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ON AN ASYMMETRIC EXTENSION OF THE BROWN-RESNICK SPATIAL COPULA: STUDY OF PROPERTIES AND ESTIMATION METHOD

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Abstract. This paper is devoted to the theoretical study of an asymmetric max-stable copula in dimension two derived from a class of copulas previously introduced in a previous work. This particular copula, constructed as an asymmetric extension of the Brown-Resnick copula, can be used to model structures of extreme spatial dependencies with asymmetry between the margins. Given the complexity of the density, particularly in higher dimensions, we recommend the use of the pairwise likelihood method for parameter estimation. This approach, which is well suited to models with complex dependencies, allows efficient inference from bivariate margins. This work thus provides a sound theoretical basis for future applications of this copula to multivariate extreme data.

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1. Introduction

The study of dependence between random variables is still a crucial issue in many fields, particularly in finance, insurance and climatology. One of the most widely used approaches at the moment, especially in the non-Gaussian framework, is copulas. Copulas are tools that can be used to describe the dependence structure independently of the margins, and therefore play a fundamental role in this study.

There are several families of copulas in the literature, but not all of them are suitable for modelling certain extreme phenomena whose behavior is unpredictable. Most are symmetrical, whereas in reality the symmetry assumption is too restrictive. To compensate for the shortcomings of these symmetrical copulas, asymmetrical copulas constructed either by marginal transformations or by other

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transformations on existing copulas have been developed. On this point, we can cite a few authors who have worked in this direction.

Rodriguez-Lallena and Ubeda-Flores [13] introduce a class of asymmetric bivariate copulas that generalizes some families of copulas already known. Kim et al [7] and Messarr and Najjari [11] extend the method of Rodriguez-Lallena and Ubeda-Flores by constructing a new family of symmetric and asymmetric copulas. Alfonsi and Brigo [1] proposed a new method for constructing asymmetric copulas based on periodic functions. Liebscher [10] in turn contributed by proposing two multivariate asymmetric copula construction methods based on the method proposed by Khoudradji in 1995 [8] in his thesis. The first method is constructed using products of copulas, while the second is a generalization of families of Archimedean copulas. Still in the history of those who contributed to the construction of asymmetric copulas, we can mention Duran [6] who proposed a method based on the product of copulas but with powerful arguments. Wu [15] proposed a new method for constructing asymmetric copulas using a mixture of basic copulas and a convex combination of asymmetric copulas that can have different tail dependencies in different directions. Di Bernardino and Rullière [2] constructed a multivariate family of copulas by generalising some known families using a Σ distortion matrix.

In a previous work, we developed a general class of asymmetric copulas, making it possible to generate a variety of copulas adapted to different contexts [**?**]. Among these, a specific copula has been constructed, inspired by the structure of the Brown-Resnick copula, but incorporating marginal transformations or mechanisms to model asymmetric dependencies. Although this copula has been introduced, its theoretical properties have not yet been explored in depth.

This paper explores the theoretical properties of this asymmetric copula, evaluates its behaviour through simulations and analyses its ability to capture extreme spatial dependence.

The aim of this approach is to demonstrate the usefulness of our copula in practical contexts and to further the understanding of asymmetric dependencies in climatology. It thus constitutes a significant contribution to the literature on asymmetric copulas by providing an in-depth analysis of a specific model and highlighting its potential for modelling climate dependencies.

2. Preliminary

2.1. **Definition of a copula.** Copulas are fundamental tools in probability theory for modelling the dependence between several random variables. They allow the marginal distributions of these variables to be appropriately related to their joint distributions. Without loss of generality, we will focus on copulas of dimension 2, i.e. bivariate copulas.

Definition 1. [12] *A copula is a function* $C : [0,1]^2 \longrightarrow [0,1]$ *satisfying the following conditions:*

i) *The boundary conditions:* $\forall u, v \in [0,1]$,

$$C(u,0) = C(0,v) = 0 \tag{1}$$

$$C(u,1) = u \tag{2}$$

$$C(1,v) = v \tag{3}$$

ii) *The 2-increasing condition:* $\forall u_1 \leq u_2$ *and* $v_1 \leq v_2$,

$$C(u_1,v_1) - C(u_2,v_1) - C(u_1,v_2) + C(u_2,v_2) \geq 0. \tag{4}$$

The notion of copula asymmetry is formally clarified in the following remark:

Remark 1. [12] *If* C *is a symmetric copula then*

$$C(u,v) = C(v,u), \quad \forall u, v \in I. \tag{5}$$

Otherwise, we say that C is asymmetric.

In the field of modelling extreme phenomena, only extreme copulas are suitable. The following definition tells us more about this notion.

