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ENUMERATION OF UNLABELED P-SERIES BY USING THE POSET MATRIX

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Abstract. We give an exact enumeration of the unlabeled P-series, a subclass of the frequently studied series-parallel posets. In this enumeration method, we determine the number of unlabeled P-series according to the number of direct terms (connected components) of the posets. Here, we use the results regarding the matrix recognition of P-series by using the poset matrix. We also give an algorithm to determine the values of the parameters involved in the enumeration formula and to compute the number of unlabeled P-series with a certain number of elements. We show that the enumeration algorithm runs in polynomial time. We include the numerical results for the numbers of unlabeled P-series up to 76 elements.

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1. Introduction

A common problem in the theory of mathematical structures is to recognize the classes of structures that satisfy some common structural properties. For a particular class of mathematical structures, conjectures for such recognitions can be made by observing various examples. In the cases of finite posets, this can be done efficiently with the help of computer codes by counting and generating, if possible, all the nonisomorphic structures belonging to the class of structures under consideration. This is one of the main reasons for which the recognitions and enumerations of several classes of finite lattices [8,10], posets [1,7,12,13], graphs [2,4], and topologies [6,10] were considered by numerous authors.

The class of series-parallel posets contains the P-series as a subclass, and the class of P-series contains the P-graphs as a subclass. Several methods for the recognitions [11,14–16] and enumerations [5,7,9,13]

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of these computationally tractable classes of posets were considered in the literature. We know that $\bar{G}(n)$, the number of n-element unlabeled P-graphs, can be given explicitly as $\bar{G}(n) = 2^{n-1}$, $n \geq 1$. On the other hand, Stanley [18] gave the generating functions for the enumeration of unlabeled series-parallel posets, see the sequence A003430 in OEIS [17], and later, El-Zahar et al. [5] gave the height counting of the unlabeled series-parallel posets by modifying the generating functions given by Stanley [18] with the height of a poset as an additional parameter. Unfortunately, neither any explicit formula for the enumeration of unlabeled P-series was obtained nor any other methods for the enumeration of these posets were considered. In this article, we give an exact enumeration method to determine $S(n)$, the number of n-element unlabeled P-series. See the integer sequences A349276 and A349488 that we contributed to OEIS [17]. Here, we compute $S(n)$ according to the number of direct terms (connected components) of the posets. For these data, see the integer sequence A350635 contributed to OEIS [17].

In Section 2, we recall some basic terminologies related to the posets and P-series. Here, we also recall common definitions related to the poset matrix and the results regarding the matrix recognition of P-series by using the poset matrix. In Section 3, we obtain the results giving an explicit formula for the enumeration of connected P-series (equivalently, connected P-graphs). In Section 4, we obtain the results regarding the enumeration of unlabeled disconnected P-series. In Section 5, we give an algorithm to determine the values of the parameters involved in the enumeration formula and to compute the numbers $S(n)$, $n \geq 2$. Here, we show that the enumeration algorithm runs in polynomial time. Also, in Section 6, we include the numerical data for $S(n)$, $1 \leq n \leq 76$, obtained by implementing the enumeration algorithm into the computer.

2. Preliminaries

2.1. **Posets and P-series.** A *poset* (*partially ordered set*) is a structure $\mathbf{A} = \langle A, \leqslant \rangle$ consisting of the nonempty set A with the order relation $\leqslant$ on A. A poset $\mathbf{A}$ is called *finite* if the underlying set A is finite. Throughout this paper, we assume that every poset is finite and nonempty. Let $\mathbf{A} = \langle A, \leqslant_A \rangle$ and $\mathbf{B} = \langle B, \leqslant_B \rangle$ be two posets. A bijective map $\phi : A \to B$ is called an *order isomorphism* if for all $x, y \in A$, $x \leqslant_A y$ if and only if $\phi(x) \leqslant_B \phi(y)$. We write $\mathbf{A} \cong \mathbf{B}$ whenever $\mathbf{A}$ and $\mathbf{B}$ are order isomorphic. Also, by a collection of isomorphic (analogously, nonisomorphic) posets, we mean that the posets are *pairwise* isomorphic (nonisomorphic). For further details on the basics of posets, we refer the readers to the classical book by Davey and Priestley [3].

Here, we use the notations $\mathbf{1}$ for the singleton poset, $\mathbf{C}_n$, $n \geq 1$, for the n-element chain posets, $\mathbf{I}_n$, $n \geq 1$, for the n-element antichain posets, $\mathbf{D}_n$, $n \geq 4$, for the n-element diamond posets, and $\mathbf{B}_{m,n}$, $m \geq 1$, $n \geq 1$, for the complete bipartite posets with m minimal elements and n maximal elements. In particular, we have $\mathbf{C}_1 \cong \mathbf{I}_1 \cong \mathbf{1}$ and $\mathbf{C}_2 \cong \mathbf{B}_{1,1}$. We also use the notations $\mathbf{A} + \mathbf{B}$ and $\mathbf{A} \oplus \mathbf{B}$ to denote, respectively, the direct sum and ordinal sum of the posets $\mathbf{A}$ and $\mathbf{B}$. Here, the posets $\mathbf{A}$ and $\mathbf{B}$ are called the *direct terms* of $\mathbf{A} + \mathbf{B}$ and the *ordinal terms* of $\mathbf{A} \oplus \mathbf{B}$. A poset having two or more direct terms is called *disconnected*, otherwise, it is called *connected*.

