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A COMPARATIVE STUDY OF THE FROBENIUS METHOD AND A LIE SYMMETRY ANALYSIS METHOD FOR SOLVING SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH REGULAR SINGULARITIES

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Abstract. This article compares two approaches for solving second-order ordinary differential equations (ODEs) with regular singularities: the Frobenius method and a method of Lie symmetry analysis of successive reduction of order. We provide a theoretical overview of each method, emphasizing their mathematical foundations and computational frameworks. The comparison is illustrated through four example ODEs with regular singularities, allowing a direct evaluation of their performance. The Frobenius method provides series expansions that can often be expressed compactly in terms of special functions, such as Bessel or hypergeometric families. In contrast, Lie symmetry analysis offers a unified and algorithmic framework that is applicable across all cases, provided the ODE admits the necessary symmetries. This difference highlights the Frobenius method's case-specific flexibility and the generality of the Lie symmetry approach.

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1. Introduction

Ordinary differential equations (ODEs) arise in numerous scientific and engineering disciplines-including physics, biology, and economics—where they model a wide spectrum of dynamic processes. A thorough understanding of ODEs is thus crucial for predicting, analyzing, and controlling real-world systems. In particular, second-order ODEs often appear in the study of mechanical vibrations, wave phenomena, and heat conduction [1–3]. Classical methods for solving these equations include characteristic equations, variation of parameters, and reduction of order, among others.

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When second-order ODEs involve variable coefficients, especially near singular points, solution techniques become more specialized. A central and time-honored approach for such equations is the Frobenius method. By considering power series expansions around a regular singular point, the method systematically derives analytic solutions, making it indispensable for equations like the Euler–Cauchy equation and a wide class of Bessel-type equations [4]. Recent work continues to extend and deepen the reach of this technique [5–7].

Alongside these developments in series-based methods, Lie symmetry analysis has emerged as a powerful, unifying framework for studying ODEs [8–13]. The method hinges on identifying continuous groups of transformations that leave a differential equation invariant-and using them to simplify the equation, often reducing its order or transforming it into a more tractable form. This strategy has proven especially effective for nonlinear and variable-coefficient ODEs.

These two methodologies—Frobenius expansions and Lie symmetry reductions—are both effective for solving second-order ODEs with variable coefficients. The Frobenius method provides series expansions that can often be expressed compactly in terms of special functions, such as Bessel or hypergeometric families [4]. While the Frobenius method requires tailoring its approach to specific cases depending on the nature of the ODE, Lie symmetry analysis offers a unified algorithmic framework that is applicable across all cases, provided the ODE admits the necessary symmetries.

In this article, we conduct a comparative study of the Frobenius method and Lie symmetry analysis for solving second-order ODEs with regular singularities. Using four representative examples from well-established texts [16,19,20], we demonstrate the applicability and effectiveness of each method.

The paper is structured as follows: Section 2 introduces the Frobenius method, detailing its application to second-order ODEs with regular singular points. In Section 3, we explore the fundamental concepts of Lie symmetry analysis, highlighting how admitted symmetries facilitate the systematic reduction of an ODE's order. Section 4 provides illustrative examples to demonstrate the practical implementation of both methods. Finally, Section 5 offers concluding remarks.

2. The Frobenius Method

The Frobenius method is a powerful technique for solving linear second-order ordinary differential equations (ODEs) with regular singular points. It adapts the standard power series solution method to handle equations that may appear to diverge at a particular point, enabling us to find solutions that are analytic in a neighborhood of that point.

Consider a second-order linear differential equation of the form:

$$y'' + p(x)y' + q(x)y = 0, \tag{1}$$

where $p(x)$ and $q(x)$ are functions of x. According to Frobenius' theorem, if the terms $(x - x_0)p(x)$ and $(x - x_0)^2 q(x)$ both have convergent power series expansions around the point x_0, then a power series solution to the differential equation can also be found around x_0.

Let us focus on finding a solution in the neighborhood of $x_0 = 0$. First, we examine the behavior of $xp(x)$ and $x^2 q(x)$ at $x_0 = 0$. It is straightforward to show that both $xp(x)$ and $x^2 q(x)$ are well-behaved (i.e., analytic) around $x_0 = 0$, which ensures that they can be expressed as convergent power series in this region.

If the conditions outlined above are satisfied, we can assume a solution to the differential equation in the form of a power series:

$$y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}, \tag{2}$$

where r is a constant to be determined, and a_n are the unknown coefficients of the series. While n is an integer, r is not restricted to integer values, and it must be determined through the process described below.

The steps to find the solution proceed as follows:

(1) *Indicial Equation:* We first derive the indicial equation by setting the coefficient of the lowest power of x (i.e., x^r) to zero. This condition provides the possible values of r, which are critical for determining the form of the solution. The value of a_0 is arbitrary, and it is determined based on the choice of r.

(2) *Recursive Relations for a_n:* Once r is found, we substitute it back into the equation and obtain recursive relations for the coefficients a_n, allowing us to compute each coefficient sequentially.

(3) *Solution Construction:* After determining the coefficients a_n for each r, the power series representation of the solution is given by:

$$y = x^r \sum_{n=0}^{\infty} a_n x^n.$$

(4) *General Solution:* The general solution to the differential equation is obtained by combining the solutions corresponding to each value of r.