Definition 2. [12] *Let* n *be a positive real constant. An extreme value copula* C^* *is a copula which satisfies the following relation :*

$$C^*(u^n, v^n) = \big(C^*(u,v)\big)^n. \tag{6}$$

2.2. **Sklar's theorem and its inversion.** To better understand the role of copulas, it is essential to consider Sklar's celebrated theorem, which is a fundamental result in copula theory. This theorem shows that it is possible to link the principle of copulas to the distribution function and marginal laws.

Theorem 1. [12] *Let* F *be a joint distribution function of margins* F_1 *and* F_2*. Then there exists a copula* $C : [0,1]^2 \longrightarrow [0,1]$ *such that for all* $(x,y) \in \mathbb{R}^2$*, we have:*

$$F(x,y) = C(F_1(x), F_2(y)). \tag{7}$$

Furthermore, if F_1 *and* F_2 *are continuous, then* C *is unique.*

Another way of expressing the copula in terms of the joint distribution and its marginals is given by the following theorem:

Theorem 2. [12]

Let F *be the bivariate distribution function of marginals* F_1 *and* F_2*. The copula* C *associated with* F *is given by:*

$$\forall (u,v) \in [0,1]^2, \; C(u,v) = F\left(F_1^{-1}(u), F_2^{-1}(v)\right). \tag{8}$$

2.3. **New family of max-stable asymmetric spatial copulas.** In [?], we constructed a new class of flexible asymmetric copulas by modifying the Darsow product operator [5]. The following definition reminds us of the formulation of this construction.

Definition 3. [16] *Let C_1 and C_2 be two symmetrical copulas and α and β be in $(0,1)$ such that $\alpha \neq \beta$. The $\overset{\alpha,\beta}{\star}$-product of C_1 and C_2 is the function $C_1 \overset{\alpha,\beta}{\star} C_2$ from I^2 to I given by:*

$$(C_1 \overset{\alpha,\beta}{\star} C_2)(u,v) = u^{1-\alpha}v^{1-\beta}\int_0^1 D_2C_1(u^\alpha,t)D_1C_2(t,v^\beta)dt. \tag{9}$$

Theorem 3. [16] *The product $C_1 \overset{\alpha,\beta}{\star} C_2$ given in (9) is an asymmetric copula.*

Using this theorem, it is now possible to construct several asymmetric copulas from two symmetric copulas. However, the copulas generated by this class are not always guaranteed to be max-stable, even if the basic copulas used are. To ensure that the copulas obtained are max-stable, a number of conditions must be met, as set out in the following proposition.

Proposition 1. [16] *Let C_1 and C_2 be two max-stable copulas such that $C = C_1 \overset{\alpha,\beta}{\star} C_2$ is well defined as in definition 3 . Then the Darsow modified copula C is max-stable if one of the copulas C_1 or C_2 is Fréchet-Hoeffding upper bound .*

Following on from this proposal, we have constructed a max-stable copula which is also an asymmetric extension of the Brown-Resnick copula. Let C^s be the copula underlying the Brown-Resnick max-stable process [18], called the Brown-Resnick copula and given in the following equation:

$$C^s(u_s,v_s) = \exp\left[\Phi\left(\frac{a(h)}{2}+\frac{1}{a(h)}\log\left(\frac{\log u_s}{\log v_s}\right)\right)\log u_s + \Phi\left(\frac{a(h)}{2}+\frac{1}{a(h)}\log\left(\frac{\log v_s}{\log u_s}\right)\right)\log v_s\right]. \tag{10}$$

where $a(h)^2 = 2\gamma(h)$ with $\gamma(h)$ the semi-variogram of a Gaussian process centred $(\Phi(.))$ at least at stationary increments and h the distance between two sites.

By applying the previous proposition, we obtain the asymmetric max-stable spatial copula given by:

$$\begin{aligned} C^s(u_s,v_s) = \exp\Bigg[&(1-\alpha)\log u_s + \alpha\Phi\left(\frac{a(h)}{2}+\frac{1}{a(h)}\log\left(\frac{\alpha\log u_s}{\beta\log v_s}\right)\right)\log u_s \\ &+ (1-\beta)\log v_s + \beta\Phi\left(\frac{a(h)}{2}+\frac{1}{a(h)}\log\left(\frac{\beta\log v_s}{\alpha\log u_s}\right)\right)\log v_s\Bigg]. \end{aligned} \tag{11}$$

As well as being asymmetrical, this copula offers flexibility in modelling spatial extremes.

Although this new copula was introduced in the previous article, it has not been studied in depth, particularly with regard to its theoretical and practical properties. The following section will therefore be devoted to a detailed analysis of this copula, in order to facilitate its use in the modelling of extreme spatial events.

Remark 2. *Limit case of the copula*

(1) *For* $(\alpha, \beta) = (1, 1)$, *we would find the classic symmetrical Brown-Resnick copula given in* (10).