A poset $\mathbf{G}$ is called a *P-graph* if it can be expressed as the ordinal sum of the antichain posets, that is, there exist the antichain posets $\mathbf{I}_{n_i}, 1 \leq i \leq m$, such that $\mathbf{G} = \mathbf{I}_{n_1} \oplus \mathbf{I}_{n_2} \oplus \cdots \oplus \mathbf{I}_{n_m}$. For every $n \geq 2$, the poset $\mathbf{I}_n$ is a disconnected P-graph. Also, the posets $\mathbf{B}_{1,2} \cong \mathbf{1} \oplus \mathbf{I}_2$ and $\mathbf{D}_5 \cong \mathbf{1} \oplus \mathbf{I}_3 \oplus \mathbf{1}$ are some connected P-graphs. Note that all the P-graphs except the antichains $\mathbf{I}_n$, $n \geq 2$, are connected posets. A poset $\mathbf{S}$ is called a *P-series* if it is either a P-graph or it can be expressed as the direct sum of the P-graphs. Thus, every P-graph is trivially a P-series. Also, the P-graphs except the antichains $\mathbf{I}_n$, $n \geq 2$, are all connected P-series. On the other hand, a poset $\mathbf{S}$ is a disconnected P-series if there exist the P-graphs $\mathbf{G}_i, 2 \leq i \leq n$, such that $\mathbf{S} = \mathbf{G}_1 + \mathbf{G}_2 + \cdots + \mathbf{G}_n$. For $m > 1$ and $n \geq 1$, the posets $\mathbf{C}_m + \mathbf{C}_n$ are disconnected P-series which are not P-graphs. A poset is called *series-parallel* if it can be expressed as the sum of the singleton posets using only the direct sum and ordinal sum of posets. Every P-series, and therefore, every P-graph is trivially series-parallel. Also, the posets $\mathbf{1} \oplus (\mathbf{1} + \mathbf{C}_2)$ and $(\mathbf{1} + \mathbf{C}_2) \oplus \mathbf{1}$ are series-parallel which are not P-series.

2.2. **Poset matrix and recognition of P-series.** Mohammad and Talukder [11] introduced the notion of poset matrix. A square $(0,1)$-matrix $M_m = [a_{ij}], 1 \leq i, j \leq m$, is called a *poset matrix* if and only if the following conditions hold:

(1) $a_{ii} = 1$ for all $1 \leq i \leq m$ i.e. M_m is reflexive,
(2) $a_{ij} = 1$ and $a_{ji} = 1$ imply $i = j$ i.e. M_m is antisymmetric,
(3) $a_{ij} = 1$ and $a_{jk} = 1$ imply $a_{ik} = 1$ i.e. M_m is transitive.

An upper (or lower) triangular $(0,1)$-matrix with entries 1s in the main diagonal is reflexive and antisymmetric clearly. Therefore, an upper (or lower) triangular $(0,1)$-matrix with entries 1s in the main diagonal is a poset matrix if it is transitive only.

Throughout this paper, we use the notations $M_{m,n}$ for an m-by-n matrix and M_m for a square matrix of order m. In particular, we use the notations I_n, O_n, and Z_n, for the identity matrix, the matrix with entries 1s only, and the matrix with entries 0s only, respectively, all of order n. We use also the notation C_n for the matrix $[c_{ij}], 1 \leq i, j \leq n$ defined as $c_{ij} = 1$ for all $i \leq j$ and $c_{ij} = 0$ otherwise. Then, for $n \geq 1$, the matrices I_n and C_n are all poset matrices because these are upper triangular and clearly transitive.

Let $M_m = [a_{ij}], 1 \leq i, j \leq m$ be a poset matrix. We associate a poset $\mathbf{A} = \langle A, \leqslant \rangle$ to M_m, where $A = \{x_1, x_2, \ldots, x_m\}$ and x_i corresponds the i-th row (or column) of M_m, by defining the order relation $\leqslant$ on A such that for all $1 \leq i, j \leq m$, we have $x_i \leqslant x_j$ if and only if $a_{ij} = 1$. Then we say that the poset matrix M_m *represents* the poset $\mathbf{A}$ and vice versa. Clearly, for every $n \geq 1$, the poset matrices I_n and C_n represent the antichain poset $\mathbf{I}_n$ and the chain poset $\mathbf{C}_n$, respectively. Also, the poset matrices L and L^t, as given in the following example, represent the complete bipartite posets $\mathbf{B}_{2,1}$ and $\mathbf{B}_{1,2}$, respectively.

Example 2.1.

$$L = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad L^t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Let M_m be a poset matrix. For some $1 \leq i, j \leq m$, the interchanges of i-th and j-th rows along with the interchanges of i-th and j-th columns in M_m is called *(i,j)-relabeling* of M_m. The following example shows a relabeling of the poset matrix L^t (Example 2.1).

Example 2.2.

$$L^t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{\text{(1,3)-relabeling}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = L^{'}$$

The following results, obtained by Mohammad and Talukder [11], give interpretations of the relabeling in a poset matrix.

Theorem 2.1. [11] *Any relabeling of a poset matrix is a poset matrix and it represents the same poset up to isomorphism.*

Theorem 2.2. [11] *Every poset matrix can be relabeled to an upper (or lower) triangular matrix with 1s in the main diagonal by a finite number of relabeling.*

Let M_m and N_n be any poset matrices. We write $M_m \oplus N_n$ and $M_m \boxplus N_n$ to denote, respectively, the direct sum and ordinal sum of the matrices M_m and N_n. Note here that the matrices M_m and N_n are called the *direct terms* of $M_m \oplus N_n$ and the *ordinal terms* of $M_m \boxplus N_n$. Throughout this paper, by a poset matrix we mean a poset matrix in upper triangular form, and by M_n we mean a poset matrix of order n.

Mohammad and Talukder [11] defined the properties of block of 0s, block of 1s, and complete blocks of 1s in a poset matrix $M_n = [a_{ij}], 1 \leq i, j \leq n$, as follows:

(1) A poset matrix M_n satisfies the property of *block of 0s* (analogously, *block of 1s*) of length r, $1 \leq r < n$, if and only if $a_{ij} = 0$ $(a_{ij} = 1)$ for all $1 \leq i \leq r$ and $r + 1 \leq j \leq n$.