In the context of the Frobenius method for solving ODEs with regular singular points, three distinct cases can arise, depending on the nature of the roots of the indicial equation:

(1) *Regular Singular Point:* This case arises when the limits p_0 and q_0 both exist and are finite as $x \to x_0$. The indicial equation will have two roots, which can be categorized into three types:

- $r_1 - r_2 \notin \mathbb{N}$ (the roots are distinct and non-integer spaced),
- $r_1 - r_2 = 0$ (the roots are equal),
- $r_1 - r_2 \in \mathbb{N}$ (the roots are integer spaced).

where, by convention, r_1 is the larger root.

(2) *Irregular Singular Point:* This case occurs when at least one of the limits referred to above fails to exist. After a suitable change of variables, a regular singularity may exist for large x, or a point at infinity. Then the same Frobenius approach can be used to solve the transformed differential equation.

The Frobenius method is extensively discussed in standard textbooks on ordinary differential equations, such as [1–3,15,17,18].

3. The Method of Successful Reduction of Order

Lie symmetry analysis offers various algorithms for solving ODEs, with comprehensive accounts available in numerous references, including [8–13,26].

Consider an nth-order ODE

$$F\left(x, y, y', \ldots, y^{(n)}\right) = 0, \tag{3}$$

where

$$y^{(k)} = \frac{d^k y}{dx^k}, \quad k = 1, 2, \ldots, n.$$

Central to methods of Lie symmetry analysis for studying ODEs is the determination of admitted symmetries.

A one-parameter Lie group of point transformations is represented by

$$\begin{aligned} \widetilde{x} &= f(x, y; \varepsilon) = x + \varepsilon \xi(x, y) + O\left(\varepsilon^2\right), \\ \widetilde{t} &= g(x, y; \varepsilon) = y + \varepsilon \eta(x, y) + O\left(\varepsilon^2\right), \end{aligned} \tag{4}$$

where ε is the group parameter, and the corresponding infinitesimal generator is defined by

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}, \tag{5}$$

for some functions ξ and η.

The Lie group (4) is admitted by the ODE (3) if and only if the nth order prolongation of (5) acting on F in (3) is zero along the solutions of (3), i.e.,

$$X^{(n)} F \mid_{F=0} = 0, \tag{6}$$

where the n order prolongation $X^{(n)}$ is given by

$$X^{(k)} = X + \eta^{(1)}\left(x, y, y'\right)\frac{\partial}{\partial y'} + \cdots + \eta^{(k)}\left(x, y, y', \ldots, y^{(k)}\right)\frac{\partial}{\partial y^{(k)}}, \tag{7}$$

where

$$\eta^{(k)}\left(x, y, y', \ldots, y^{(k)}\right) = D_x \eta^{(k-1)} - y^{(k)} D_x \xi, \quad k = 1, 2, \ldots, n$$

with

$$\eta^{(0)} = \eta(x, y),$$

and D_x is the total differential operator defined by

$$D_x = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + y'' \frac{\partial}{\partial y'} + \cdots . \tag{8}$$

A straightforward algorithm, Lie's algorithm, is employed to find symmetries of the ODE (3). The infinitesimal criterion (6) provides an overdetermined set of linear partial differential equations which are solved for the infinitesimals ξ and η. The infinitesimal generators span a Lie algebra of symmetries of (3).

Many computer algebra packages (e.g., Mulie, Dimsym, MathLie, Spde, Symgrp, Relie, etc.) exist that enable users to efficiently perform the cumbersome calculations involved in the application of Lie's algorithm (see, for example, [21–25]). With such packages the computation of symmetries is rendered almost automatic. These packages also significantly simplify the process of using the symmetries of differential equations.

The Lie symmetries of a given ODE can be used for various tasks, including integration of the ODE, which is the focus of the article. A particularly useful integration routine of Lie symmetry analysis is the integration via a method of successive reduction of order [8–11].

If a second order ODE

$$y'' = f\left(x, y, y'\right) \tag{9}$$

admits a two-parameter Lie group of transformations, then one can construct the general solution of (9) through a reduction to two quadratures in two successive steps.

Let X_1, X_2 be infinitesimal generators of point symmetries of (9) such that

$$[X_1, X_2] = \lambda X_1 \text{ for some constant } \lambda, \tag{10}$$

which means that the vector space spanned by X_1 and X_2 is a two-dimensional Lie algebra.

If

$$p = P(x, y) \quad \text{and} \quad q = Q\left(x, y, y'\right), \tag{11}$$

with $\frac{\partial q}{\partial y'} \neq 0$, are invariants of $X_1^{(1)}$, i.e.,

$$X_1 p = 0, \quad X_1^{(1)} q = 0,$$

then in the (p, q)-variables Eq. (9) reduces to

$$\frac{dq}{dp} = H(p, q), \tag{12}$$

for some function $H(p, q)$. Remarkably, (12) admits the symmetry X_2 written in the variables p and q. i.e.,

$$X_2^{(1)} = \alpha(p) \frac{\partial}{\partial p} + \beta(p, q) \frac{\partial}{\partial q}, \tag{13}$$

where the coefficients α and β are determined as follows:

$$\alpha(p) = X_2 p,\ \beta(p, q) = X_2^{(1)} q. \tag{14}$$

Therefore (12) can be integrated, using the symmetry (13) if necessary, to obtain a solution

$$\phi(p, q, K) = 0. \tag{15}$$

In terms of x and y, equation (15) is a first order ODE that admits the symmetry X_1.

This means that integration of any second-order ODE that admits two symmetry X_1 and X_2, and for which the Lie bracket satisfies the condition (10), reduces to the integration of two first-order ODEs, each of which has an admitted symmetry.

