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A ZADEH'S MAX-MIN COMPOSITION OPERATOR FOR TWO GENERALIZED 3-DIMENSIONAL QUADRATIC FUZZY SETS

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Abstract. We extended the scope to consider operations between general quadratic fuzzy sets, whose maximum membership values are not necessarily equal to 1. We have proven that the max-min composition between two fuzzy sets with peak values h_1 and h_2, where $0 < h_1 < h_2 < 1$, yields a result with a maximum value equal to h_1. Furthermore, we demonstrated that the result of such an operation does not preserve the form of a quadratic fuzzy set, highlighting a significant structural distinction when extending beyond normalized fuzzy numbers. The study of general quadratic fuzzy sets-rather than traditional quadratic fuzzy numbers-opens avenues for broader applications. In particular, our results are expected to contribute significantly to fields requiring high precision and flexibility, such as fuzzy decision-making systems, high-dimensional fuzzy control systems, information fusion and sensor data integration, and fuzzy image processing and segmentation.

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Key words and phrases. max-min composition operator; generalized 3-dimensional quadratic fuzzy sets; α-cut.

1. Introduction

Although various types of fuzzy sets and fuzzy numbers are utilized in fuzzy theory, quadratic fuzzy numbers remain among the most extensively studied and applied. Alongside triangular fuzzy numbers, quadratic fuzzy numbers represent a classical area of research and have been widely employed in numerous fields, including fuzzy decision making [1], fuzzy numerical computation [2], fuzzy regression [3], and engineering design and optimization [4].

In fuzzy theory, operators constitute one of the most fundamental and essential components. Unlike classical sets, fuzzy sets are not defined deterministically; as a result, a variety of operators have been developed to address different analytical needs and practical applications. In addition to the basic operator introduced by Professor Lotfi A. Zadeh, the founder of fuzzy theory, other notable operators

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include the Gödel operator [5] and the Łukasiewicz operator [6], both commonly used in fuzzy logic and fuzzy conditional expressions. Further, the Mamdani operator [7] is widely applied in fuzzy inference systems, while the Sugeno operator [8] and the Dubois-Prade fuzzy synthesis operator [9] are employed in various fuzzy reasoning tasks. The Dombi operator [10], a generalized T-norm and S-norm operator, also plays an important role in this context. These operators have been developed and refined to suit specific domains, and among them, the Zadeh max-min composition operator is particularly notable for its broad applicability in fuzzy control systems [11], decision making and inference [12], and fuzzy database systems.

In our previous work [13], we examined the max-min operator applied to quadratic fuzzy exponents in one dimension. As the complexity and ambiguity of fuzzy systems increased, the necessity for dimensional expansion became evident. This led to the extension of the one-dimensional results to two dimensions, as presented in [14]. In one dimension, the membership function of a quadratic fuzzy number is represented by a quadratic function. In contrast, in two dimensions, it is described by a quadratic surface function. Specifically, when a two-dimensional quadratic fuzzy exponent is intersected by a vertical plane passing through its vertex, the resulting cross-section corresponds to the original one-dimensional quadratic fuzzy number. It is important to emphasize that dimensional expansion is not simply achieved by adding an extra variable or component. Rather, it necessitates a redefinition of operations, transforming the fuzzy operation defined on the real line $\mathbb{R}$ into an operation on the two-dimensional space $\mathbb{R}^2$. Furthermore, when the two-dimensional operation is restricted back to one dimension, it is essential that the original one-dimensional results are preserved, ensuring consistency and compatibility between the dimensions.

In [15], we further extended our research from the two-dimensional space $\mathbb{R}^2$ to the three-dimensional space $\mathbb{R}^3$. While a true graphical representation of a quadratic fuzzy number in 3D would require four-dimensional space—three spatial dimensions plus the membership function value—we addressed this limitation by expressing the membership function as color intensity, enabling visualization within a conventional three-dimensional space. When a 3D graph is sliced by a plane passing through the center of its domain, the resulting cross-sectional view yields a two-dimensional graph, where the membership values appear as color gradients on the cut surface. If this cross-sectional representation is plotted as a 3D surface graph of a function defined on a 2D domain, it matches the previously derived 2D result. This correspondence validates the consistency of our dimensional extension. However, the transition from 2D to 3D is not a trivial extension achieved by simply introducing an additional variable. Instead, it requires the formulation of a new parametric operation that governs the behavior of fuzzy operators in three dimensions while ensuring that, when restricted to a lower dimension, the operation faithfully reproduces the 2D results. This careful design is critical to maintaining both mathematical integrity and interpretational coherence across dimensions.

Previous studies have primarily focused on quadratic fuzzy numbers with a maximum membership value of 1. In such cases, the results of operations also yield fuzzy numbers with a peak value of 1, typically attained at a single point. However, in this study, we extended the scope to consider operations between general quadratic fuzzy sets, whose maximum membership values are not necessarily equal to 1. While the operation between two fuzzy numbers with peak membership 1 maintains this maximum, we have proven that the max-min composition between two fuzzy sets with peak values h_1 and h_2, where $0 < h_1 < h_2 < 1$, yields a result with a maximum value equal to h_1. Furthermore, we demonstrated that the result of such an operation does not preserve the form of a quadratic fuzzy set, highlighting a significant structural distinction when extending beyond normalized fuzzy numbers.

