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ITERATIVE METHODS FOR APPROXIMATING TWO ASYMPTOTICALLY NONEXPANSIVE NONSELF-MAPPINGS IN CAT(0) SPACES

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Abstract. In this manuscript, we investigate the approximation of common fixed points for two nonself asymptotically nonexpansive mappings within the framework of CAT(0) spaces. We establish both weak and strong convergence theorems for a two-step iterative scheme that is tailored to the geometric structure of CAT(0) spaces.

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1. Introduction

Fixed point theory is a fundamental topic in mathematics with broad applications in engineering, economics, and biology. Fixed points of nonexpansive mappings and their generalizations play a key role in solving problems in image restoration, signal processing, machine learning, and motion control. The theory also provides powerful tools for analyzing various types of differential equations of the form

$$0 \in \frac{du}{dt} + T(t)u$$

can be tackled with fixed point approach (see, e.g., Bruck [1]).

In this paper, $\mathbb{N}$ stands for the set of natural numbers. We will also denote by $F(T) := \{x \in K : Tx = x\}$ the set of fixed points of T, and by $\mathbb{F} := F(T_1) \cap F(T_2)$, the set of common fixed points of two mappings T_1 and T_2.

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A mapping T is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$, and for all $x, y \in K$ and all $n \in \mathbb{N}$,

$$d(T^n x, T^n y) \leq k_n d(x, y).$$

A mapping T is called uniformly $L-$Lipschitzian if for some $L > 0$,

$$d(T^n x, T^n y) \leq Ld(x, y)$$

for all $x, y \in K$ and $n \in \mathbb{N}$.

A mapping T is said to be nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in K$.

A mapping $T : X \to X$ is called semi-compact if, for every bounded sequence $\{x_n\} \subset X$ such that $\{Tx_n\}$ converges, there exists a subsequence $\{x_{n_k}\}$ that converges in X. This condition is weaker than compactness but still imposes a kind of continuity behavior on bounded sequences.

Let $P : X \to K$ be a nonexpansive retraction of X onto K.

A nonself-mapping $T : K \to X$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$, and

$$d(T(PT)^{n-1}x, T(PT)^{n-1}y) \leq k_n d(x, y)$$

for all $x, y \in K$ and $n \in \mathbb{N}$. Similarly, T is called uniformly $L-$Lipschitzian if for some $L > 0$,

$$d(T(PT)^{n-1}x, T(PT)^{n-1}y) \leq Ld(x, y)$$

for all $x, y \in K$ and $n \in \mathbb{N}$.

In what follows, we fix $x_1 \in K$ as the starting point of the process under consideration and take sequences $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$. Agarwal, O'Regan and Sahu [3] introduced the iteration process

$$\begin{aligned} y_n &= (1 - \beta_n)x_n + \beta_n T^n x_n, \\ x_{n+1} &= (1 - \alpha_n)T^n x_n + \alpha_n T^n y_n, , \quad n \in \mathbb{N}, \end{aligned} \tag{1.1}$$

It has been shown that this process is independent of the classical Mann and Ishikawa iteration schemes, and it converges faster than both. Clearly, the process above deals with a single self-mapping T^n.

The case involving two mappings in iteration processes has also remained under study. Das and Debata [4] proposed and investigated a two-mapping scheme. See also Takahashi and Tamura [5], and Khan and Takahashi [6], for further developments.

The two-mapping case that is, the task of approximating common fixed points of two-mappings has notable importance because of its direct connection with minimization problems. For more details, see Takahashi [7]. A significant generalization of the class of nonexpansive self-mappings is the class of asymptotically nonexpansive self-mappings, introduced by Goebel and Kirk [8]. The concept of asymptotically nonexpansive nonself-mappings was later introduced by Chidume, Ofoedu and Zegeye [2] in 2003, as a further generalization.

In fact, they studied the iteration process for these mappings

$$x_{n+1} = P\left((1-\alpha_n)x_n + \alpha_n T((PT)^{n-1}x_n)\right), \quad n \in \mathbb{N}. \tag{1.2}$$

The study of nonself asymptotically nonexpansive mappings has attracted considerable attention in the literature (see, e.g., [9–13]).

As a matter of fact, if T is a self-mapping, then P is an identity mapping. In addition, if $T\colon K \to X$ is asymptotically nonexpansive and $P\colon X \to K$ is a nonexpansive retraction, then $PT\colon K \to K$ is asymptotically nonexpansive. For all $x, y \in K$ and $n \in \mathbb{N}$, it follows that

$$\begin{aligned} \|(PT)^n x - (PT)^n y\| &= \|P(T(PT)^{n-1}x) - P(T(PT)^{n-1}y)\| \\ &\leq \|T(PT)^{n-1}x - T(PT)^{n-1}y\| \\ &\leq k_n\|x-y\|. \end{aligned} \tag{1.3}$$

It has been shown that PT is an asymptotically nonexpansive. However, the converse statement does not necessarily hold. To address this limitation, Zhou et al. [14] introduced the following generalized definition.