4. Illustrative Examples

In the comparative analysis between the Frobenius method the method of successive reduction of Lie symmetries, we consider the following ordinary differential equations (ODEs):

$$2x^2y'' - xy' + (1 + x)y = 0, \tag{16}$$

$$x^2y'' + 3xy' + (1 - 2x)y = 0, \tag{17}$$

$$x^4y'' + y = 0, \tag{18}$$

$$xy'' + 3y' - xy = 0. \tag{19}$$

These ODEs are taken, respectively, from [19, p. 281], [16, p. 374], [16, p. 388], and [20, p. 473]. Solutions of such equations often involve both exponential-type and special functions. In particular, the following Bessel functions frequently appear in the solution process [4]:

- $J_\nu(x)$: Bessel function of the first kind,
- $Y_\nu(x)$: Bessel function of the second kind,
- $I_\nu(x)$: modified Bessel function of the first kind,
- $K_\nu(x)$: modified Bessel function of the second kind.

It is noteworthy that many of the classical Bessel functions can be represented as special cases or particular values of the generalized hypergeometric function ${}_iF_j$. In particular, the Bessel functions of the first kind, $J_\nu(x)$, and the modified Bessel functions of the first kind, $I_\nu(x)$, can each be written as a series expansion that matches the ${}_0F_1$ or ${}_0F_2$ form (depending on the parameters). Similarly, the Bessel functions of the second kind, $Y_\nu(x)$, and the modified Bessel functions of the second kind, $K_\nu(x)$, can be related to linear combinations of these hypergeometric expansions, along with additional logarithmic terms in some representations.

4.1. **Solution of (16) Using the Frobenius Method.** For the ODE (16), it is easy to show that $x = 0$ is a regular singular point of Eq. (16). Further, $xp(x) = -1/2$ and $x^2q(x) = (1+x)/2$. Thus $p_0 = -1/2,\ q_0 = 1/2$.

To solve Eq. (16) we assume that there is a solution of the form

$$y = x^r\left(a_0 + a_1x + \cdots + a_nx^n + \cdots\right) = x^r\sum_{n=0}^{\infty} a_nx^n = \sum_{n=0}^{\infty} a_nx^{r+n}. \tag{20}$$

Then y' and y'' are given by

$$y' = \sum_{n=0}^{\infty} a_n(r+n)x^{r+n-1} \tag{21}$$

and

$$y'' = \sum_{n=0}^{\infty} a_n(r+n)(r+n-1)x^{r+n-2}. \tag{22}$$

By substituting the expressions for y, y', and y'' in Eq. (16), we obtain

$$\begin{aligned} 2x^2y'' - xy' + (1+x)y &= \sum_{n=0}^{\infty} 2a_n(r+n)(r+n-1)x^{r+n} \\ - \sum_{n=0}^{\infty} a_n(r+n)x^{r+n} &+ \sum_{n=0}^{\infty} a_nx^{r+n} + \sum_{n=0}^{\infty} a_nx^{r+n+1}. \end{aligned} \tag{23}$$

The last term in Eq. (23) can be rewritten as $\sum_{n=1}^{\infty} a_{n-1}x^{r+n}$, so by combining the terms in Eq. (23), we obtain

$$\begin{aligned} 2x^2y'' - xy' + (1+x)y &= a_0[2r(r-1) - r + 1]x^r \\ + \sum_{n=1}^{\infty} &\left\{[2(r+n)(r+n-1) - (r+n) + 1]a_n + a_{n-1}\right\} x^{r+n} = 0. \end{aligned} \tag{24}$$

If Eq. (24) is to be satisfied for all x, the coefficient of each power of x in Eq. (24) must be zero. From the coefficient of x^r we obtain, since $a_0 \neq 0$,

$$2r(r-1) - r + 1 = 2r^2 - 3r + 1 = (r-1)(2r-1) = 0. \tag{25}$$

Equation (25) is called the indicial equation for Eq. (16). The roots of the indicial equation are

$$r_1 = 1, \quad r_2 = 1/2. \tag{26}$$

These values of r are called the exponents at the singularity for the regular singular point $x = 0$. They determine the qualitative behavior of the solution (20) in the neighborhood of the singular point.

Now we return to Eq. (24) and set the coefficient of x^{r+n} equal to zero. This gives the relation

$$[2(r+n)(r+n-1) - (r+n) + 1]a_n + a_{n-1} = 0 \tag{27}$$

or

$$\begin{aligned} a_n &= -\frac{a_{n-1}}{2(r+n)^2 - 3(r+n) + 1} \\ &= -\frac{a_{n-1}}{[(r+n) - 1][2(r+n) - 1]}, \quad n \geq 1. \end{aligned} \tag{28}$$

For each root r_1 and r_2 of the indicial equation, we use the recurrence relation (28) to determine a set of coefficients $a_1, a_2, \ldots$. For $r = r_1 = 1$, Eq. (28) becomes

$$a_n = -\frac{a_{n-1}}{(2n+1)n}, \quad n \geq 1.$$

Thus

$$a_0 = \frac{-1}{(2n+1)n} \cdot \frac{-1}{(2n-1)(n-1)} \cdot \frac{-1}{(2n-3)(n-2)} \cdots \frac{-a_0}{(3)1}.$$

In general, we have

$$a_n = \frac{(-1)^n}{[3 \cdot 5 \cdot 7 \cdot \cdot (2n+1)]n!} a_0, \quad n \geq 4. \tag{29}$$

