_+}{\partial y^2} \in C^{2\alpha}([0,\delta]; C([0,1])) \text{ and } \frac{\partial^2 u_-}{\partial x^2}, \frac{\partial^2 u_-}{\partial y^2} \in BUC^{2\alpha}(]-\infty,0]; C([0,1]))$$

if $\frac{\partial^2}{\partial y^2} f_-, g_+^\delta(0,.) - g_-(0,.) \in C_0^{2\alpha}([0,1])$.

Example 2. *Let us set for some* $0 < 2\alpha < 1$ *that*

$$E = C_0^{2\alpha}(0,1) = \left\{\psi \in C^{2\alpha}(0,1) : \psi(0) = 0\right\}$$

and

$$\begin{cases} D(B) = \left\{\psi, \psi' \in C^{1+2\alpha}(0,1) : \psi(0) = \psi'(0) = 0\right\} := C_0^{1+2\alpha}(0,1) \\ B\psi = -\psi'. \end{cases}$$

Since $A = -B^2$, *then*

$$\begin{cases} A\psi = -\psi'' \\ D(A) = \left\{\psi \in C^{2+2\alpha}(0,1) : \psi(0) = \psi'(0) = \psi''(0) = 0\right\} := C_0^{2+2\alpha}(0,1) \end{cases}$$

Furthermore, we have $\overline{D(A)} = \overline{D(B)} = h_0^{2\alpha}(0,1) \neq E$, *where*

$$h^{2\alpha}(0,1) = \left\{ f : [0,1] \to \mathbb{C} : \lim_{\epsilon \to 0} \sup_{\substack{x,y \in [0,1] \\ x \neq y}} \frac{|f(x) - f(y)|}{|x-y|^{2\alpha}} = 0 \right\},$$

with $\|f\|_{h^{2\alpha}(0,1)} = \|f\|_{\|f\|_{C^{2\alpha}(0,1)}}$. *The Hypothesis* (H) *holds through a direct computation of its resolvent (see* [4, *Example 14.2, p. 318*]*). Using* [11, *Example 1.25, p. 37 and Example 5.15, p. 142*] *or* [12, *Corollary 1.2.19, p. 32*], *we can characterize the real interpolation spaces* $D_B(2\alpha, +\infty)$ *as*

$$\left(C_0^{1+2\alpha}(0,1), C_0^{2\alpha}(0,1)\right)_{1-2\alpha,\infty} = \left(C_0^{2\alpha}(0,1), C_0^{1,2\alpha}(0,1)\right)_{2\alpha,\infty} = C_0^{4\alpha}(0,1),$$

As immediate consequences of Theorems 1 and 2, the main regularity result is given in the following proposition

Proposition 2. *Let* $g_+^\delta \in C^{2\alpha}\left([0,\omega]; C_0^{2\alpha}(0,1)\right)$,

$$g_- \in BUC^{2\alpha}\left(]-\infty,0]; C_0^{2\alpha}(0,1)\right) \text{ with } g_-(-\infty, y) = 0.$$

For $f_- \in C_0^{2+2\alpha}(0,1)$ *and* $f_+^\delta \in C_0^{1+2\alpha}(0,1)$, *we have*

1) *The problem*

$$\begin{cases} \dfrac{\partial^2 u_\pm}{\partial x^2}(x,y) - \dfrac{\partial^2 u_\pm}{\partial y^2}(x,y) = g_\pm^\delta(x,y) \text{ in } (]-\infty,0[\cup]0,\delta[) \times]0,1[\\ \dfrac{\partial u_+}{\partial x}(\delta,y) = f_+^\delta(y) \text{ and } \lim\limits_{x\to-\infty} u_-(x,y) = f_-(y) \quad y \in]0,1[\\ u_\pm(x,0) = \dfrac{\partial u_\pm}{\partial y}(x,0) = \dfrac{\partial^2 u_\pm}{\partial y^2}(x,0) = 0 \quad x \in]-\infty,\delta] \end{cases}$$

has a unique strict solution i.e.,

$$\begin{cases} u_- \in BUC^2\left(-\infty,0; C_0^{2\alpha}(0,1)\right) \cap BUC\left(-\infty,0; C_0^{2+2\alpha}(0,1)\right), \\ u_+ \in C^2\left(0,\delta; C_0^{2\alpha}(0,1)\right) \cap C\left(0,\delta; C_0^{2+2\alpha}(0,1)\right), \end{cases}$$

if $\dfrac{\partial}{\partial y} f_+^\delta, \dfrac{\partial^2}{\partial y^2} f_-, g_+^\delta(0,.) - g_-(0,.) \in h_0^{2\alpha}(0,1)$.

2) *The strict solution satisfies this property of maximal regularity*

$$\begin{cases} \dfrac{\partial^2}{\partial x^2} u_+, \dfrac{\partial^2}{\partial \theta^2} u_+ \in C^{2\alpha}\left([0,\delta]; C_0^{2\alpha}(0,1)\right) \\ \dfrac{\partial^2}{\partial x^2} u_-, \dfrac{\partial^2}{\partial \theta^2} u_- \in BUC^{2\alpha}\left(]-\infty,0]; C_0^{2\alpha}(0,1)\right) \end{cases}$$

if $\dfrac{\partial}{\partial y} f_+^\delta, \dfrac{\partial^2}{\partial y^2} f_-, g_+^\delta(0,.) - g_-(0,.) \in C_0^{4\alpha}(0,1)$.

Conclusion

In conclusion, this study focuses on the analysis of a transmission problem in Hölder spaces. Here, the solution does not necessarily vanish at infinity, unlike the classical results obtained in L^p spaces, where the condition $u(-\infty) = 0$ is imposed. This difference led us to examine some terms related to the nonhomogeneous Dirichlet boundary condition, which play a central role in this context.

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