(2) *Also, if* $(\alpha, \beta) \in \{(0, i), (i, 0)\}$,$i \in [0, 1]$, *then our copula reduces to an independent copula* $\Pi(u, v) = uv$.

3. Main Results

3.1. **Copula asymmetry.** The asymmetry of a copula is essential because it influences the way in which the dependence between variables manifests itself in different regions of the distribution. In many cases, the dependence is not symmetrical: in finance, assets are often more correlated in times of crisis, while in climatology, extreme precipitation can be influenced differently depending on the season or climate. Formally, this asymmetry can also be manifested by the non-tradability of its margins, i.e. the order of the variables influences the value of the copula. The following proposition guarantees that the Brown-Resnick extended copula is asymmetric.

Proposition 2. *Let* α *and* β *be two different real numbers* $(\alpha \neq \beta)$ *such that* $\alpha; \beta \in [0, 1]$. *Then the copula* C^s *defined in* (11) *is an asymmetric copula.*

Proof. We have:

$$\begin{aligned} C^s(u_s, v_s) = \exp\Bigg[&(1-\alpha)\log u_s + \alpha\Phi\left(\frac{a(h)}{2} + \frac{1}{a(h)}\log\left(\frac{\alpha \log u_s}{\beta \log v_s}\right)\right)\log u_s \\ &+ (1-\beta)\log v_s + \beta\Phi\left(\frac{a(h)}{2} + \frac{1}{a(h)}\log\left(\frac{\beta \log v_s}{\alpha \log u_s}\right)\right)\log v_s\Bigg]. \end{aligned} \tag{12}$$

If we swap the margins u_s and v_s, we obtain:

$$\begin{aligned} C^s(v_s, u_s) = \exp\Bigg[&(1-\alpha)\log v_s + \alpha\Phi\left(\frac{a(h)}{2} + \frac{1}{a(h)}\log\left(\frac{\alpha \log v_s}{\beta \log u_s}\right)\right)\log v_s \\ &+ (1-\beta)\log u_s + \beta\Phi\left(\frac{a(h)}{2} + \frac{1}{a(h)}\log\left(\frac{\beta \log u_s}{\alpha \log v_s}\right)\right)\log u_s\Bigg]. \end{aligned} \tag{13}$$

Since $\alpha \neq \beta$, then it is clear that $C^s(u_s, v_s) \neq C^s(v_s, u_s)$. From the remark 1, we conclude that C^s is an asymmetric copula. □

3.2. **Max-stability of the copula.** A copula that is not max-stable does not respect the dependency structure specific to extreme values. Indeed, the limiting distributions of maxima (as in the analysis of extreme events) obey a property called max-stability. If a copula is not max-stable, it cannot correctly represent the dependence between maxima of random variables. Conventional copulas (such as the normal copula) model the central dependence well, but often underestimate the dependence in the tails, where the extremes are located. This can lead to significant errors in risk modelling. The copula we describe here, in addition to being spatial and asymmetric, is also max-stable.

Proposition 3. *If* C^s *is the extended Brown-Resnick copula defined in* (11), *then* C^s *is max-stable.*

Proof. Let n be a strictly positive real number. We have:

$$\begin{aligned}C^s(u_s^n, v_s^n) &= \exp\Bigg[(1-\alpha)\log u_s^n + \alpha\Phi\left(\frac{a(h)}{2} + \frac{1}{a(h)}\log\left(\frac{\alpha\log u_s^n}{\beta\log v_s^n}\right)\right)\log u_s^n \\ &\quad + (1-\beta)\log v_s^n + \beta\Phi\left(\frac{a(h)}{2} + \frac{1}{a(h)}\log\left(\frac{\beta\log v_s^n}{\alpha\log u_s^n}\right)\right)\log v_s^n\Bigg] \\ &= \exp\Bigg[n(1-\alpha)\log u_s + n\alpha\Phi\left(\frac{a(h)}{2} + \frac{1}{a(h)}\log\left(\frac{\alpha\log u_s}{\beta\log v_s}\right)\right)\log u_s \\ &\quad + n(1-\beta)\log v_s + n\beta\Phi\left(\frac{a(h)}{2} + \frac{1}{a(h)}\log\left(\frac{\beta\log v_s}{\alpha\log u_s}\right)\right)\log v_s\Bigg]\end{aligned} \tag{14}$$