Definition 1.1 ([14]). *Let K be a nonempty subset of a real normed linear space E. Let $P\colon X \to K$ be the nonexpansive retraction of X into K.*

(i) *A nonself-mapping $T\colon K \to X$ is called asymptotically nonexpansive with respect to P if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that*

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq k_n\|x-y\|, \quad \forall x, y \in K,\, n \in \mathbb{N}.$$

(ii) *A nonself-mapping $T\colon K \to X$ is said to be* uniformly L-Lipschitzian with respect to P *if there exists a constant $L \geq 0$ such that*

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq L\|x-y\|, \quad \forall x, y \in K,\, n \in \mathbb{N}.$$

Furthermore, by studying the following iterative process:

$$x_1 \in K, \quad x_{n+1} = P((1-\alpha_n)x_n + \alpha_n T(PT_1)^{n-1}x_n), \quad n \in \mathbb{N}, \tag{1.4}$$

where $\{\alpha_n\}$ be a sequence in $[0,\,1]$ satisfying the condition $0 < a \leq \alpha_n \leq b < 1,\ n \geq 1$, for some constant a, b.

Zhou et al. [14] established both strong and weak convergence theorems for common fixed points of asymptotically nonexpansive nonself-mappings with respect to P in uniformly convex Banach spaces. As a result, the main findings of Chidume, Ofoedu and Zegeye [2] were deduced.

It is important to note that a topological space X is said to have the topological fixed point property (TFPP) if every continuous mapping $T : X \to X$ has a fixed point.

It is well-known that this property is a topological invariant. Suppose X and Y are two homeomorphic spaces, with X having the fixed point property. Let $H : X \to Y$ be a homeomorphism, such that $H(X) = Y$, and suppose $F : Y \to Y$ is continuous. Define $G = H^{-1} \circ F \circ H : X \to X$, which is also continuous. Then, there exists $x \in X$ such that $Gx = x$, and consequently, for $y = Hx$, we have $y = Fy$.

Retractions also preserve the fixed point property. A subset K of X is said to be a retract if there exists a continuous mapping $P : X \to K$ such that $Px = x$ for all $x \in K$, and P is called a retraction [2]. Note that if P is a retraction, then $P^2 = P$ and $Pz = z$ for every $z \in \text{Range}(P)$.

Suppose Y is a retract of X and let $F : Y \to X$ be continuous. Using any retraction $R : X \to Y$, the mapping F can be extended to a continuous mapping $G = F \circ R : X \to X$. Any fixed point of G must also be a fixed point of F. Therefore, if X has the TFPP, the same holds for all its retracts. For more details on TFPP, refer to Goebel [16].

Chidume, Ofoedu and Zegeye [2] obtained the following strong and weak convergence theorems for asymptotically nonexpansive nonself-mappings.

Theorem 1.2 ([2])**.** *Let X be a real uniformly convex Banach space and let C be a nonempty closed convex subset of X. Let $T : C \to X$ be a completely continuous and asymptotically nonexpansive map with sequence $\{k_n\} \subset [1, \infty)$ such that*

$$\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$$

and $F(T) \neq \emptyset$. Let $\{\alpha_n\} \subset (0, 1)$ be such that $\varepsilon \leq 1 - \alpha_n \leq 1 - \varepsilon$ for all $n \geq 1$ and some $\varepsilon > 0$. From an arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by (1.4). Then $\{x_n\}$ converges strongly to some fixed point of T.

Theorem 1.3 ([2])**.** *Let X be a real uniformly convex Banach space which has a Fréchet differentiable norm and let C be a nonempty closed convex subset of X. Let $T : C \to X$ be an asymptotically nonexpansive map with sequence $\{k_n\} \subset [1, \infty)$ such that*

$$\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$$

and $F(T) \neq \emptyset$. Let $\{\alpha_n\} \subset (0, 1)$ be such that $\varepsilon \leq 1 - \alpha_n \leq 1 - \varepsilon$ for all $n \geq 1$ and some $\varepsilon > 0$. From an arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by (1.4). Then $\{x_n\}$ converges weakly to some fixed point of T.

In 2006, Wang [11] generalized the iteration process (1.4) as follows: for $x_1 \in C$,

$$\begin{aligned} y_n &= P\left((1-\beta_n)x_n + \beta_n T_2(PT_2)^{n-1}x_n\right), \\ x_{n+1} &= P\left((1-\alpha_n)x_n + \alpha_n T_1(PT_1)^{n-1}y_n\right), \quad n \geq 1, \end{aligned} \tag{1.5}$$

where $T_1, T_2 : C \to X$ are asymptotically nonexpansive nonself-mappings and $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $[0, 1)$. He studied the strong and weak convergence of the iterative scheme (1.5) under proper conditions. Meanwhile, the results of [11] generalized the results of [2].

In 2009, Thianwan [13] proposed a new two-step iteration scheme for two asymptotically nonexpansive nonself-mappings in a uniformly convex Banach space. Both weak and strong convergence results were established for this scheme.

Let X be a normed space, $C \subset X$ be a nonempty convex subset, and $P : X \to C$ be a nonexpansive retraction of X onto C. Let $T_1, T_2 : C \to X$ be given mappings. Then for an arbitrary $x_1 \in C$, the following iteration scheme is studied:

$$\begin{aligned} y_n &= P\left((1-\beta_n)x_n + \beta_n T_2 (PT_2)^{n-1} x_n\right), \\ x_{n+1} &= P\left((1-\alpha_n)y_n + \alpha_n T_1 (PT_1)^{n-1} y_n\right), \quad n \geq 1, \end{aligned} \tag{1.6}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are appropriate real sequences in $[0, 1)$.