(2) A poset matrix M_n satisfies the property of *complete blocks of 1s* of length $\{r_1, r_2, \ldots, r_m\}$, where $0 \leq r_1 < r_2 < \cdots < r_m < n$, if and only if for all $1 \leq i < j \leq n$,

$$a_{ij} = \begin{cases} 1, & \text{if } 1 \leq i \leq r_k \text{ and } r_k + 1 \leq j \leq n \; (1 \leq k \leq m), \\ 0, & \text{otherwise.} \end{cases}$$

A length l in the property of complete blocks of 1s is called *nonzero* if and only if $l \neq \{0\}$. Obviously, for every $n \geq 2$, both the poset matrices I_n and C_n satisfy the properties of block of 0s and block of 1s, respectively, of the lengths $1, 2, \ldots, n-1$. Also, for every $n \geq 1$, matrix I_n satisfies the property of complete blocks of 1s of length $\{0\}$, and for every $n \geq 2$, matrix C_n satisfies the property of complete blocks of 1s of length $\{1, 2, \ldots, n-1\}$. In particular, the matrices $1 \oplus C_2$, $C_2 \oplus 1$, and I_3, as in the following example, satisfy the property of block of 0s of length 1, length 2, and lengths 1, 2, respectively.

Example 2.3.

$$1 \oplus C_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad C_2 \oplus 1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that for any relabeling, a matrix M_n can satisfy one of the two properties at a time but not both the properties together. We have the following immediate consequences.

Theorem 2.3. [11] *For $n \geq 2$,*

(1) *a poset matrix M_n satisfies the property of block of 0s of lengths $n_1, n_2, \ldots, n_m$ if and only if $M_n = M_{n_1} \oplus M_{n_2-n_1} \oplus \cdots \oplus M_{n-n_m}$.*

(2) *a poset matrix M_n satisfies the property of block of 1s of lengths $n_1, n_2, \ldots, n_m$ if and only if $M_n = M_{n_1} \boxplus M_{n_2-n_1} \boxplus \cdots \boxplus M_{n-n_m}$.*

We observe that all the poset matrices in Example 2.3 represent disconnected posets. However, we have the following results regarding the matrix recognitions of the connected and disconnected posets in general.

Theorem 2.4. *Let M_n represent the poset $\mathbf{P} \ncong \mathbf{1}$. Then*

(1) *$\mathbf{P}$ is connected if M_n can be relabeled in such a form that it satisfies the property of block of 1s.*

(2) *$\mathbf{P}$ is disconnected if and only if M_n can be relabeled in such a form that it satisfies the property of block of 0s.*

Proof. Proofs follow by Theorem 2.3 and the definitions of connected posets and disconnected posets. □

Our method for the enumeration of P-series is based mainly on the following results regarding the recognition of P-series by using the poset matrix.

Theorem 2.5. [11] *Let M_n represent the poset $\mathbf{P}$. Then $\mathbf{P}$ is a P-series if and only if M_n can be relabeled in such a form that either M_n satisfies the property of complete blocks of 1s or M_n satisfies the property of block of 0s, and every direct term of M_n satisfies the property of complete blocks of 1s.*

In general, the converse of the result obtained in the first part of Theorem 2.4 is not true. Because, the 4-element N-shaped poset is connected but the poset matrix that represents this poset does not satisfy the property of block of 1s for any labeling. However, the following result regarding the recognition of connected P-series (equivalently, connected P-graphs) shows that the converse of the aforesaid result holds in the cases of connected P-series.

Corollary 2.1. *Let M_n represent the poset $\mathbf{P} \ncong \mathbf{1}$. Then*

(1) **P** *is a connected* P*-series (equivalently, a connected* P*-graph) if and only if* M_n *can be relabeled in such a form that it satisfies the property of complete blocks of 1s of some nonzero lengths.*

(2) **P** *is a disconnected* P*-series if and only if* M_n *can be relabeled in such a form that it satisfies the property of block of 0s and every direct term satisfies the property of complete blocks of 1s of some nonzero lengths.*

Proof. Proofs follow by Theorem 2.4 and Theorem 2.5. □

3. Enumeration of connected P-series

Let $G(n)$ be the number of unlabeled connected P-series with n elements. Since the singleton poset **1** is connected, we have $G(1) = 1$. Let M_n, $n \geq 2$, represent a connected P-series, that is, a connected P-graph. By Corollary 2.1, the matrix M_n satisfies the property of complete blocks of 1s of some nonzero lengths. Then we observe the following.

(1) An M_2 can satisfy the property of complete blocks of 1s of nonzero length $\{1\}$ only. Then M_2 represents the connected P-series $\mathbf{C}_2$ only. Thus, $G(2) = 1$.

(2) An M_3 can satisfy the property of complete blocks of 1s of nonzero lengths $\{1\}$, $\{2\}$, and $\{1, 2\}$. Thus, M_3 represents 3 connected P-series all of which are nonisomorphic. This gives $G(3) = 3$.

(3) All M_4 that satisfy the property of complete blocks of 1s are given in Table 1. This shows that M_4 represents 7 connected P-series all of which are nonisomorphic. Thus, $G(4) = 7$.

This intuition gives that an M_n, $n \geq 2$ can satisfy the property of complete blocks of 1s of nonzero lengths equal to a nonempty subset of $\{1, 2, \ldots, n-1\}$ such that in every case it represents a single connected P-series. This result gives an explicit formula for the enumeration of n-element unlabeled connected P-series (equivalently, connected P-graphs) for all $n \geq 2$.

Theorem 3.1. *For* $n \geq 2$*, let* $G(n)$ *be the number of* n*-element unlabeled connected* P*-series. Then* $G(n) = 2^{n-1} - 1$.