By factoring by n inside the exponential function, we obtain:

$$\begin{aligned}C^s(u_s^n, v_s^n) &= \exp\Bigg[n\Bigg((1-\alpha)\log u_s + \alpha\Phi\left(\frac{a(h)}{2} + \frac{1}{a(h)}\log\left(\frac{\alpha\log u_s}{\beta\log v_s}\right)\right)\log u_s \\ &\quad + (1-\beta)\log v_s + \beta\Phi\left(\frac{a(h)}{2} + \frac{1}{a(h)}\log\left(\frac{\beta\log v_s}{\alpha\log u_s}\right)\right)\log v_s\Bigg)\Bigg] \\ &= \exp\Bigg[(1-\alpha)\log u_s + \alpha\Phi\left(\frac{a(h)}{2} + \frac{1}{a(h)}\log\left(\frac{\alpha\log u_s}{\beta\log v_s}\right)\right)\log u_s \\ &\quad + (1-\beta)\log v_s + \beta\Phi\left(\frac{a(h)}{2} + \frac{1}{a(h)}\log\left(\frac{\beta\log v_s}{\alpha\log u_s}\right)\right)\log v_s\Bigg]^n. \\ C^s(u_s^n, v_s^n) &= \big(C^s(u_s, v_s)\big)^n.\end{aligned} \tag{15}$$

So according to the definition 2, the copula C^s is indeed max-stable. □

3.3. **Copula density.** The density of a copula is an important element in the modelling process, particularly in the parameter estimation stage for the maximum likelihood method. In the absence of numerical estimation, we give here the closed expression of the Brown-Resnick extended copula density.

Proposition 4. *Let C^s be the Brown-Resnick extended copula defined in (11). Then the density c^s associated with C^s is given by:*

$$\begin{aligned}c^s(u_s, v_s) =& \frac{C^s(u_s, v_s)}{u_s v_s}\Bigg[\left((1-\beta) + \frac{\beta}{a(h)}\phi(y) + \beta\Phi(y) - \frac{\alpha\log u_s}{a(h)\log v_s}\phi(x)\right) \\ &\times\left((1-\alpha) + \frac{\alpha}{a(h)}\phi(x) + \alpha\Phi(x) - \frac{\beta\log v_s}{a(h)\log u_s}\phi(y)\right) \\ &+ \frac{\alpha(x-a(h))}{a^2(h)\log v_s}\phi(x) + \frac{\beta(y-a(h))}{a^2(h)\log u_s}\phi(y)\Bigg]\end{aligned} \tag{16}$$

with: $x = \dfrac{a(h)}{2} + \dfrac{1}{a(h)}\log\left(\dfrac{\alpha\log u_s}{\beta\log v_s}\right)$; $y = \dfrac{a(h)}{2} + \dfrac{1}{a(h)}\log\left(\dfrac{\beta\log v_s}{\alpha\log u_s}\right)$

$\Phi(.)$ et $\phi(.)$ are the distribution function and the density of the standard normal distribution respectively.

3.4. **Dependency function and dependency measures.**

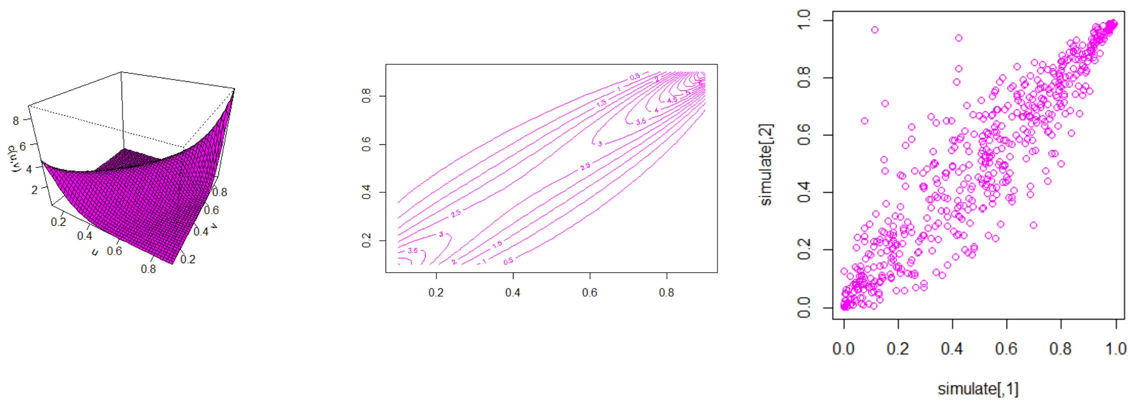


Figure 1. Visualization of the Brown-Resnick extended copula: density, contour lines, and scatter plot for $a(h) = 0.46$, $\alpha = 0.99$ and $\beta = 0.96$.