The iterative scheme (1.6) is called the projection-type Ishikawa iteration for two asymptotically nonexpansive nonself- mappings. Note that if $T_1 = T_2$ and $\beta_n = 0$ for all $n \geq 1$, then the iteration scheme (1.6) reduces to the simpler form (1.4), as previously introduced.

In 2023, Kratuloek et al. [33] studied and approximated common fixed points of two asymptotically nonexpansive nonself-mappings in the context of CAT(0) spaces. They provided three examples and conducted numerical experiments to demonstrate the implementation of the approximation schemes.

Let K be a nonempty closed convex subset of a CAT(0) space X with retraction P. Let $T_1, T_2 : K \to X$ be two nonself asymptotically nonexpansive mappings with respect to P. For $x_1 \in K$, the iteration scheme is given by

$$\begin{aligned} y_n &= (1-\beta_n)x_n \oplus \beta_n (PT_1)^n x_n, \\ x_{n+1} &= (1-\alpha_n)(PT_1)^n y_n \oplus \alpha_n (PT_2)^n y_n, \quad n \geq 1, \end{aligned} \tag{1.7}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$.

Inspired and motivated by these facts, we introduce and study a new class of iterative schemes in this paper. The scheme is defined as follows:

Let X be a complete CAT(0) space, $C \subset X$ be a nonempty convex subset, and $P : X \to C$ be a nonexpansive retraction of X onto C, and $T_1, T_2 : C \to X$ given mappings. Then for an arbitrary $x_1 \in C$, the following iteration scheme is studied:

$$\begin{aligned} y_n &= P((1-\beta_n)x_n \oplus \beta_n T_2 (PT_2)^{n-1} x_n), \\ x_{n+1} &= P((1-\alpha_n)y_n \oplus \alpha_n T_1 (PT_1)^{n-1} y_n), \quad n \geq 1, \end{aligned} \tag{1.8}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are appropriate real sequences in $[0, 1)$.

The iterative scheme (1.8) is called the projection-type Ishikawa iteration for two asymptotically nonexpansive nonself-mappings.

The purpose of this paper is to construct an iteration scheme for approximating common fixed points of two asymptotically nonexpansive nonself-mappings and to prove some strong and weak convergence theorems for such mappings in a $\mathrm{CAT}(0)$ space.

2. Preliminaries

Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ is a map $\eta\colon [0, l] \to X$ such that $\eta(0) = x$, $\eta(l) = y$, and $d(\eta(t), \eta(t')) = |t - t'|$ for all $t, t' \in [0, l]$. The image α of η is called a geodesic segment joining x and y, denoted by $[x, y]$.

A space (X, d) is a geodesic space if every two points in X can be joined by a geodesic. If each pair has exactly one geodesic, the space is uniquely geodesic. A subset $Y \subseteq X$ is convex if it includes all geodesic segments joining its points.

A geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) consists of three points and a geodesic segment between each pair. A comparison triangle $\bar{\Delta}(x_1, x_2, x_3)$ in $\mathbb{R}^2$ satisfies $d_{\mathbb{R}^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

Definition 2.1. *A geodesic space (X, d) is called a $\mathrm{CAT}(0)$ space if for any triangle $\Delta \subset X$ and any $x, y \in \Delta$, the inequality*

$$d(x, y) \leq d(\bar{x}, \bar{y})$$

holds, where $\bar{x}, \bar{y}$ are corresponding points in the comparison triangle.

It is well-known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a $\mathrm{CAT}(0)$ space. Other examples of $\mathrm{CAT}(0)$ spaces include pre-Hilbert spaces, $\mathbb{R}$-trees (see Bridson and Haefliger [17]), Euclidean buildings (see Brown [18]), the complex Hilbert ball with a hyperbolic metric (see Goebel and Reich [19]), and many others.

Definition 2.2. *A geodesic triangle $\Delta(p, q, r)$ in (X, d) is said to satisfy the $\mathrm{CAT}(0)$ inequality if for any $u, v \in \Delta(p, q, r)$ and for their comparison points $\bar{u}, \bar{v} \in \bar{\Delta}(\bar{p}, \bar{q}, \bar{r})$, one has*

$$d(u, v) \leq d_{\mathbb{R}^2}(\bar{u}, \bar{v}).$$

Based on Definition 2.1, one can see that all geodesic triangles in a $\mathrm{CAT}(0)$ space satisfy the $\mathrm{CAT}(0)$ inequality. For further details on $\mathrm{CAT}(0)$ spaces, we refer the readers to standard texts such as Bridson and Haefliger [17]. It is well-known that every $\mathrm{CAT}(0)$ space is uniquely geodesic.

Note that if x, y_1, y_2 are points of a $\mathrm{CAT}(0)$ space and if y_0 is the midpoint of the segment $[y_1, y_2]$ (we write $y_0 = \frac{1}{2}y_1 \oplus \frac{1}{2}y_2$), then the $\mathrm{CAT}(0)$ inequality implies

$$d(x, y_0)^2 \leq \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2. \tag{2.1}$$

In fact, a geodesic metric space is a $\mathrm{CAT}(0)$ space if and only if it satisfies inequality (2.1) above. This inequality is known as the CN inequality of Bruhat and Tits [20]. As a consequence of this inequality, CAT(0) spaces possess many interesting properties such as a non-convex function (resp. set) can be viewed as a convex function (resp. set) [21].