Multiplying the numerator and denominator of the right side of Eq. (29) by $2 \cdot 4 \cdot 6 \cdots 2n = 2^n n!$, we can rewrite a_n as

$$a_n = \frac{(-1)^n 2^n}{(2n+1)!} a_0, \quad n \geq 1.$$

Hence, if we omit the constant multiplier a_0, one solution of Eq. (16) is

$$\begin{aligned} y_1(x) &= x\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{(2n+1)!} x^n\right], \\ &= \frac{x}{\sqrt{2x}} \sum_{n=0}^{\infty} \frac{(-1)^n \left\{\sqrt{2x}\right\}^{2n+1}}{(2n+1)!} = \sqrt{\frac{x}{2}} \sin\sqrt{2x} = x^{\frac{3}{4}} J_{\frac{1}{2}}\left(\sqrt{2x}\right), \quad x > 0. \end{aligned} \tag{30}$$

To determine the radius of convergence of the series in Eq. (30) we use the ratio test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| = \lim_{n\to\infty} \frac{2|x|}{(2n+2)(2n+3)} = 0$$

for all x. Thus the series converges for all x. Corresponding to the second root $r = r_2 = \frac{1}{2}$, we proceed similarly. From Eq. (28) we have

$$a_n = -\frac{a_{n-1}}{2n\left(n - \frac{1}{2}\right)} = -\frac{a_{n-1}}{n(2n-1)}, \quad n \geq 1.$$

Hence

$$a_n = \frac{-1}{n(2n-1)} \cdot \frac{-1}{(n-1)(2n-3)} \cdot \frac{-1}{(n-2)(2n-5)} \cdots \frac{-a_0}{(1)(1)}$$

and, in general,

$$a_n = \frac{(-1)^n}{n![1 \cdot 3 \cdot 5 \cdots (2n-1)]} a_0, \quad n \geq 4. \tag{31}$$

Just as in the case of the first root r_1, we multiply the numerator and denominator by $2 \cdot 4 \cdot 6 \cdots 2n = 2^n n!$. Then we have

$$a_n = \frac{(-1)^n 2^n}{(2n)!} a_0, \quad n \geq 1.$$

Again omitting the constant multiplier a_0, we obtain the second solution

$$y_2(x) = x^{1/2}\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n 2^n}{(2n)!}x^n\right],$$

$$= \sqrt{x}\sum_{n=0}^{\infty}\frac{(-1)^n\left\{\sqrt{2x}\right\}^{2n}}{(2n)!} = \sqrt{x}\cos\sqrt{2x} = x^{\frac{3}{4}}Y_{\frac{1}{2}}\left(\sqrt{2x}\right), \quad x>0. \tag{32}$$

As before, we can show that the series in Eq. (32) converges for all x. Since the leading terms in the series solutions y_1 and y_2 are x and $x^{1/2}$, respectively, it follows that the solutions are linearly independent. Hence the general solution of Eq. (16) is

$$y = c_1 y_1(x) + c_2 y_2(x), \quad x>0. \tag{33}$$

4.2. **Solution of (16) Using Lie Symmetry Analysis.** The nonlinear equation (16) admits eight Lie point symmetries, two of which have the infinitesimal generators given by:

$$X_1 = e^{i\sqrt{2}\sqrt{x}}\sqrt{x}\frac{\partial}{\partial y}, \quad X_2 = y\frac{\partial}{\partial y}, \tag{34}$$

with the commutation relation:

$$[X_1, X_2] = X_1. \tag{35}$$

The first prolongation of X_1 is:

$$X_1^{(1)} = e^{i\sqrt{2}\sqrt{x}}\left[\sqrt{x}\frac{\partial}{\partial y} + \left(\frac{1}{2\sqrt{x}} + \frac{i}{\sqrt{2}}\right)\right]\frac{\partial}{\partial y'}. \tag{36}$$

We solve the corresponding characteristic equations

$$\frac{dx}{0} = \frac{dy}{\eta} = \frac{dy'}{\eta^{(1)}}, \tag{37}$$

where

$$\eta = e^{i\sqrt{2}\sqrt{x}}\sqrt{x}, \quad \eta^{(1)} = e^{i\sqrt{2}\sqrt{x}}\left(\frac{1}{2\sqrt{x}} + \frac{i}{\sqrt{2}}\right),$$

and obtain invariants

$$p = x, \quad q = 2y' - \frac{y}{x} - \frac{i\sqrt{2}\,y}{\sqrt{x}}. \tag{38}$$

Therefore,

$$\frac{dq}{dp} = \frac{D_x q}{D_x p} = 2y'' + y\left(\frac{1}{x^2} + \frac{i}{\sqrt{2}\,x^{3/2}}\right) - y'\left(\frac{1}{x} + \frac{i\sqrt{2}}{\sqrt{x}}\right). \tag{39}$$

Substituting y'' from equation (16) into equation (39) and expressing the resulting equation in terms of p and q using (38), we obtain the following first-order ODE:

$$\frac{dq}{dp} + \frac{iq}{\sqrt{2}\sqrt{p}} = 0. \tag{40}$$

From the first prolongation of X_2, given by

$$X_2^{(1)} = y\frac{\partial}{\partial y} + y'\frac{\partial}{\partial y'}, \tag{41}$$

and using the expression for p and q from (38), we obtain the following results:

$$X_2^{(1)} p\Big|_{(38)} = 0, \quad X_2^{(1)} q\Big|_{(38)} = q, \tag{42}$$

which shows that $X_2^{(1)}$ projects onto the (p, q)-plane as:

$$Y = q\partial_q. \tag{43}$$

Equation (40) admits this symmetry, and while the symmetry (43) can be used for further integration if needed, equation (40) is a variables-seperable ODE. Solving it, we find the solution:

$$q = Ke^{-i\sqrt{2}\sqrt{p}}, \tag{44}$$

where K is an arbitrary constant.