Proof. By Corollary 2.1, $G(n)$, $n \geq 2$, equals the number of distinct M_n that satisfies the property of complete blocks of 1s of all possible nonzero lengths. We observe that M_n can satisfy the property of complete blocks of 1s of nonzero lengths equal to a nonempty subset of $S = \{1, 2, \ldots, n-1\}$. Since there are $2^{n-1} - 1$ nonempty subsets of S, an M_n can satisfy the property of complete blocks of 1s of nonzero lengths in $2^{n-1} - 1$ ways. Thus, M_n represents $2^{n-1} - 1$ connected P-series. To show that all the connected P-series represented by M_n are nonisomorphic, let M_n satisfy the property of complete blocks of 1s of lengths $\{n_1, n_2, \ldots, n_m\}$. Then M_n satisfies the property of block of 1s of lengths n_1, n_2, $\ldots$, n_m such that the ordinal terms are $I_{n_i - n_{i-1}}$, $1 \leq i \leq m+1$, where we assume $n_0 = 0$ and $n_{m+1} = n$. Since for every $1 \leq i \leq m+1$, the ordinal term $I_{n_i - n_{i-1}}$ represents a single poset, and the ordinal sum of the poset matrices is not commutative, in this case, M_n represents a single connected P-series. Since

TABLE 1. All M_4 that satisfy the property of complete blocks of 1s.

$$1 \boxplus I_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad I_2 \boxplus I_2 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$I_3 \boxplus 1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad 1 \boxplus I_2 \boxplus 1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$1 \boxplus 1 \boxplus I_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad I_2 \boxplus 1 \boxplus 1 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$1 \boxplus 1 \boxplus 1 \boxplus 1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

all the $2^{n-1}-1$ lengths satisfied by M_n are different, M_n represents $2^{n-1}-1$ nonisomorphic connected P-series. Therefore, we have $G(n)$ as follows:

$$G(n) = 2^{n-1} - 1, \text{ where } n \geq 2. \tag{1}$$

□

4. ENUMERATION OF DISCONNECTED P-SERIES

Let $D(n)$, $n \geq 1$, be the numbers of n-element unlabeled disconnected P-series. Then $D(1) = 0$. For every $n \geq 2$, let M_n represent a disconnected P-series, that is, a P-series with two or more direct terms. By Corollary 2.1, the matrix M_n satisfies the property of block of 0s such that every direct term satisfies the property of complete blocks of 1s of nonzero lengths. We observe the following.

(1) An M_2 can satisfy the property of block of 0s of length 1 only. Then both the direct terms are M_1. Thus, $D(2) = 1$.

(2) An M_3 can satisfy the property of block of 0s of length 1, length 2, and lengths 1, 2. Here, the matrices M_3 that satisfy the property of block of 0s of length 1 and length 2 represent the isomorphic P-series. Therefore, $D(3) = 2$.

(3) All M_4 that satisfy the property of block of 0s and represent nonisomorphic P-series are given in Table 2. Here,

  (a) the matrices M_4 that satisfy the property of block of 0s of length 1 and length 3 represent isomorphic P-series.

  (b) the matrices M_4 that satisfy the property of block of 0s of lengths 1, 2 and lengths 1, 3 represent isomorphic P-series.

  Therefore, $D(4) = 6$.

Table 2. All M_4 that satisfy the property of block of 0s and represent nonisomorphic P-series.

$$1 \oplus (1 \boxplus I_2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad 1 \oplus (I_2 \boxplus 1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$1 \oplus C_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad C_2 \oplus C_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$1 \oplus 1 \oplus C_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad 1 \oplus 1 \oplus 1 \oplus 1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since the direct sum of the posets is commutative, the matrices M_n that satisfy the property of block of 0s of different lengths can represent isomorphic P-series. Therefore, in the case of enumeration of M_n that represent nonisomorphic disconnected P-series, to exclude the cases when the matrices M_n satisfy the property of block of 0s and represent isomorphic P-series, we restrict firstly the lengths in the property of block of 0s to be *nondecreasing inter-distant* (as defined below). In addition, we count the number of *repetitions* of the consecutive direct terms of *same order* and reduce the number of M_n accordingly.

Definition 4.1. The lengths $n_1, n_2, \ldots, n_m$, where $1 \leq m \leq n-1$, chosen as a subcollection of the integers $1, 2, \ldots, n-1$ are called *nondecreasing inter-distant* if the following condition holds:

$$n_1 \leq n_2 - n_1 \leq \cdots \leq n_{i+1} - n_i \leq \cdots \leq n - n_m.$$

For example, all the nondecreasing inter-distant lengths $l(m, j)$, $1 \leq m \leq 5$, $1 \leq j \leq p_m$, for some integers p_m, are given in Table 3.

TABLE 3. Nondecreasing inter-distant lengths $l(m, j)$, $1 \leq m \leq 5$, $1 \leq j \leq p_m$, for some integer $p_m \leq \binom{5}{m}$.

m	j	$l(m, j)$
1	1	1
1	2	2
1	3	3
2	1	1, 2
2	2	1, 3
2	3	2, 4
3	1	1, 2, 3
3	2	1, 2, 4
4	1	1, 2, 3, 4
5	1	1, 2, 3, 4, 5

For all $1 \leq m \leq n-1$, it is clear that $p_m \leq \binom{n_m}{m}$. By inspection, we have $p \leq n^2 \leq \binom{n_m}{m}$. However, the following result gives an upper bound of n_m.

Lemma 4.1. *Let the lengths* $n_1, n_2, \ldots, n_m$ *be nondecreasing inter-distant. Then* $n_m \leq \lfloor \frac{mn}{m+1} \rfloor$ *for every* $1 \leq m \leq n-1$.

Proof. The proof follows from the fact that $n_m + \frac{n_m}{m} \leq n$ holds for all $1 \leq m \leq n-1$. □

Now we establish the result regarding the enumeration of M_n that satisfy the property of block of 0s of the nondecreasing inter-distant lengths $n_1, n_2, \ldots, n_m$, and represent nonisomorphic disconnected P-series.