The following discussion focuses on the notion of asymptotic centers and associated results. Let K be a nonempty closed convex subset of a CAT(0) space X, and let $\{x_n\}$ be a bounded sequence in X. For each $x \in X$, the asymptotic radius of the sequence $\{x_n\}$ at the point x is defined by

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x_n, x).$$

The following is the formal definition of $r \equiv r(K, \{x_n\}) := \inf \{r(x, \{x_n\}) : x \in K\}$, and $A \equiv A(K, \{x_n\}) := \{x \in K : r(x, \{x_n\}) = r\}$. The quantity r is called the asymptotic radius, and the set A is called the asymptotic center of the sequence $\{x_n\}$ relative to K.

It is well known that if X is a complete CAT(0) space and $K \subseteq X$ is a closed convex subset, then the asymptotic center $A(K, \{x_n\})$ consists of exactly one point.

A sequence $\{x_n\}$ in a CAT(0) space X is said to be $\Delta-$convergent to a point $x \in X$ if x is the unique asymptotic center of every subsequence of $\{x_n\}$. A bounded sequence $\{x_n\}$ is said to be regular with respect to K if for every subsequence $\{x'_n\} \subset \{x_n\}$, the following equality holds:

$$r(K, \{x_n\}) = r(K, \{x'_n\}).$$

The concept of $\Delta-$convergence is formally defined as follows.

Definition 2.3 ([34, 40]). *A sequence $\{x_n\}$ in a* CAT(0) *space X is said to be $\Delta-$convergent to a point $x \in X$ if x is the unique asymptotic center of every subsequence $\{u_n\}$ of $\{x_n\}$. In the present setting, we write $\Delta - \lim_{n\to\infty} x_n = x$, and refer to x as the $\Delta-$limit of the sequence $\{x_n\}$.*

Some brilliant known results in CAT(0) spaces can be found in previous studies [22–31] and references therein.

We now collect some elementary facts about CAT(0) spaces which will be used in the proofs of our main results. The following lemma can be found in [23, 35, 40, 41].

Lemma 2.4 ([23]). *If K is a closed convex subset of a complete* CAT(0) *space and $\{x_n\}$ is a bounded sequence in K, then the asymptotic center of $\{x_n\}$ is in K.*

Lemma 2.5 ([35]). *Let (X, d) be a* CAT(0) *space.*

(i) *For $x, y \in X$ and $u \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that*

$$d(x, z) = ud(x, y) \quad \text{and} \quad d(y, z) = (1 - u)d(x, y). \tag{2.2}$$

We use the notation $(1 - u)x \oplus uy$ for the unique point z satisfying (2.2).

(ii) *For $x, y, z \in X$ and $u \in [0, 1]$, we have*

$$d((1 - u)x \oplus uy, z) \leq (1 - u)d(x, z) + ud(y, z).$$

(iii) *For $x, y, z \in X$ and $u \in [0,1]$, we have*

$$d((1-u)x \oplus uy, z)^2 \leq (1-u)d(x,z)^2 + ud(y,z)^2 - u(u-1)d(x,y)^2.$$

Lemma 2.6 ([40]). *Every bounded sequence in a complete* CAT(0) *space has a* $\Delta-$*convergent subsequence.*

Theorem 2.7 ([41]). *Let (X,d) be a complete* CAT(0) *space and $\{x_n\} \subset X$ be a bounded sequence. Suppose that $\{x_n\}$ $\Delta-$converges to both x and y. Then $x = y$.*

Lemma 2.8 ([32]). *Let $\{\alpha_n\}, \{\beta_n\}$ be two sequences such that*

(i) $0 \leq \alpha_n, \beta_n < 1$,

(ii) $\beta_n \to 0$ *and* $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$.

Let $\{\gamma_n\}$ be a nonnegative real sequence such that $\sum_{n=1}^{\infty} \alpha_n \beta_n (1-\beta_n)\gamma_n$ is bounded. Then, $\{\gamma_n\}$ has a subsequence which converges to zero.

The existence of fixed points for asymptotically nonexpansive mappings in CAT(0) spaces was proved by Kirk [40] as the following result.

Theorem 2.9 ([40]). *Let K be a nonempty bounded closed and convex subset of a complete* CAT(0) *space X and let $t : K \to K$ be asymptotically nonexpansive. Then t has a fixed point.*

Theorem 2.10 ([37]). *Let X be a complete* CAT(0) *space and K be a nonempty bounded closed and convex subset of X, and $t : K \to K$ be an asymptotically nonexpansive mapping. Then $I - t$ is demiclosed at* 0.

Corollary 2.11 ([35]). *Let K be a closed and convex subset of a complete* CAT(0) *space X and let $t : K \to X$ be an asymptotically nonexpansive mapping. Let $\{x_n\}$ be a bounded sequence in K such that $\lim_{n\to\infty} d(tx_n, x_n) = 0$ and $\triangle-\lim_{n\to\infty} x_n = w$. Then $tw = w$.*

Lemma 2.12 ([39]). *Let X be a complete* CAT(0) *space and let $x \in X$. Suppose $\{\alpha_n\}$ is a sequence in $[a,b]$ for some $a, b \in (0,1)$, and $\{x_n\}, \{y_n\}$ are sequences in X such that $\limsup_{n\to\infty} d(x_n, x) \leq r$, $\limsup_{n\to\infty} d(y_n, x) \leq r$, and $\lim_{n\to\infty} d((1-\alpha_n)x_n \oplus \alpha_n y_n, x) = r$ for some $r \geq 0$. Then $\lim_{n\to\infty} d(x_n, y_n) = 0$.*

The following property can be found in [15].