By expressing (44) in terms of x and y through (38), we obtain the first-order linear ODE:

$$y' - y\left(\frac{1}{2x} + \frac{i}{\sqrt{2}\sqrt{x}}\right) = \tfrac{K}{2}e^{-i\sqrt{2}\sqrt{x}}, \tag{45}$$

which is solved to give:

$$y = e^{-i\sqrt{2x}}\sqrt{x}\left(C_1 + C_2 e^{2i\sqrt{2x}}\right), \tag{46}$$

where C_1 and C_2 are arbitrary constants. This solution is the desired solution to the second-order ODE (16).

4.3. **Solution of (17) Using the Frobenius Method.** Consider the problem of solving the equation (17), repeated here:

$$x^2y'' + 3xy' + (1 - 2x)y = 0, \quad x > 0. \tag{47}$$

For this ODE, we have

$$p(x) = \frac{3}{x}, \quad q(x) = \frac{1-2x}{x^2},$$

so that $p_0 = 3$ and $q_0 = 1$. Therefore, the indicial equation is

$$r^2 + 2r + 1 = 0 \tag{48}$$

with roots $r_1 = r_2 = -1$.

Substituting

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{49}$$

into equation (47) leads to the recursive relation

$$a_n = \frac{2a_{n-1}}{n^2}\,, n \geq 1$$

which leads to

$$a_n = \frac{2^n a_0}{(n!)^2}.$$

The solution $y_1(x)$ becomes

$$y_1(x) = \frac{a_0}{x}\left\{\sum_{n=0}^{\infty}\frac{2^n x^n}{(n!)^2}\right\} = \frac{a_0}{x}I_0\left(2\sqrt{2x}\right) = \frac{a_0}{x}{}_0F_1(;1;2x).$$

By Frobenius approach, the second solution in a repeated root case takes the form

$$y_2(x) = y_1\ln x + \sum_{n=1}^{\infty} b_n x^{n-1}$$

which leads to

$$y_2(x) = y_1\ln x + -4 - 3x - \frac{22}{27}x^2 - \frac{25}{516}x^3 + \cdots$$

which can be repackaged as a modified second Bessel function of order zero

$$\frac{1}{x}K_0\left(2\sqrt{2x}\right) = \frac{\pi}{2x}I_0\left(2\sqrt{2x}\right)\left(1 - \frac{1}{\pi}\ln(2x)\right)$$

to relate it with the Lie symmetry method output below.

4.4. **Solution of (47) Using Lie Symmetry Analysis.** The nonlinear equation (47) admits two Lie point symmetries with the infinitesimal generators:

$$X_1 = \frac{\mathrm{K}_0\left(2\sqrt{2}\sqrt{x}\right)}{x}\frac{\partial}{\partial y}, \quad X_2 = y\frac{\partial}{\partial y}, \tag{50}$$

that have the commutation relation:

$$[X_1, X_2] = X_1. \tag{51}$$

The first prolongation of X_1 is given by:

$$X_1^{(1)} = \frac{\mathrm{K}_0\left(2\sqrt{2}\sqrt{x}\right)}{x}\frac{\partial}{\partial y} - \left(\frac{K_0(2\sqrt{2}\sqrt{x})}{x^2} + \frac{\sqrt{2}K_1(2\sqrt{2}\sqrt{x})}{x^{3/2}}\right)\frac{\partial}{\partial y'}. \tag{52}$$

From the invariants obtained by solving the corresponding characteristic equations:

$$p = x, \quad q = y' + y\left(\frac{1}{x} + \frac{\sqrt{2}K_1(2\sqrt{2}\sqrt{x})}{\sqrt{x}K_0(2\sqrt{2}\sqrt{x})}\right), \tag{53}$$

we derive the following:

$$\begin{aligned}\frac{dq}{dp} &= y'' + y'\left(\frac{1}{x} + \frac{\sqrt{2}K_1(2\sqrt{2}\sqrt{x})}{\sqrt{x}K_0(2\sqrt{2}\sqrt{x})}\right)\\ &\quad + \frac{y}{x^2}\left(\frac{2xK_1^2(2\sqrt{2}\sqrt{x})}{K_0^2(2\sqrt{2}\sqrt{x})} - \frac{\sqrt{2}\sqrt{x}K_1(2\sqrt{2}\sqrt{x})}{K_0(2\sqrt{2}\sqrt{x})} - (1+2x)\right)\end{aligned} \tag{54}$$

By substituting for y'' from equation (47) and simplifying using (53), we obtain a first-order ODE:

$$\frac{dq}{dp} = \frac{q}{p}\left(\sqrt{2}\sqrt{p}\,\frac{K_1(2\sqrt{2}\sqrt{p})}{K_0(2\sqrt{2}\sqrt{p})} - 2\right). \tag{55}$$

Projecting $X_2^{(1)}$ onto the (p, q)-plane, we find the symmetry:

$$X_2^{(1)} = q\frac{\partial}{\partial q}, \tag{56}$$

which is a symmetry of (55). Integrating (55) leads to the solution:

$$q = \frac{C_1}{p^2 K_0(2\sqrt{2}\sqrt{p})}, \tag{57}$$

where C_1 is an arbitrary constant.