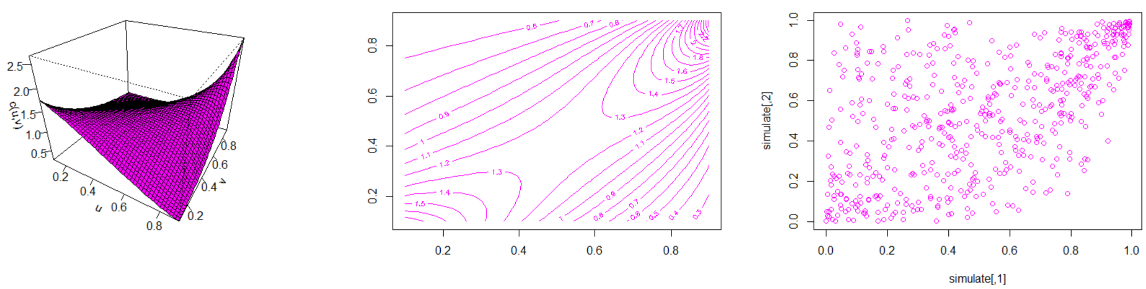


Figure 2. Visualization of the Brown-Resnick extended copula: density, contour lines, and scatter plot for $a(h) = 1$, $\alpha = 0.8$ and $\beta = 0.5$.

3.4.1. *Pickands dependency function.* Another way of representing the dependency structure of bivariate limit laws is via a special function called the Pickands dependency function $A : [0, 1] \to [1/2, 1]$. This function can therefore be used to represent any extreme value copula. In the bivariate case, the following theorem reminds us of the details of this function.

Theorem 4. [12] *If C is an extreme value copula, then*

$$C(u, v) = \exp\left(\log(uv) A\left(\frac{\log u}{\log(uv)} \right) \right), \tag{17}$$

for an appropriate choice of the A function.
In particular, the following constraints must be satisfied:

(1) $A(0) = A(1) = 1$.
(2) $\max\{t, 1-t\} \leq A(t) \leq 1, \quad \forall\, t \in [0, 1]$.
(3) *A is convex.*

The Pickands function completely characterizes bivariate extremal dependence. It is equal to 1 in case of independence and equal to $\max(t, 1-t)$ in case of perfect dependence.
For our max-stable copula given in (11), the pickands dependence function can be summarised in the following proposition.

Proposition 5. *Let C^s be the extreme spatial copula defined in (11), then the Pickands dependence function is defined by:*

$$A(t) = \big[(1-\alpha) + \alpha\Phi(x)\big]t + \big[(1-\beta) + \beta\Phi(y)\big](1-t), \quad t \in [0,1]. \tag{18}$$

where $x = \dfrac{a(h)}{2} + \dfrac{1}{a(h)} \log\left(\dfrac{\alpha t}{\beta(1-t)}\right)$ *and* $y = \dfrac{a(h)}{2} + \dfrac{1}{a(h)} \log\left(\dfrac{\beta(1-t)}{\alpha t}\right)$

Proof. Let C^s be the Brown-Resnick copula denoted by:

$$\begin{aligned} C^s(u_s, v_s) = \exp\Bigg[&(1-\alpha)\log u_s + \alpha\Phi\left(\frac{a(h)}{2} + \frac{1}{a(h)}\log\left(\frac{\alpha \log u_s}{\beta \log v_s}\right)\right)\log u_s \\ &+ (1-\beta)\log v_s + \beta\Phi\left(\frac{a(h)}{2} + \frac{1}{a(h)}\log\left(\frac{\beta \log v_s}{\alpha \log u_s}\right)\right)\log v_s\Bigg]. \end{aligned} \tag{19}$$

Let $u_s = s$ and $v_s = 1-s$. Then equation (19) can be rewritten:

$$\begin{aligned} C^s(s, 1-s) = \exp\Bigg[&(1-\alpha)\log s + \alpha\Phi\left(\frac{a(h)}{2} + \frac{1}{a(h)}\log\left(\frac{\alpha \log s}{\beta \log(1-s)}\right)\right)\log s \\ &+ (1-\beta)\log v_s + \beta\Phi\left(\frac{a(h)}{2} + \frac{1}{a(h)}\log\left(\frac{\beta \log(1-s)}{\alpha \log s}\right)\right)\log(1-s)\Bigg]. \end{aligned} \tag{20}$$

Next, let's ask:

$$t = \frac{\log s}{\log s + \log(1-s)} \text{ and } \quad 1-t = \frac{\log(1-s)}{\log s + \log(1-s)} \tag{21}$$

From (21), we deduce that:

$$\log s = t\big(\log s + \log(1-s)\big) \text{ and } \quad \log(1-s) = (1-t)\big(\log s + \log(1-s)\big) \tag{22}$$

By replacing the expressions for $\log s$ and $\log(1-s)$ in (23), we obtain:

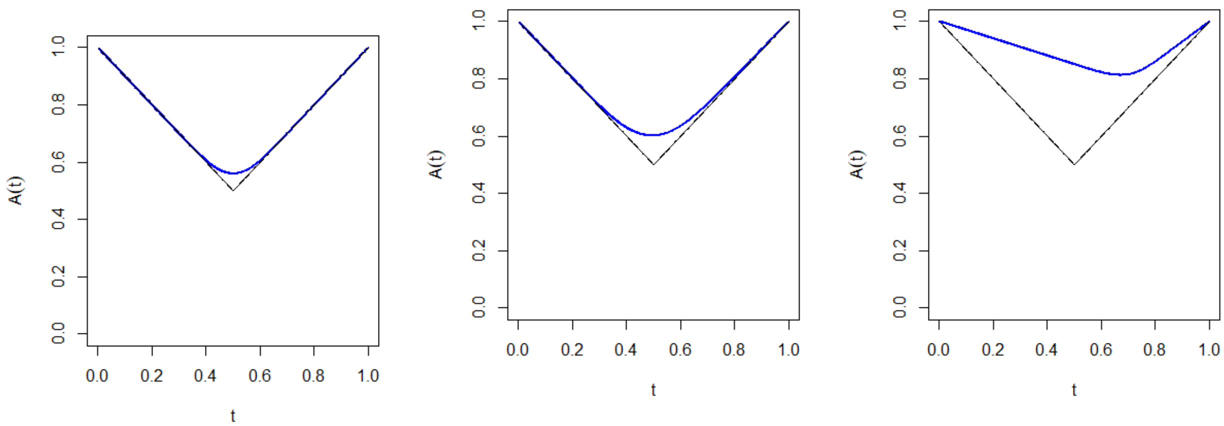
$$\begin{aligned} C^s(s, 1-s) &= \exp\Bigg[(1-\alpha)t\big(\log s + \log(1-s)\big) + \alpha\Phi\left(\frac{a(h)}{2} + \frac{1}{a(h)}\log\left(\frac{\alpha t}{\beta(1-t)}\right)\right) \\ &\quad \times t\big(\log s + \log(1-s)\big) + (1-\beta)(1-t)(\log s + \log(1-s)) \\ &\quad + \beta\Phi\left(\frac{a(h)}{2} + \frac{1}{a(h)}\log\left(\frac{\beta(1-t)}{\alpha t}\right)\right)(1-t)(\log s + \log(1-s))\Bigg] \\ &= \exp\Bigg[\Bigg((1-\alpha)t + \alpha\Phi\left(\frac{a(h)}{2} + \frac{1}{a(h)}\log\left(\frac{\alpha t}{\beta(1-t)}\right)\right)t + (1-\beta)(1-t) \\ &\quad + \beta\Phi\left(\frac{a(h)}{2} + \frac{1}{a(h)}\log\left(\frac{\beta(1-t)}{\alpha t}\right)\right)(1-t)\Bigg)\Bigg(\log(s(1-s))\Bigg)\Bigg] \\ &= \exp\Big[\log\big(s(1-s)\big)A(t)\Big]. \end{aligned} \tag{23}$$

with

$$A(t) = (1-\alpha)t + \alpha\Phi\left(\frac{a(h)}{2} + \frac{1}{a(h)}\log\left(\frac{\alpha t}{\beta(1-t)}\right)\right)t + (1-\beta)(1-t)$$
$$+ \beta\Phi\left(\frac{a(h)}{2} + \frac{1}{a(h)}\log\left(\frac{\beta(1-t)}{\alpha t}\right)\right)(1-t). \tag{24}$$

The function $A(t)$ expressed in (24), normally satisfies the three conditions given in theorem 4. So A is effectively a function of Pickands. □

The following figure shows how the parameters α and β influence the dependency structure. The asymmetry can also be seen.



(a): $\alpha = 1$, $\beta = 1$ and $a(h) = 0.3$ (b): $\alpha = 0.99$, $\beta = 0.96$ and $a(h) = 0.3$ (c): $\alpha = 0.3$, $\beta = 0.7$ and $a(h) = 0.3$

FIGURE 3. Visualization of the Pickands dependence function $A(t)$ for different values of α, β and $a(h)$.

3.4.2. *Dependency measures.* In this section, we discuss some non-linear dependence measures that can be expressed in terms of copulas. These are Spearman's rho and Kendall's tau, which are measures that capture dependence over the entire distribution, and tail dependences, which focus essentially on dependence at the tail of the distribution. The latter are crucial in modelling extreme value dependence.

3.4.2.1 Spearman's Rho and Kendal's Tau.

Definition 4. [3, 9] *Let C be a bivariate copula. Spearman's rho and Kendall's rate as a function of the copula C are given respectively by:*

$$\rho_S = 12\iint_{[0,1]^2} C(u,v)dudv - 3 \tag{25}$$

and

$$\tau_K = 4\iint_{[0,1]^2} C(u,v)dC(u,v) - 1. \tag{26}$$

If we replace our copula (11) in the equations (25) and (26), it would be difficult to calculate the integrals and give explicit forms in view of the complexity of the extended Brown-Resnick copula. In this case, the numerical approach is sometimes recommended to analyse the dependence capacity of a copula. With the function *integral2* integrated in the package *pracma* of the R software, the numerical calculation can be done without any problem. As an example, Table 1 below summarizes the evolution of Spearman's Rho as a function of the parameters α and β of our copula.