Lemma 2.13. *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative numbers such that*

$$a_{n+1} \leq (1+b_n)a_n \quad \text{for all } n \geq 1.$$

If $\sum_{n=1}^{\infty} b_n$ converges, then $\lim_{n\to\infty} a_n$ exists. In particular, if there is a subsequence of $\{a_n\}$ which converges to 0, then $\lim_{n\to\infty} a_n = 0$.

One important concept in the analysis of weak convergence in metric spaces is Opial's condition. This condition provides a useful criterion for distinguishing the weak limit of a sequence from any other point in the space, and it is often employed to guarantee the convergence of various iterative schemes in fixed point theory.

We now recall the formal definition of Opial's condition in the setting of CAT(0) spaces.

Definition 2.14 ([40]). *Let (X, d) be a CAT(0) space. Then X is said to satisfy Opial's condition if for every sequence $\{x_n\} \subset X$ that converges weakly to a point $x \in X$, and for every point $y \in X$ with $y \neq x$, the following inequality holds:*

$$\liminf_{n\to\infty} d(x_n, x) < \liminf_{n\to\infty} d(x_n, y).$$

3. Main Results

In this section, we establish strong and weak convergence results for the iterative scheme defined by (1.8), with respect to a common fixed point of two asymptotically nonexpansive nonself-mappings in CAT(0) spaces. In preparation for the proofs of the main theorems, we present the following preliminary lemmas.

Lemma 3.1. *Let X be a complete CAT(0) space and let C be a nonempty closed convex nonexpansive retract of X, with P as a nonexpansive retraction. Let $T_1, T_2 : C \to X$ be two asymptotically nonexpansive nonself-mappings of C with sequences $\{k_n\}, \{\ell_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty, \sum_{n=1}^{\infty}(\ell_n - 1) < \infty, k_n \to 1, \ell_n \to 1$ as $n \to \infty$, and assume that $\mathbb{F} \neq \emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1)$. From an arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ using (1.8). If $q \in \mathbb{F}$, then $\lim_{n\to\infty} d(x_n, q)$ exists.*

Proof. Let $q \in \mathbb{F}$. Set $k_n = 1 + u_n$ and $\ell_n = 1 + v_n$. Since $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $\sum_{n=1}^{\infty}(\ell_n - 1) < \infty$, it follows that $\sum_{n=1}^{\infty} u_n < \infty$ and $\sum_{n=1}^{\infty} v_n < \infty$. In equation (1.8), we have using the iteration and properties of the retraction P, we have

$$\begin{aligned} d(y_n, q) &= d(P((1-\beta_n)x_n \oplus \beta_n T_2(PT_2)^{n-1}x_n), P(q)) \\ &\leq d((1-\beta_n)x_n \oplus \beta_n T_2(PT_2)^{n-1}x_n, q) \\ &\leq (1-\beta_n)d(x_n, q) + \beta_n d(T_2(PT_2)^{n-1}x_n, q) \\ &\leq (1-\beta_n)d(x_n, q) + \beta_n(1+v_n)d(x_n, q) \\ &= (1-\beta_n)d(x_n, q) + (\beta_n + \beta_n v_n)\, d(x_n, q) \\ &\leq (1+v_n)d(x_n, q). \end{aligned} \tag{3.1}$$

Similarly, for the next iteration:

$$\begin{aligned} d(x_{n+1}, q) &= d(P\left((1-\alpha_n)y_n \oplus \alpha_n T_1(PT_1)^{n-1}y_n\right), P(q)) \\ &\leq d((1-\alpha_n)y_n \oplus \alpha_n T_1(PT_1)^{n-1}y_n, q) \end{aligned} \tag{3.2}$$

$$\le (1-\alpha_n)d(y_n,q)+\alpha_n d(T_1(PT_1)^{n-1}y_n,q)$$
$$\le (1-\alpha_n)d(y_n,q)+\alpha_n(1+u_n)d(y_n,q)$$
$$=(1+u_n)d(y_n,q)$$
$$\le (1+u_n)(1+v_n)d(x_n,q)$$
$$=(1+u_n+v_n+u_nv_n)d(x_n,q)$$
$$=d(x_n,q)+(u_n+v_n+u_nv_n)d(x_n,q).$$

Hence, we have the recursive inequality:

$$\begin{aligned} d(x_{n+1},q) &\le d(x_n,q)+(u_n+v_n+u_nv_n)d(x_n,q) \\ &= (1+u_n+v_n+u_nv_n)d(x_n,q) \\ &\le e^{\sum_{n=1}^{\infty}(u_n+v_n+u_nv_n)}d(x_1,q). \end{aligned}$$

Since $\sum_{n=1}^{\infty}(u_n+v_n+u_nv_n)<\infty$, then $\{x_n\}$ is bounded. It implies that there exists a constant $M>0$ such that $d(x_n,q)\le M$ for all $n\ge 1$. So, $d(x_{n+1},q)\le d(x_n,q)+(u_n+v_n+u_nv_n)M$.