By transforming this solution through (53), we obtain the first-order linear ODE in x and y:

$$y' + \left(\frac{1}{x} + \frac{\sqrt{2}\,K_1(2\sqrt{2}\sqrt{x})}{\sqrt{x}\,K_0(2\sqrt{2}\sqrt{x})}\right) y = \frac{C_1}{x^2\,K_0(2\sqrt{2}\sqrt{x})}. \tag{58}$$

This linear equation is solved to yield the solution:

$$y = \frac{\kappa_1 I_0(2\sqrt{2}\sqrt{x}) + \kappa_2 K_0(2\sqrt{2}\sqrt{x})}{x}, \tag{59}$$

where κ_1 and κ_2 are arbitrary constants. This solution satisfies the original second-order ODE (47).

4.5. **Solution of (18) Using the Frobenius Method.** Let us consider equation (18), which we restate below:

$$x^4 y'' + y = 0. \tag{60}$$

Next, we solve equation (60) using the Frobenius method. First, we introduce the substitution $\omega = 1/x$, which implies:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{d\omega}\frac{d\omega}{dx} = -\omega^2 \frac{dy}{d\omega}, \\
\frac{d}{dx}\left(\frac{dy}{dx}\right) &= \frac{d\omega}{dx}\frac{d}{d\omega}\left(-\omega^2 \frac{dy}{d\omega}\right) \\
&= \omega^2 \left[2\omega \frac{dy}{d\omega} + \omega^2 \frac{d^2 y}{d\omega^2}\right].
\end{aligned}$$

Substituting this into the original equation (60), we obtain:

$$y'' + \frac{2}{\omega} y' + y = 0. \tag{61}$$

At $\omega = 0$, we find:

$$p_0 = \lim_{\omega \to 0} \omega \cdot \frac{2}{\omega} = 2, \quad q_0 = \lim_{\omega \to 0} \omega^2 \cdot 1 = 0.$$

Thus, the indicial equation is:

$$r(r-1) + 2r = r(r+1) = 0. \tag{62}$$

Next, we proceed with the series solution. Expanding the equation:

$$\sum_{n=0}^{\infty} n(n-1) a_n \omega^{n-1} + \sum_{n=0}^{\infty} 2n a_n \omega^{n-1} + \sum_{n=0}^{\infty} a_n \omega^{n+1} = 0. \tag{63}$$

We combine the terms:

$$\sum_{n=0}^{\infty} \{n(n-1) + 2n\} a_n \omega^{n-1} + \sum_{n=2}^{\infty} a_{n-2} \omega^{n-1} = 0. \tag{64}$$

This leads to:

$$0 \cdot a_0/\omega + 2a_1 + \sum_{n=2}^{\infty} \left[n(n+1)a_n + a_{n-2}\right] \omega^{n-1} = 0. \tag{65}$$

From here, we deduce that a_0 is an arbitrary constant and $a_1 = 0$, implying that all odd-index terms vanish: $a_{2n+1} = 0$. Thus, the even-index terms are given by:

$$a_{2n} = \frac{-a_{2n-2}}{(2n+1)2n} = \frac{(-1)^n a_0}{(2n+1)!}. \tag{66}$$

The solution is then:

$$y_1(\omega) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n \omega^{2n}}{(2n+1)!} = a_0 \left(1 - \frac{\omega^2}{6} + \frac{\omega^4}{120} - \frac{\omega^6}{5040} + \cdots \right). \tag{67}$$

Alternatively, this can be expressed as:

$$y_1(\omega) = \frac{a_0}{\omega} \sum_{n=0}^{\infty} \frac{(-1)^n \omega^{2n+1}}{(2n+1)!} = a_0 x \sin\left(\frac{1}{x}\right). \tag{68}$$

Next, we find the second solution $y_2(\omega)$:

$$y_2(\omega) = C y_1(\omega) \ln \omega + \sum_{n=0}^{\infty} b_n \omega^{n-1}. \tag{69}$$

To compute the derivatives:

$$y_2'(\omega) = C y_1' \ln \omega + \frac{C y_1}{\omega} + \sum_{n=0}^{\infty} (n-1) b_n \omega^{n-2}, \tag{70}$$

$$y_2''(\omega) = C y_1'' \ln \omega + \frac{2C y_1'}{\omega} - \frac{C y_1}{\omega^2} + \sum_{n=0}^{\infty} (n-1)(n-2) b_n \omega^{n-3}. \tag{71}$$

Substituting into the equation:

$$2C y_1' + C \frac{y_1}{\omega} + \sum_{n=0}^{\infty} \left[(n-1)(n-2) + 2(n-1)\right] b_n \omega^{n-2} + \sum_{n=0}^{\infty} b_n \omega^n = 0. \tag{72}$$

This simplifies to:

$$2C y_1' + \frac{C y_1}{\omega} + \sum_{n=2}^{\infty} \left[n(n-1) b_n + b_{n-2}\right] \omega^{n-2} = 0. \tag{73}$$

From here, we deduce that both b_0 and b_1 are arbitrary. Comparing coefficients:

$$\omega^{-1} : C = 0 \quad \text{and} \quad b_n = \frac{-b_{n-2}}{n(n-1)}, \quad n \geq 2.$$

The solution for the b_n's is:

$$b_{2n} = \frac{(-1)^n b_0}{(2n)!}, \quad b_{2n+1} = \frac{(-1)^n b_1}{(2n+1)!}. \tag{74}$$

Thus, the second solution is:

$$y_2(\omega) = \frac{b_0}{\omega}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n \omega^{2n}}{(2n)!}\right] + \frac{b_1}{\omega}\left[\omega + \sum_{n=1}^{\infty} \frac{(-1)^n \omega^{2n+1}}{(2n+1)!}\right]. \tag{75}$$