Theorem 4.1. *For $n \geq 2$, let M_n satisfy the property of block of 0s of nondecreasing inter-distant lengths n_1, n_2, $\ldots$, n_m, where t_k, $1 \leq k \leq q$, for some $q \leq m+1$, be the number of the k-th group of consecutive lengths of equal inter-distance r_k, such that the direct term M_{r_i} represents $G(r_i)$ nonisomorphic connected P-series for every $1 \leq i \leq m+1$. Then $\bar{D}(n)$, the number of nonisomorphic disconnected P-series represented by M_n, can be given as $\bar{D}(n) = \prod_{k=1}^{q} \binom{G(r_k)+t_k}{1+t_k}$, $n \geq 2$.*

Proof. For $n \geq 2$, let M_n satisfy the property of block of 0s of nondecreasing inter-distant lengths n_1, n_2, $\ldots$, n_m, where for some $q \leq m+1$,

$$
\begin{aligned}
r_1 &= n_1 - n_0 = n_2 - n_1 = \cdots = n_{t_1+1} - n_{t_1}, \\
r_2 &= n_{t_1+2} - n_{t_1+1} = \cdots = n_{t_1+t_2+2} - n_{t_1+t_2+1}, \\
&\vdots \\
r_q &= n_{t_1+\cdots+t_{q-1}+q} - n_{t_1+\cdots+t_{q-1}+q-1} = \cdots = n - n_m,
\end{aligned}
$$

such that $r_1 < r_2 < \cdots < r_q$ and $m = t_1 + \cdots + t_q + q - 1$. Here, we assume $n_0 = 0$ and $n_{m+1} = n$. This shows that for every $1 \leq k \leq q$, all the $t_k + 1$ consecutive direct terms equal M_{r_k} which represents $G(r_k)$ nonisomorphic connected P-series. Therefore, the direct sum $M_{(t_k+1)r_k}$ represents the nonisomorphic P-series having direct terms as a subcollection of the $t_k + 1$ posets each of which is chosen from one of the same $t_k + 1$ collections of $G(r_k)$ nonisomorphic connected P-series. This implies that $\bar{D}((t_k+1)r_k)$, the number of nonisomorphic disconnected P-series represented by $M_{(t_k+1)r_k}$, equals the number of combinations of $t_k + 1$ items chosen from $G(r_k) + t_k$ distinct items. Therefore, for every $1 \leq k \leq q$, the number $\bar{D}((t_k+1)r_k)$ can be given as follows:

$$\bar{D}((t_k+1)r_k) = \binom{G(r_k)+t_k}{1+t_k}. \tag{2}$$

Since $r_1 < r_2 < \cdots < r_q$, for every $1 \leq k \leq q$, the direct term $M_{(t_k+1)r_k}$ represents nonisomorphic P-series of distinct orders. This shows that M_n represents the nonisomorphic P-series having the direct terms as a subcollection of q P-series each of which is chosen from one of the q collections of $\bar{D}((t_k+1)r_k)$ nonisomorphic P-series. Therefore, $\bar{D}(n)$ equals the number of combinations of q items each of which is chosen from one of the q disjoint sets of $\bar{D}((t_k+1)r_k)$ distinct items. Therefore, $\bar{D}(n)$ can be given as follows:

$$\bar{D}(n) = \bar{D}((t_1+1)r_1) \times \cdots \times \bar{D}((t_q+1)r_q) = \prod_{k=1}^{q} \bar{D}((t_k+1)r_k).$$

Then by using the equation (2), we have

$$\bar{D}(n) = \prod_{k=1}^{q} \binom{G(r_k)+t_k}{1+t_k}, \text{ where } n \geq 2. \tag{3}$$

□

Now we establish the result regarding the computation of $D(n)$, the number of n-element unlabeled disconnected P-series for $n \geq 2$.

Theorem 4.2. *For $n \geq 2$, let M_n satisfy the property of block of 0s of nondecreasing inter-distant lengths $l(m,j)$, where t_{mjk} be the number of the k-th group of consecutive lengths of equal inter-distance r_{mjk}, such that the direct terms $M_{r_{mjk}}$ represents $G(r_{mjk})$ nonisomorphic connected P-series for every $1 \leq k \leq q_{mj}$, $1 \leq j \leq p_m$, and $1 \leq m \leq n-1$, for some $p_m \leq n^2$ and $q_{mj} \leq m+1$. Then we have $D(n) = \sum_{m=1}^{n-1} \sum_{j=1}^{p_m} \prod_{k=1}^{q_{mj}} \binom{2^{r_{mjk}-1}+t_{mjk}-1}{t_{mjk}+1}$, $n \geq 2$.*

Proof. Let $\bar{S}(m,j)$ be the number of M_n that satisfy the property of block of 0s of the nondecreasing inter-distant lengths $l(m,j)$: $n_{1j}, n_{2j}, \ldots, n_{mj}$ such that t_{mjk} be the number of the k-th group of consecutive lengths of equal inter-distance r_{mjk}, for every $1 \leq k \leq q_{mj}$, $1 \leq j \leq p_m$, and $1 \leq m \leq n-1$. Then we have

$$r_{mj1} = n_{ij} - n_{(i-1)j},\ 1 \leq i \leq t_{mj1}+1,$$
$$r_{mj2} = n_{ij} - n_{(i-1)j},\ t_{mj1}+2 \leq i \leq t_{mj2}+1,$$
$$\vdots$$
$$r_{mjq} = n_{ij} - n_{(i-1)j},\ t_{mj(q-1)}+2 \leq i \leq t_{mjq}+1,$$

such that $r_{mj1} < r_{mj2} < \cdots < r_{mjq_{mj}}$, where we assume $n_{0j} = 0$ and $n_{(m+1)j} = n$. Then the direct terms are $M_{r_{mjk}}$, $1 \leq i \leq t_{mjk}+1$, $1 \leq k \leq q_{mj}$. By hypothesis, $M_{r_{mjk}}$ represents $G(r_{mjk})$ nonisomorphic connected P-series for every $1 \leq i \leq t_{mjk}+1$ and $1 \leq k \leq q_{mj}$. Therefore, by Theorem 4.1, we have $\bar{S}(m,j)$ as follows:

$$\bar{S}(m,j) = \prod_{k=1}^{q_{mj}} \binom{G(r_{mjk})+t_{mjk}}{1+t_{mjk}}. \tag{4}$$