It follows from Lemma 2.13 that $\lim_{n\to\infty} d(x_n,q)$ exists. □

Lemma 3.2. *Let X be a complete* CAT(0) *space and let C be a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1,T_2:C\to X$ be two asymptotically nonexpansive nonself-mappings of C with sequences $\{k_n\},\{\ell_n\}\subset[1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n-1)<\infty$, $\sum_{n=1}^{\infty}(\ell_n-1)<\infty$, $k_n\to 1, \ell_n\to 1$ as $n\to\infty$, and assume that $\mathbb{F}\neq\emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[\varepsilon,1-\varepsilon]$ for some $\varepsilon\in(0,1)$. From an arbitrary $x_1\in C$, define the sequence $\{x_n\}$ by (1.8). Then $\lim_{n\to\infty} d(x_n,T_1x_n)=\lim_{n\to\infty} d(x_n,T_2x_n)=0$.*

Proof. Let $q\in\mathbb{F}$. Set $k_n=1+u_n$, $\ell_n=1+v_n$. By Lemma 3.1, we see that $\lim_{n\to\infty} d(x_n,q)$ exists. Assume that $\lim_{n\to\infty} d(x_n,q)=c$. Using (3.1), we have

$$d(y_n,q)\le(1+v_n)d(x_n,q). \tag{3.3}$$

Taking the lim sup on both sides of inequality (3.3), we obtain

$$\limsup_{n\to\infty} d(y_n,q)\le c. \tag{3.4}$$

In addition, since $d(T_1(PT_1)^{n-1}y_n,q)\le k_nd(y_n,q)$, taking the lim sup on both sides, we have

$$\limsup_{n\to\infty} d(T_1(PT_1)^{n-1}y_n,q)\le c. \tag{3.5}$$

From (1.8), we also have

$$\begin{aligned} d(x_{n+1}, q) &\le (1-\alpha_n)d(y_n, q) + \alpha_n d(T_1(PT_1)^{n-1}y_n, q) \\ &\le (1 + u_n + v_n + u_n v_n)d(x_n, q) \\ &= d(x_n, q) + (u_n + v_n + u_n v_n)d(x_n, q). \end{aligned} \tag{3.6}$$

Since $\sum_{n=1}^{\infty}(u_n + v_n + u_n v_n) < \infty$ and $\lim_{n\to\infty} d(x_{n+1}, q) = c$, letting $n \to \infty$ in inequality (3.6), we have

$$\lim_{n\to\infty}((1-\alpha_n)d(y_n, q) + \alpha_n d(T_1(PT_1)^{n-1}y_n, q)) = c. \tag{3.7}$$

By using (3.4), (3.5), (3.7) and Lemma 2.12, we have

$$\lim_{n\to\infty} d(T_1(PT_1)^{n-1}y_n, y_n) = 0. \tag{3.8}$$

In addition, $d(T_2(PT_2)^{n-1}x_n, q) \le \ell_n d(x_n, q)$, and taking the $\limsup$ on both sides, we obtain

$$\limsup_{n\to\infty} d(T_2(PT_2)^{n-1}x_n, q) \le c. \tag{3.9}$$

Using (1.8), we have

$$\begin{aligned} d(x_{n+1}, q) &\le (1-\alpha_n)d(y_n, q) + \alpha_n d(T_1(PT_1)^{n-1}y_n, q) \\ &\le (1-\alpha_n)d(y_n, q) + \alpha_n d(T_1(PT_1)^{n-1}y_n, y_n) + \alpha_n d(y_n, q) \\ &\le d(y_n, q) + d(T_1(PT_1)^{n-1}y_n, y_n). \end{aligned} \tag{3.10}$$

Taking the $\liminf$ on both sides of inequality (3.10), and using (3.8) and the fact that $\lim_{n\to\infty} d(x_{n+1}, q) = c$, we obtain

$$\liminf_{n\to\infty} d(y_n, q) \ge c. \tag{3.11}$$

It follows from (3.4) and (3.11) that

$$\lim_{n\to\infty} d(y_n, q) = c.$$

This implies that

$$\begin{aligned} c &= \lim_{n\to\infty} d(y_n, q) \\ &\le \lim_{n\to\infty}((1-\beta_n)d(x_n, q) + \beta_n d(T_2(PT_2)^{n-1}x_n, q)) \\ &\le \lim_{n\to\infty} d(x_n, q) = c, \end{aligned}$$

and so

$$\lim_{n\to\infty}((1-\beta_n)d(x_n, q) + \beta_n d(T_2(PT_2)^{n-1}x_n, q)) = c.$$

Using (3.9) and Lemma 2.12, we obtain

$$\lim_{n\to\infty} d(T_2(PT_2)^{n-1}x_n, x_n) = 0. \tag{3.12}$$

From $y_n = P((1-\beta_n)x_n \oplus \beta_n T_2(PT_2)^{n-1}x_n)$ and (3.12), we have

$$\begin{aligned} d(y_n, x_n) &= d(P((1-\beta_n)x_n \oplus \beta_n T_2(PT_2)^{n-1}x_n), x_n) \\ &\leq (1-\beta_n)d(x_n, x_n) + \beta_n d(T_2(PT_2)^{n-1}x_n, x_n) \\ &= \beta_n d(T_2(PT_2)^{n-1}x_n, x_n) \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.13}$$