Finally, we arrive at the general solution:

$$y(x) = x\left[b_0 \cos\frac{1}{x} + b_1 \sin\frac{1}{x}\right]. \tag{76}$$

4.6. **Solution of (60) Using Lie Symmetry Analysis.** Equation (60) admits a two-dimensional symmetry Lie algebra spanned by the infinitesimal generators:

$$X_1 = e^{i/x} x \frac{\partial}{\partial y}, \quad X_2 = y\frac{\partial}{\partial y}, \tag{77}$$

with the commutation relation:

$$[X_1, X_2] = X_1. \tag{78}$$

The first prolongation of X_1 is:

$$X_1^{(1)} = e^{i/x} x \frac{\partial}{\partial y} + e^{i/x}\frac{(x-i)}{x}\frac{\partial}{\partial y'}. \tag{79}$$

From this, we obtain the invariants:

$$p = x, \quad q = \frac{xy' - y}{x} + \frac{iy}{x^2}. \tag{80}$$

Using these invariants, we write the following expression for $\frac{dq}{dp}$:

$$\frac{dq}{dp} = \frac{D_x q}{D_x p} = y\left(\frac{1}{x^2} - \frac{2i}{x^3}\right) + y'\left(\frac{i}{x^2} - \frac{1}{x}\right) + y''. \tag{81}$$

Substituting for y'' from equation (60) into the above and using (80), we arrive at the first-order ODE:

$$\frac{dq}{dp} = \frac{q(i-p)}{p^2}. \tag{82}$$

This equation admits the symmetry given by the projection of X_2 onto the (p, q)-plane:

$$X_2^{(1)} = q\frac{\partial}{\partial q}. \tag{83}$$

Integrating equation (82), which is separable, we obtain:

$$q = \frac{C_1 e^{-i/p}}{p}, \tag{84}$$

where C_1 is an arbitrary constant. Substituting for p and q using (80), we obtain the first-order ODE in terms of x and y:

$$y' = \frac{C_1 e^{-i/x}}{x} + \frac{u(x-i)}{x^2}. \tag{85}$$

This is a separable equation, and its solution is:

$$y = x\left(\kappa_1 e^{i/x} + \kappa_2 e^{-i/x}\right), \tag{86}$$

where κ_1 and κ_2 are arbitrary constants. This solution is the desired solution to the second-order ODE (60).

4.7. **Solution of (19) Using the Frobenius Method.** We seek a series solution about the regular singular point $x = 0$ for the differential equation:

$$xy'' + 3y' - xy = 0. \tag{87}$$

The given equation has $p(x) = 3/x$ and $q(x) = -1$, which indicates that $x = 0$ is a regular singular point. To confirm, we compute the limits:

$$p_0 = \lim_{x\to 0} xp(x) = 3, \quad q_0 = \lim_{x\to 0} x^2 q(x) = 0.$$

The indicial equation is derived from the general form of the equation, and it takes the form:

$$r(r-1) + 3r = r^2 + 2r = r(r+2) = 0,$$

which has roots $r_1 = 0$ and $r_2 = -2$.

Using the root $r_1 = 0$, we obtain the first solution in the form of a power series:

$$y_1(x) = a_0 \sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2n}}{(n+1)!n!} = a_0 \sum_{n=0}^{\infty} \frac{\left(\frac{x^2}{4}\right)^{n}}{(n+1)!n!}. \tag{88}$$

This can be rewritten as:

$$y_1(x) = \frac{a_0}{x} I_1(x) = a_0 \left\{ {}_0F_1\left(;2;\frac{x^2}{4}\right)\right\}.$$

Since $r_1 - r_2 = 2$ is a positive integer, the second solution $y_2(x)$ is of the form:

$$y_2(x) = Cy_1(x)\ln x + \sum_{n=0}^{\infty} b_n x^{n-2}. \tag{89}$$

Substitute the expansion for $y_2(x)$ into equation (87) and simplify:

$$\begin{aligned}\left\{xy_1'' + 3y_1' - xy_1\right\} C\ln x + 2Cx^{-1}y_1 + 2Cy_1' + \sum_{n=0}^{\infty}(n-2)(n-3)b_n x^{n-3} \\ + \sum_{n=0}^{\infty} 3(n-2)b_n x^{n-3} - \sum_{n=0}^{\infty} b_n x^{n-1} = 0.\end{aligned} \tag{90}$$

Since the factor in braces is zero (because $y_1(x)$ is a solution to equation (87)), we combine the summations and simplify:

$$2Cx^{-1}y_1(x) + 2Cy_1'(x) - b_1 x^{-2} + \sum_{n=2}^{\infty}\left[n(n-2)b_n - b_{n-2}\right] x^{n-3} = 0. \tag{91}$$

Substituting the series expansions for $y_1(x)$ and $y_1'(x)$ and writing out the first few terms of the summation in (91) leads to

$$- b_1 x^{-2} + (2C - b_0)\, x^{-1} + (3b_3 - b_1) + \left(\frac{3}{4}C + 8b_4 - b_2\right) x + (15b_5 - b_3)\, x^2 + \left(\frac{5}{96}C + 24b_6 - b_4\right) x^3 + \cdots = 0. \tag{92}$$