Since the equation (4) holds for all lengths $l(m,j)$, where $1 \leq j \leq p_m$ and $1 \leq m \leq n-1$, we have

$$D(n) = \sum_{m=1}^{n-1} \sum_{j=1}^{p_m} \bar{S}(m,j) = \sum_{m=1}^{n-1} \sum_{j=1}^{p_m} \prod_{k=1}^{q_{mj}} \binom{G(r_{mjk})+t_{mjk}}{1+t_{mjk}}, \text{ where } n \geq 2.$$

Then by using the equation (1), we have

$$D(n) = \sum_{m=1}^{n-1} \sum_{j=1}^{p_m} \prod_{k=1}^{q_{mj}} \binom{2^{r_{mjk}-1}+t_{mjk}-1}{t_{mjk}+1}, \text{ where } n \geq 2. \tag{5}$$

□

5. Enumeration algorithm

Recall that we did not specify the parameters p_m, q_{mj}, $1 \leq m \leq n-1$, $1 \leq j \leq p_m$, as in the equation (5), explicitly. For given n, we have $1 \leq p \leq n^2$ and $1 \leq q \leq m+1$, where $1 \leq m \leq n-1$. Here, we give an algorithm for determining mainly the parameters p_m, q_{mj}, $1 \leq m \leq n-1$, $1 \leq j \leq p_m$, and for computing $D(n)$, the number of n-element unlabeled disconnected P-series, as in the equation (5). In the algorithm below, we recall also the equation (1), and finally, determine $S(n)$, the number of n-element unlabeled P-series, for $1 \le n \le 76$, see the sequence A349276 in OEIS [17].

Algorithm 5.1. To compute $S(n)$, the number of unlabeled P-series with $n \ge 2$ elements.

(1) Compute $G(n)$, the number of n-element unlabeled connected P-series, by using the equation (1).
(2) Initialize $D(n) = 0$, where $D(n)$ is the number of n-element unlabeled disconnected P-series.
(3) Repeat (a) for every $m \le n-1$.
  (a) Repeat (i) to (iv) for every distinct nondecreasing inter-distant lengths $l(m,j)$ as is constructed in (i). (Here, the total number of repetitions equals the parameter p_m in the equation (5)).
    (i) Construct j-th nondecreasing inter-distant lengths $l(m,j)$ consisting of m integers chosen from the integers less than or equal to $n-1$.
    (ii) Initialize $\bar{S}(m,j) = 1$ as given in the equation (4).
    (iii) Compute t_{mjk} and repeat (α) for every distinct r_{mjk} in the lengths $l(m,j)$. (Here, the total number of distinct r_{mjk} equals the parameter q_{mj} in the equation (4)).
      (α) Update $\bar{S}(m,j)$ with $\bar{S}(m,j) \times \binom{2^{r_{mjk}-1}+t_{mjk}-1}{t_{mjk}+1}$.
    (iv) Increase $D(n)$ by $\bar{S}(m,j)$.
(4) Return the sum of $G(n)$ and $D(n)$.

Lemma 5.1. *Algorithm 5.1 is a polynomial time algorithm.*

Proof. The constructions of the nondecreasing inter-distant lengths $l(m,j)$ in the step (i) have complexities equivalent to $\mathcal{O}(m(n-1))$. Then $1 \le m \le n-1$ implies $\mathcal{O}(m(n-1)) \approx \mathcal{O}(n^2)$. Since we have $1 \le t_{mjk}, q_{mj} \le m+1$ and $t_{mjk} \propto \frac{1}{q_{mj}}$, the computations of $\bar{S}(m,j)$ in the step (iii) have complexities equivalent to $\mathcal{O}(m+1) \approx \mathcal{O}(n)$. Since $1 \le p_m \le n^2$, the repetitions in the step (a) increase the complexities to $n^2(\mathcal{O}(n^2)+\mathcal{O}(n)) \approx \mathcal{O}(n^4)$. Finally, the repetitions in the step (3) increase the complexities to $(n-1)(\mathcal{O}(n^4)) \approx \mathcal{O}(n^5)$. This shows that Algorithm 5.1 runs in polynomial time. □

6. Numerical results

We implemented the enumeration algorithm on an Intel CORE-i7 (3.6 GHz) personal computer and determined the numbers $S(n)$, $n \le 76$, see Table 4. To compute $S(n)$, the machine took about 1 minute for $n \le 30$ and about 2 minutes for $n \le 55$. A modified version of the computer codes consisting of the basic operations with the numbers greater than the maximum of unsigned 64-bit integers was used to determine $S(n)$, $56 \le n \le 76$. The same machine then took about 5 days to compute the numbers $S(n)$,

$56 \leq n \leq 76$. The numbers $S(n), n \leq 7$, were verified by the direct counting of the Hasse diagrams of the posets. Also, in Table 5 to Table 10, we include the numbers $D(n)$, $2 \leq n \leq 50$, obtained according to the number of direct terms $d = m + 1$, $2 \leq d \leq 50$, see the equation (5). Note that in the cases of Table 6 to Table 10, we omitted some rows of the tables, because the numbers in these rows become fixed and can be found from the preceding tables.