In addition,

$$\begin{aligned} d(T_1(PT_1)^{n-1}x_n, x_n) &\leq d(T_1(PT_1)^{n-1}x_n, y_n) + d(y_n, x_n) \\ &\leq d(T_1(PT_1)^{n-1}x_n, T_1(PT_1)^{n-1}y_n) + d(T_1(PT_1)^{n-1}y_n, y_n) + \\ &\quad d(y_n, x_n) \\ &\leq k_n d(x_n, y_n) + d(T_1(PT_1)^{n-1}y_n, y_n) + d(y_n, x_n). \end{aligned} \tag{3.14}$$

Thus, it follows from (3.8) and (3.13) that

$$\lim_{n\to\infty} d(T_1(PT_1)^{n-1}x_n, x_n) = 0. \tag{3.15}$$

By using (1.8), we have

$$\begin{aligned} d(x_{n+1}, x_n) &\leq (1-\alpha_n)d(y_n, x_n) + \alpha_n d(T_1(PT_1)^{n-1}y_n, x_n) \\ &\leq (1-\alpha_n)d(y_n, x_n) + \alpha_n d(T_1(PT_1)^{n-1}y_n, y_n) + \alpha_n(y_n, x_n) \\ &\leq d(y_n, x_n) + d(T_1(PT_1)^{n-1}y_n, y_n). \end{aligned}$$

It follows from (3.8) and (3.13) that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{3.16}$$

Using (3.15) and (3.16), we have

$$\begin{aligned} d(x_{n+1}, T_1(PT_1)^{n-1}x_{n+1}) &\leq d(x_{n+1}, x_n) + d(T_1(PT_1)^{n-1}x_{n+1},\ T_1(PT_1)^{n-1}x_n) + \\ &\quad d(T_1(PT_1)^{n-1}x_n, x_n) \\ &\leq d(x_{n+1}, x_n) + k_n d(x_{n+1}, x_n) + d(T_1(PT_1)^{n-1}x_n, x_n) \end{aligned}$$

Therefore,

$$d(x_{n+1}, T_1(PT_1)^{n-1}x_{n+1}) \to 0 \ \text{ as } n \to \infty. \tag{3.17}$$

In addition,

$$\begin{aligned} d(x_{n+1}, T_1(PT_1)^{n-2}x_{n+1}) &\leq d(x_{n+1}, x_n) + d(x_n, T_1(PT_1)^{n-2}x_n) + \\ &\quad d(T_1(PT_1)^{n-2}x_{n+1}, T_1(PT_1)^{n-2}x_n) \\ &\leq d(x_{n+1}, x_n) + d(x_n, T_1(PT_1)^{n-2}x_n) + \xi d(x_{n+1}, x_n), \end{aligned} \tag{3.18}$$

where $\xi = \sup\{k_n : n \geq 1\}$. It follows from (3.16) and (3.17) that

$$\lim_{n\to\infty} d(x_{n+1}, T_1(PT_1)^{n-2}x_{n+1}) = 0. \tag{3.19}$$

We define $(PT_1)^{1-1}$ as the identity map on C. Therefore, by inequalities (3.17) and (3.19), we obtain

$$\begin{aligned}
d(x_{n+1}, T_1x_{n+1}) &\leq d(x_{n+1}, T_1(PT_1)^{n-1}x_{n+1}) + d(T_1(PT_1)^{n-1}x_{n+1}, T_1x_{n+1}) \\
&= d(x_{n+1}, T_1(PT_1)^{n-1}x_{n+1}) + d(T_1(PT_1)^{1-1}(PT_1)^{n-1}x_{n+1}, T_1(PT_1)^{1-1}x_{n+1}) \\
&\leq d(x_{n+1}, T_1(PT_1)^{n-1}x_{n+1}) + \xi d((PT_1)^{n-1}x_{n+1}, x_{n+1}) \\
&= d(x_{n+1}, T_1(PT_1)^{n-1}x_{n+1}) + \xi d((PT_1)(PT_1)^{n-2}x_{n+1}, Px_{n+1}) \\
&\leq d(x_{n+1}, T_1(PT_1)^{n-1}x_{n+1}) + \xi d(T_1(PT_1)^{n-2}x_{n+1}, x_{n+1}) \to 0.
\end{aligned}$$

This implies that $\lim_{n\to\infty} d(x_n, T_1x_n) = 0$. Similarly, we may show that $\lim_{n\to\infty} d(x_n, T_2x_n) = 0$. Hence, the proof is complete. □

Theorem 3.3. *Let X be a complete* CAT(0) *space and let C be a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2 : C \to X$ be two asymptotically nonexpansive nonself-mappings of C with sequences $\{k_n\}, \{\ell_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(\ell_n - 1) < \infty$, $k_n \to 1$, $\ell_n \to 1$ as $n \to \infty$, and assume that $\mathbb{F} \neq \emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$. Let $\{x_n\}$ and $\{y_n\}$ be the sequences defined by (1.8). If one of T_1 or T_2 is completely continuous, then $\{x_n\}$ and $\{y_n\}$ converge strongly to a common fixed point of T_1 and T_2.*

Proof. By Lemma 3.1, the sequence $\{x_n\}$ is bounded. In addition, by Lemma 3.2, we have

$$\lim_{n\to\infty} d(x_n, T_1x_n) = 0 \text{ and } \lim_{n\to\infty} d(x_n, T_2x_n) = 0,$$

which implies that $\{T_1x_n\}$ and $\{T_2x_n\}$ are also bounded.