TABLE 1. Values of ρ_S as a function of α and β for $a(h) = 0.5$

$\alpha\backslash\beta$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0	0	0	0	0	0	0	0	0	0	0	0
0.1	0	0.072	0.098	0.111	0.120	0.126	0.130	0.133	0.136	0.138	0.139
0.2	0	0.098	0.148	0.180	0.202	0.218	0.231	0.241	0.249	0.256	0.262
0.3	0	0.111	0.180	0.227	0.263	0.290	0.313	0.331	0.346	0.359	0.371
0.4	0	0.120	0.202	0.263	0.310	0.349	0.381	0.408	0.431	0.451	0.469
0.5	0	0.126	0.218	0.290	0.349	0.398	0.439	0.475	0.506	0.534	0.558
0.6	0	0.130	0.231	0.313	0.381	0.439	0.490	0.534	0.574	0.609	0.640
0.7	0	0.133	0.241	0.331	0.408	0.475	0.534	0.587	0.634	0.677	0.716
0.8	0	0.136	0.249	0.346	0.431	0.506	0.574	0.634	0.689	0.740	0.786
0.9	0	0.138	0.256	0.359	0.451	0.534	0.609	0.677	0.740	0.797	0.851
1.0	0	0.139	0.262	0.371	0.469	0.558	0.640	0.716	0.786	0.851	0.911

By looking at the different values of ρ_S, we can see that the copula studied is flexible and can model different dependency structures.

3.4.2.2 Upper and lower tail dependency. The indices of the upper λ_U and lower λ_L tails are therefore instruments for measuring the tail of the distribution, unlike the ρ_S and τ_K that we saw earlier. Their expressions in terms of any copula are given in the following theorem.

Lemma 1. [3] *Let U and V be uniform random variables on $[0,1]$ and C the associated copula. Then for all $u, v \in I$, we have:*

$$\mathbb{P}(U \leq u | V \leq v) = \frac{C(u,v)}{v} \tag{27}$$

and

$$\mathbb{P}(U > u | V > v) = \frac{1 - u - v + C(u,v)}{1 - v}. \tag{28}$$

Theorem 5. [3] *Let X and Y be two random variables with joint distribution function F and C the copula associated with F.If the limits of the equations (27) and (28) exist then*

$$\lambda_U = 2 - \lim_{u \to 1^-} \frac{1 - C(u,u)}{1-u} \qquad \text{and} \qquad \lambda_L = \lim_{u \to 0^+} \frac{C(u,u)}{u}. \tag{29}$$

Applying the theorem 5, the upper tail indices of the copula studied here is given in the following proposition.

Proposition 6. *Let X and Y be two continuous random variables and C^s the copula associated with X and Y as defined in (11). Then the indices of the upper and lower tails are defined respectively by:*

$$\lambda_U = 2 + \alpha + \beta - \alpha\Phi\left(\frac{a(h)}{2} + \frac{1}{a(h)} \log\left(\frac{\alpha(h)}{\beta}\right)\right) - \beta\Phi\left(\frac{a(h)}{2} + \frac{1}{a(h)} \log\left(\frac{\beta}{\alpha}\right)\right). \tag{30}$$

And

$$\lambda_L = 0. \tag{31}$$

where $\Phi(.)$ denotes the standard normal distribution, $a(h) = \sqrt{2\gamma(h)}$, with $\gamma(h)$ the semi-variogram, α and β real numbers in $[0,1]$.

Proof. Let C^s be the Brown-Resnick copula defined by:

$$\begin{aligned} C^s(u_s, v_s) = \exp\Bigg[&(1-\alpha)\log u_s + \alpha\Phi\left(\frac{a(h)}{2} + \frac{1}{a(h)} \log\left(\frac{\alpha \log u_s}{\beta \log v_s}\right)\right) \log u_s \\ &+ (1-\beta)\log v_s + \beta\Phi\left(\frac{a(h)}{2} + \frac{1}{a(h)} \log\left(\frac{\beta \log v_s}{\alpha \log u_s}\right)\right) \log v_s\Bigg]. \end{aligned}$$