Setting the coefficients of the powers of x equal to zero gives the following recurrence relations:

$$\begin{aligned}
-b_1 = 0 \quad &\Longrightarrow \quad b_1 = 0,\\
2C - b_0 = 0 \quad &\Longrightarrow \quad b_0 = 2C \quad \text{(where } C \text{ is arbitrary)},\\
3b_3 - b_1 = 0 \quad &\Longrightarrow \quad b_3 = \frac{1}{3}b_1 = 0,\\
8b_4 - b_2 + \frac{3}{4}C = 0 \quad &\Longrightarrow \quad b_4 = \frac{b_2 - \frac{3}{4}C}{8} = \frac{1}{8}b_2 - \frac{3}{32}C \quad \text{(where } b_2 \text{ is arbitrary)},\\
15b_5 - b_3 = 0 \quad &\Longrightarrow \quad b_5 = \frac{1}{15}b_3 = 0,\\
24b_6 - b_4 + \frac{5}{96}C = 0 \quad &\Longrightarrow \quad b_6 = \frac{b_4 - \frac{5}{96}C}{24} = \frac{1}{192}b_2 - \frac{7}{1152}C.
\end{aligned}$$

Substituting these values for b_n into equation (89) yields the second solution:

$$\begin{aligned}
y_2(x) = \quad & C\left\{y_1(x)\ln x + 2x^{-2} - \frac{3}{32}x^2 - \frac{7}{1152}x^4 + \cdots\right\}\\
& + b_2\left\{1 + \frac{1}{8}x^2 + \frac{1}{192}x^4 + \cdots\right\}.
\end{aligned} \tag{93}$$

Since the factor multiplying b_2 is the first solution $y_1(x)$, we obtain a second linearly independent solution by choosing $C = 1$ and $b_2 = 0$:

$$y_2(x) = y_1(x)\ln x + 2x^{-2} - \frac{3}{32}x^2 - \frac{7}{1152}x^4 + \cdots. \tag{94}$$

Thus, the general solution to the differential equation (87) is given by the linear combination of the two independent solutions $y_1(x)$ and $y_2(x)$:

$$y(x) = C_1 y_1(x) + C_2 y_2(x),$$

where C_1 and C_2 are arbitrary constants.

4.8. **Solution of (87) Using Lie Symmetry Analysis.** Equation (87) admits a two-dimensional Lie algebra of symmetries spanned by the infinitesimal generators

$$X_1 = \frac{J_1(ix)}{x}\frac{\partial}{\partial y}, \quad X_2 = y\,\frac{\partial}{\partial y}, \tag{95}$$

which satisfy the commutation relation

$$[X_1, X_2] = X_1. \tag{96}$$

From the first prolongation of X_1,

$$X_1^{(1)} = \frac{J_1(ix)}{x}\frac{\partial}{\partial y} + \left[\frac{i(J_0(ix) - J_2(ix))}{2x} - \frac{J_1(ix)}{x^2}\right]\frac{\partial}{\partial y'}, \tag{97}$$

we find invariants

$$p = x, \quad q = \frac{y' I_1(x) - y I_2(x)}{I_1(x)}. \tag{98}$$

Therefore,

$$\frac{dq}{dp} = \frac{D_x q}{D_x p} = y'' - \frac{y' I_2(x)}{I_1(x)} - \frac{u\left(I_1^2(x) + I_3(x)I_1(x) - I_2(x)(I_0(x) + I_2(x))\right)}{2I_1^2(x)}. \tag{99}$$

Rewriting equation (99) in terms of the new variables p and q, after substituting the original ODE (87), and using the invariants (98), we obtain the first-order ODE

$$\frac{dq}{dp} \ + \ q\left(\frac{1}{p} \ + \ \frac{I_0(p)}{I_1(p)}\right) \ = \ 0. \tag{100}$$

Equation (100) admits

$$X_2^{(1)} = q\frac{\partial}{\partial q}, \tag{101}$$

which is the projetion of X_2 onto the (p, q)-plane. The solution of Equation (100)

$$q = \frac{C_1}{p^2 I_1(p)}, \tag{102}$$

where C_1 is an arbitrary constant, is transformed through (98) into the first-order ODE

$$y' = \frac{C_1}{x^2 I_1(x)} + \frac{y I_2(x)}{I_1(x)}. \tag{103}$$

Integrating (103), we obtain

$$y = \frac{1}{x}\left(C_1 K_1(x) + C_2 I_1(x)\right), \tag{104}$$

where C_2 is another arbitrary constant. The solution (104) is the desired solution of the second-order ODE (87).

5. Concluding Remarks

The Frobenius method is a robust and systematic approach for solving linear second-order ordinary differential equations (ODEs) with variable coefficients, particularly around regular singular points. It provides a structured framework for constructing solutions, making it a valuable tool in mathematical physics for addressing many important differential equations. The Frobenius method treats equations differently depending on the nature of the singular points, specifically distinguishing between regular and irregular singular points.

In contrast, the method of successive reduction based on Lie symmetry analysis eliminates the need for such distinctions. The primary requirement for this approach is that the ODE admits two symmetries X_i and X_j such that their commutator satisfies $[X_i, X_j] = \lambda X_j$, where λ is a constant. This method is entirely algorithmic, systematically reducing a second-order ODE to two first-order ODEs, each of which admits a Lie point symmetry.

In this article, we applied the Frobenius method and the Lie symmetry analysis method of successive reduction of order to typical second-order ODEs with regular singularities. Through the analysis of four representative examples, we demonstrated that while the Frobenius method is a reliable and well-established technique, the Lie symmetry analysis method offers a more versatile and efficient alternative. Its ability to provide closed-form solutions through a unified and algorithmic approach simplifies the solving process and enhances its applicability across a broader range of ODEs.

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