If T_1 is completely continuous, then there exists a subsequence $\{T_1x_{n_j}\} \subset \{T_1x_n\}$ such that

$$T_1x_{n_j} \to q \quad \text{as } j \to \infty.$$

It follows from Lemma 3.2 that $\lim_{j\to\infty} d(x_{n_j}, T_1x_{n_j}) = \lim_{j\to\infty} d(x_{n_j}, T_2x_{n_j}) = 0$. By the continuity of T_1 and Theorem 2.10, we obtain $\lim_{j\to\infty} d(x_{n_j}, q) = 0$, and hence $q \in \mathbb{F}$. Furthermore, by Lemma 3.1, we have that $\lim_{n\to\infty} d(x_n, q)$ exists. Thus, $\lim_{n\to\infty} d(x_n, q) = 0$. From (3.13), we also have $\lim_{n\to\infty} d(y_n, x_n) = 0$, which implies that $\lim_{n\to\infty} d(y_n, q) = 0$. Thus, the proof is complete. □

Theorem 3.4. *Let X be a complete* CAT(0) *space and let C be a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2 : C \to X$ be two asymptotically nonexpansive nonself-mappings of C with sequences $\{k_n\}, \{\ell_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(\ell_n - 1) < \infty$, $k_n \to 1$, $\ell_n \to 1$ as $n \to \infty$, and assume that $\mathbb{F} \neq \emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[\varepsilon, 1-\varepsilon]$ for* *some* $\varepsilon \in (0,1)$. *Let* $\{x_n\}$ *and* $\{y_n\}$ *be the sequences defined by* (1.8). *If one of* T_1 *or* T_2 *is semi-compact, then* $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to a common fixed point of* T_1 *and* T_2.

Proof. Since one of T_1 or T_2 is semi-compact, and $\{x_n\}$ is bounded with

$$\lim_{n\to\infty} d(x_n, T_1x_n) = \lim_{n\to\infty} d(x_n, T_2x_n) = 0,$$

there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $x_{n_j} \to q$ strongly in X. It follows from Theorem 2.10 that $q \in \mathbb{F}$. By Lemma 3.1, $\lim_{n\to\infty} d(x_n, q)$ exists.

Since a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converges strongly to q, we conclude that the full sequence $\{x_n\}$ converges strongly to the common fixed point $q \in \mathbb{F}$.

From (3.13), we have $\lim_{n\to\infty} d(y_n, x_n) = 0$, and it follows that $\lim_{n\to\infty} d(y_n, q) = 0$. Therefore, the theorem is proved. □

Finally, we prove that the iterative scheme (1.8) converges weakly in the case of two asymptotically nonexpansive nonself-mappings defined on a complete CAT(0) space that satisfies Opial's condition.

Theorem 3.5. *Let* X *be a uniformly convex complete* CAT(0) *space which satisfies Opial's condition and let* C *be a nonempty closed convex nonexpansive retract of* X *with* P *as a nonexpansive retraction. Let* $T_1, T_2 : C \to X$ *be two asymptotically nonexpansive nonself-mappings of* C *with sequences* $\{k_n\}, \{\ell_n\} \subset [1,\infty)$ *such that* $\sum_{n=1}^{\infty}(k_n - 1) < \infty,\ \sum_{n=1}^{\infty}(\ell_n - 1) < \infty,\ k_n \to 1,\ \ell_n \to 1$ *as* $n \to \infty$, *and assume that* $\mathbb{F} \neq \emptyset$. *Suppose that* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are real sequences in* $[\varepsilon, 1-\varepsilon]$ *for some* $\varepsilon \in (0,1)$. *Let* $\{x_n\}$ *and* $\{y_n\}$ *be the sequences defined by* (1.8). *Then* $\{x_n\}$ *and* $\{y_n\}$ *converge weakly to a common fixed point of* T_1 *and* T_2.

Proof. It follows from Lemma 3.2 that

$$\lim_{n\to\infty} d(x_n, T_1x_n) = \lim_{n\to\infty} d(x_n, T_2x_n) = 0.$$

Since X is uniformly convex and $\{x_n\}$ is bounded, we may assume that $x_n \rightharpoonup u$ weakly as $n \to \infty$, without loss of generality. By Theorem 2.10, we have $u \in \mathbb{F}$.

Suppose that subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ converge weakly to u and v, respectively. According to Theorem 2.10, we have that u and v belong to the set $\mathbb{F}$. By Lemma 3.1, $\lim_{n\to\infty} d(x_n, u)$ and $\lim_{n\to\infty} d(x_n, v)$ exist. It follows from Theorem 2.7 that $u = v$. Therefore, $\{x_n\}$ converges weakly to a common fixed point of T_1 and T_2.

Moreover, $\lim_{n\to\infty} d(y_n, x_n) = 0$ as proved in Lemma 3.2, and since $x_n \rightharpoonup u$ weakly as $n \to \infty$, it follows that $y_n \rightharpoonup u$ weakly as $n \to \infty$. This concludes the proof of the theorem. □

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