Posing $u_s = v_s$, we obtain:

$$\begin{aligned} C^s(u_s, u_s) = \exp\Bigg[&\log u_s \Bigg((1-\alpha) + \alpha\Phi\left(\frac{a(h)}{2} + \frac{1}{a(h)} \log\left(\frac{\alpha}{\beta}\right)\right) \\ &+ (1-\beta) + \beta\Phi\left(\frac{a(h)}{2} + \frac{1}{a(h)} \log\left(\frac{\beta}{\alpha}\right)\right)\Bigg)\Bigg]. \end{aligned} \tag{32}$$

simplify, we obtain

$$C^s(u_s, u_s) = u_s^{2-\alpha-\beta+\alpha\Phi\left(\frac{a(h)}{2} + \frac{1}{a(h)} \log\left(\frac{\alpha}{\beta}\right)\right) + \beta\Phi\left(\frac{a(h)}{2} + \frac{1}{a(h)} \log\left(\frac{\beta}{\alpha}\right)\right)}. \tag{33}$$

Passing now to the limit, we have:

$$\begin{aligned} \lambda_U &= 2 - \lim_{u_s \to 1^-} \frac{1 - C(u_s, u_s)}{1 - u_s} \\ &= 2 - \lim_{u_s \to 1^-} \frac{1 - u_s^{2-\alpha-\beta+\alpha\Phi\left(\frac{a(h)}{2} + \frac{1}{a(h)} \log\left(\frac{\alpha}{\beta}\right)\right) + \beta\Phi\left(\frac{a(h)}{2} + \frac{1}{a(h)} \log\left(\frac{\beta}{\alpha}\right)\right)}}{1 - u_s}. \end{aligned} \tag{34}$$

Using the hospital rule, we can easily calculate the limit and therefore:

$$\lambda_U = 2 + \alpha + \beta - \alpha\Phi\left(\frac{a(h)}{2} + \frac{1}{a(h)} \log\left(\frac{\alpha}{\beta}\right)\right) - \beta\Phi\left(\frac{a(h)}{2} + \frac{1}{a(h)} \log\left(\frac{\beta}{\alpha}\right)\right).$$

This therefore corresponds to the result given in (30).

In a similar way, and taking into account the formula for λ_L given in (29), we obtain $\lambda_L = 0$. □

3.5. **Statistical inference.** Statistical inference here consists of estimating the parameters of a model in a general way. In the case of max-stable copulas, parameter estimation is based on methods adapted to the structural constraints of these models. Unlike other families of copulas, max-stable copulas do not always have explicit, closed and simple expressions, and the associated densities quickly become unusable in dimensions greater than 2, i.e. in high dimensions. This makes classical maximum likelihood estimation impossible in practice.

In our case, the copula studied is an asymmetric extension in dimension 2 of the Brown-Resnick copula. It therefore inherits the complexity of max-stable processes, while incorporating an asymmetric dependency. In dimension 2, we were able to obtain the analytical expression of the copula and its density. This allows us to use the maximum likelihood method for parameter estimation without any problems. However, in high dimension, it is difficult to obtain the closed form of the asymmetric multivariate Brown-Resnick copula and even more difficult to talk about its multivariate density.

For these multiple reasons, we adopt the composite estimation method, and more precisely the pairwise likelihood, which is a well-established technique in the literature [17]. This approach consists of approximating the full likelihood by the product of the bivariate likelihoods for all pairs of variables. That is:

$$\ell_{pair}(\theta) = \sum_{i<j} \sum_{k=1}^{n} \log c_\theta^{(i,j)}(u_{ki}, u_{kj}), \tag{35}$$

where $c_\theta^{(i,j)}$ denotes the bivariate density of the Brown-Resnick extended copula for the pair (i, j) , calculated from the pseudo-observations (u_{ki}, u_{kj}), and $\theta = (a(h), \alpha, \beta)$ is the vector of parameters to be estimated.

The estimate is then obtained by maximising this composite log-likelihood

$$\hat{\theta} = \arg\max \ell_{pair}. \tag{36}$$

In practice, this maximisation is possible thanks to numerical optimisation algorithms such as L-BFGS-B and GA (Genetic Algorithm), which make it possible to respect the bounds imposed on the parameters $a(h)$, α and β of our model.

4. Conclusion

In this paper, we present a new asymmetric max-stable spatial copula in dimension two, constructed as an extension of the classical Brown-Resnick spatial copula. This copula can be used to model extreme asymmetric spatial dependence structures that might be encountered in phenomena observed in climate or financial data.

The theoretical analysis carried out established the main properties of the copula, notably its max-stability, its ability to capture dependencies in the tails and its structural link with the Brown-Resnick process. Given the inherent complexity of the copula and multivariate density in this framework, we chose the pairwise likelihood method for parameter estimation, an approach that is now well established for complex extreme dependence models.

This contribution provides an additional tool for modelling spatially dependent extreme phenomena, with increased flexibility thanks to the built-in asymmetry. Interesting prospects include the study of generalisation to higher dimensions, as well as the application of this copula to real datasets, in order to empirically evaluate its performance compared with existing classical models.

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