TABLE 4. The number of unlabeled P-series, $S(n)$, $1 \leq n \leq 76$.

n	$S(n)$	n	$S(n)$
1	1	39	35832848639728
2	2	40	78705877884915
3	5	41	172713052281618
4	13	42	378658685153078
5	31	43	829444630192847
6	76	44	1815327343588985
7	178	45	3969733570967104
8	423	46	8673949863105406
9	988	47	18937880726772891
10	2312	48	41315603669403295
11	5361	49	90068720253991344
12	12427	50	196209285382157748
13	28626	51	427128874227627952
14	65813	52	929180107820533570
15	150700	53	2019993307095465670
16	344232	54	4388500556899660906
17	783832	55	9528080978492183705
18	1780650	56	20673992475866448294
19	4034591	57	44831081784801194655
20	9121571	58	97157449500339784571
21	20576349	59	210436395165112089588
22	46322816	60	455532200360723949309
23	104079338	61	985544410433148763070
24	233421517	62	2131068542916883839418
25	522574991	63	4605616404036914986284
26	1167974002	64	9948373437303420329108
27	2606282841	65	21478059545386107220988
28	5806953923	66	46346950727795354034183
29	12919314397	67	99961971368012285152972
30	28702716868	68	215496487079393129129125
31	63682839588	69	464345657776589836023901
32	141111193270	70	1000097485812093813586914
33	312292169989	71	2153013980763130917951360
34	690306198843	72	4632960798441274853250227
35	1524130470505	73	9965090320649581955683375
36	3361399303025	74	21415423429824053319007625
37	7405463570514	75	46025078414400880740770376
38	16298002803048	76	98911833079304855821827814

TABLE 5. The number of unlabeled disconnected P-series, $D(n)$, $2 \leq n \leq 18$, according to the number direct terms d, $2 \leq d \leq 18$.

$d \diagdown n$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
2	1	1	4	10	28	67	167	388	908	2053	4629	10246	22566	49159	106567	229384	491656
3		1	1	4	11	31	80	213	534	1343	3291	7980	19040	44984	104988	242884	556791
4			1	1	4	11	32	83	226	580	1504	3796	9536	23583	5,900	140496	338304
5				1	1	4	11	32	84	229	593	1550	3957	10062	25244	62948	155431
6					1	1	4	11	32	84	230	596	1563	4003	10223	25770	64637
7						1	1	4	11	32	84	230	597	1566	4016	10269	25931
8							1	1	4	11	32	84	230	597	1567	4019	10282
9								1	1	4	11	32	84	230	597	1567	4020
10									1	1	4	11	32	84	230	597	1567
11										1	1	4	11	32	84	230	597
12											1	1	4	11	32	84	230
13												1	1	4	11	32	84
14													1	1	4	11	32
15														1	1	4	11
16															1	1	4
17																1	1
18																	1
$D(n)$:	1	2	6	16	45	115	296	733	1801	4338	10380	24531	57622	134317	311465	718297	1649579

TABLE 6. The number of unlabeled disconnected P-series, $D(n)$, $19 \leq n \leq 27$, according to the number of direct terms d, $2 \leq d \leq 14$.

$d \diagdown n$	19	20	21	22	23	24	25	26	27
2	1048585	2228489	4718602	9961994	20971531	44041227	92274700	192940044	402653197
3	1266751	2861202	6422009	14329484	31805747	70252549	154488108	338336382	738193444
4	807237	1912072	4494918	10497181	24356462	56184170	128879083	294109263	667901109
5	381277	928112	2244879	5394318	12886346	30605995	72302182	169918416	397378534
6	160626	396947	974151	2377592	5769408	13929199	33459298	79999191	190404747
7	65163	162315	402178	990017	2424639	5906454	14321530	34565725	83075361
8	25977	65324	162841	403867	995248	2440550	5953753	14459900	34963876
9	10285	25990	65370	163002	404393	996937	2445781	5969664	14507254
10	4020	10286	25993	65383	163048	404554	997463	2447470	5974895
11	1567	4020	10286	25994	65386	163061	404600	997624	2447996
12	597	1567	4020	10286	25994	65387	163064	404613	997670
13	230	597	1567	4020	10286	25994	65387	163065	404616
14	84	230	597	1567	4020	10286	25994	65387	163065
$D(n)$:	3772448	8597284	19527774	44225665	99885035	225032910	505797776	1134419571	2539173978

TABLE 7. The number of unlabeled disconnected P-series, $D(n)$, $28 \leq n \leq 34$, according to the number of direct terms d, $2 \leq d \leq 18$.

$d \setminus n$	28	29	30	31	32	33	34
2	838864909	1744830478	3623886862	7516192783	15569272847	32212254736	66572025872
3	1605012086	3478467868	7516176493	16195589395	34807097956	74625024265	159629552218
4	1509867489	3398600871	7619289505	17016923183	37870246585	83994797573	185706360360
5	924973200	2143477586	4946037062	11366662041	26020964794	59348001466	134881344515
6	451241226	1064970949	2503508837	5862757276	13679302057	31804663743	73695371416
7	198847778	474135412	1126358220	2666381958	6290646019	14793003401	34678077433
8	84205462	202013276	482899046	1150352137	2731408818	6465188021	15257303293
9	35102561	84605296	203150904	486095681	1159235316	2755839396	6531740146
10	14523165	35149915	84744047	203551123	487235394	1162441405	2764762205
11	5976584	14528396	35165826	84791401	203689874	487635691	1163581580
12	2448157	5977110	14530085	35171057	84807312	203737228	487774442
13	997683	2448203	5977271	14530611	35172746	84812543	203753139
14	404617	997686	2448216	5977317	14530772	35173272	84814232
15	163064	404617	997687	2448219	5977330	14530818	35173433
16	65387	163064	404617	997687	2448220	5977333	14530831
17	25994	65387	163064	404617	997687	2448220	5977334
18	10286	25994	65387	163064	404617	997687	2448220
$D(n)$:	5672736196	12650878942	28165845957	62609097765	138963709623	307997202694	681716264252

TABLE 8. The number of unlabeled disconnected P-series, $D(n)$, $35 \leq n \leq 40$, according to the number of direct terms d, $2 \leq d \leq 21$.

$d \setminus n$	35	36	37	38	39	40
2	137438953489	283467907089	584115552274	1202590973970	2473901162515	5085241540627
3	340734006527	725849342543	1543324783860	3275628128836	6940666889448	14683061004856
4	409351418116	899770401861	1972412861594	4312761385740	9407206034047	20472269741866
5	305512784640	689767269943	1552503917727	3483993382458	7796348096160	17399073347974
6	170201538669	391842528114	899355963864	2058114658969	4696439658814	10687375059445
7	81047757227	188867721744	438881655169	1017064309232	2350701482265	5419121237462
8	35902678657	84251790220	197186885086	460325889435	1071959991411	2490307177236
9	15436987973	36383784779	85529867179	200557072734	469150541905	1094913052502
10	6556324949	15504111139	36565492083	86017890423	201858086319	472594806691
11	2767970824	6565259359	15528745095	36632808617	86200321470	202348701371
12	1163981877	2769111090	6568468524	15537682523	36657456541	86267697710
13	487821796	1164120628	2769511387	6569608790	15540891793	36666394606
14	203758370	487837707	1164167982	2769650138	6570009087	15542032059
15	84814758	203760059	487842938	1164183893	2769697492	6570147838
16	35173479	84814919	203760585	487844627	1164189124	2769713403
17	14530834	35173492	84814965	203760746	487845153	1164190813
18	5977334	14530835	35173495	84814978	203760792	487845314
19	2448220	5977334	14530835	35173496	84814981	203760805
20	997687	2448220	5977334	14530835	35173496	84814982
21	404617	997687	2448220	5977334	14530835	35173496
$D(n)$:	1506950601322	3327039564658	7336744093779	16160563849577	35557970732785	78156122071028

TABLE 9. The number of unlabeled disconnected P-series, $D(n)$, $41 \leq n \leq 45$, according to the number of direct terms d, $2 \leq d \leq 23$.

$d \diagdown n$	41	42	43	44	45
2	10445360463892	21440477265940	43980465111061	90159954526229	184717953466390
3	31015389975772	65420940806699	137805456299215	289904563759646	609129439696065
4	44454957781390	96331619708056	208331276709173	449693716677808	968931536739116
5	38728538760019	85990928093694	190473579856294	420938987475544	928208260681351
6	24255843356810	54908954740940	123990449527175	279310303001575	627730601961794
7	12461643067304	28586943323221	65423791190219	149385168779836	340337759138687
8	5771925395845	13347881861258	30800377251935	70921565825199	162968649866921
9	2549629583160	5924315615251	13737078604595	31788839948556	73418617030197
10	1103971401864	2573304750371	5985827821321	13895997702443	32197202864856
11	473904783197	1107445726719	2582461315091	6009816527499	13958490498912
12	202531369392	474396293016	1108758936042	2585946920021	6019011268778
13	86292349183	202598762187	474579032326	1109250730904	2587261213457
14	36669603876	86301287368	202623414395	474646429244	1109433489576
15	15542432356	36670744142	86304496638	202632352580	474671081588
16	6570195192	15542571107	36671144439	86305636904	202635561850
17	2769718634	6570211103	15542618461	36671283190	86306037201
18	1164191339	2769720323	6570216334	15542634372	36671330544
19	487845360	1164191500	2769720849	6570218023	15542639603
20	203760808	487845373	1164191546	2769721010	6570218549
21	84814982	203760809	487845376	1164191559	2769721056
22	35173496	84814982	203760809	487845377	1164191562
23	14530835	35173496	84814982	203760809	487845377
$D(n)$:	171613540653843	376459661897527	825046583681744	1806531250566778	3952141384922689

Table 10. The number of unlabeled disconnected P-series, $D(n)$, $46 \leq n \leq 50$, according to the number of direct terms d, $2 \leq d \leq 26$.

$d \diagdown n$	46	47	48	49	50
2	378232002052118	774056185954327	1583296748191767	3236962232172568	6614661961089048
3	1278365515033104	2679876336571571	5611907339790849	11739852145251492	24535235453168114
4	2084098123128673	4475326479602254	9594966779907938	20540133808336393	43906745522488463
5	2042448364261787	4485095705244385	9829718631564444	21502614372408198	46951849531703518
6	1407602179101315	3149475326523435	7031990909757367	15668563529981774	34843255271767539
7	773694738380576	1755136990705480	3973363951584615	8977075395155599	20242448421527101
8	373729014087492	855379360957198	1954029064304096	4455470993905336	10140647950148663
9	169244336709722	389423469869103	894442022352300	2050808406537735	4694185790941035
10	74462548616777	171899823803530	396146215824697	911383714980428	2093314021306824
11	32359132509046	74880000922228	172970781542159	398880942497005	918335798591009
12	13982605095485	32422030986395	75043211526947	173392205098663	399964009836122
13	6022500819359	13991813705348	32446192869064	75106267106545	173555925826013
14	2587753092221	6023815450370	13995304547710	32455406209237	75130445717662
15	1109500887334	2587935855633	6024307351523	13996619276266	32458897446169
16	474680019773	1109525539678	2588003253544	6024490115887	13997111182819
17	202636702116	474683229043	1109534477863	2588027905888	6024557513798
18	86306175952	202637102413	474684369309	1109537687133	2588036844073
19	36671346455	86306223306	202637241164	474684769606	1109538827399
20	15542641292	36671351686	86306239217	202637288518	474684908357
21	6570218710	15542641818	36671353375	86306244448	202637304429
22	2769721069	6570218756	15542641979	36671353901	86306246137
23	1164191563	2769721072	6570218769	15542642025	36671354062
24	487845377	1164191563	2769721073	6570218772	15542642038
25	203760809	487845377	1164191563	2769721073	6570218773
26	84814982	203760809	487845377	1164191563	2769721073
$D(n)$:	8638765491016575	18867511982595228	41174866181047968	89787245277280689	195646335428736437

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