2. Preliminaries

We defined the 2-dimensional quadratic fuzzy numbers on $\mathbb{R}^2$ as a generalization of quadratic fuzzy numbers on $\mathbb{R}$. We defined the parametric operations between two 2-dimensional quadratic fuzzy numbers using region valued α-cuts in $\mathbb{R}^2$. We define α-cut and α-set of the fuzzy set A on $\mathbb{R}$ with the membership function $\mu_A(x)$.

Definition 2.1. An *α-cut* of the fuzzy number A is defined by $A_\alpha = \{x \in \mathbb{R} \mid \mu_A(x) \geq \alpha\}$ if $\alpha \in (0, 1]$ and $A_0 = \text{cl}\{x \in \mathbb{R} \mid \mu_A(x) > \alpha\}$. For $\alpha \in (0, 1)$, the set $A^\alpha = \{x \in X \mid \mu_A(x) = \alpha\}$ is said to be the *α-set* of the fuzzy set A, A^0 is the boundary of $\{x \in \mathbb{R} \mid \mu_A(x) > \alpha\}$ and $A^1 = A_1$.

Definition 2.2. [16] A fuzzy set A with a membership function

$$\mu_A(x, y) = \begin{cases} h - \left(\frac{(x-x_1)^2}{a^2} + \frac{(y-y_1)^2}{b^2}\right), & b^2(x-x_1)^2 + a^2(y-y_1)^2 \leq ha^2b^2, \\ 0, & \text{otherwise}, \end{cases}$$

where $a, b > 0$ and $0 < h < 1$ is called the *the generalized 2-dimensional quadratic fuzzy set* and denoted by $[[a,\ x_1,\ h,\ b,\ y_1]]^2$.

The α-cut A_α of a generalized 2-dimensional quadratic fuzzy set $A = [[a, x_1, h, b, y_1]]^2$ is an interior of ellipse in an xy-plane including the boundary

$$\begin{aligned} A_\alpha &= \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; b^2(x-x_1)^2 + a^2(y-y_1)^2 \leq a^2b^2(h-\alpha)\right\} \\ &= \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; \frac{(x-x_1)^2}{a^2(h-\alpha)} + \frac{(y-y_1)^2}{b^2(h-\alpha)} \leq 1\right\}. \end{aligned}$$

Definition 2.3. [17] Let A be a continuous convex fuzzy number defined on $\mathbb{R}^2$ and $A^\alpha = \{(x, y) \in \mathbb{R}^2 | \mu_A(x, y) = \alpha\}$ be the α-set of A. Then for all $\alpha \in (0, 1)$, there exist continuous functions $f_1^\alpha(t)$ and $f_2^\alpha(t)$ defined on $[0, 2\pi]$ such that

$$A^\alpha = \{(f_1^\alpha(t), f_2^\alpha(t)) \in \mathbb{R}^2 | 0 \leq t \leq 2\pi\}.$$

Definition 2.4. [16] Let A and B be convex fuzzy numbers defined on $\mathbb{R}^2$ and

$$A^\alpha = \{(f_1^\alpha(t), f_2^\alpha(t)) \in \mathbb{R}^2 | 0 \le t \le 2\pi\},$$

$$B^\alpha = \{(g_1^\alpha(t), g_2^\alpha(t)) \in \mathbb{R}^2 | 0 \le t \le 2\pi\}$$

be the α-sets of A and B, respectively. For $\alpha \in (0,1)$, we define that the parametric addition $A(+)_pB$, parametric subtraction $A(-)_pB$, parametric multiplication $A(\cdot)_pB$ and parametric division $A(/)_pB$ of two fuzzy numbers A and B are fuzzy numbers that have their α-sets as follows.

(1) $A(+)_pB$: $(A(+)_pB)^\alpha = \{(f_1^\alpha(t) + g_1^\alpha(t), f_2^\alpha(t) + g_2^\alpha(t)) \in \mathbb{R}^2 | 0 \le t \le 2\pi\}$

(2) $A(-)_pB$: $(A(-)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \in \mathbb{R}^2 | 0 \le t \le 2\pi\}$, where

$$x_\alpha(t) = \begin{cases} f_1^\alpha(t) - g_1^\alpha(t+\pi), & \text{if } 0 \le t \le \pi \\ f_1^\alpha(t) - g_1^\alpha(t-\pi), & \text{if } \pi \le t \le 2\pi \end{cases}$$

and

$$y_\alpha(t) = \begin{cases} f_2^\alpha(t) - g_2^\alpha(t+\pi), & \text{if } 0 \le t \le \pi \\ f_2^\alpha(t) - g_2^\alpha(t-\pi), & \text{if } \pi \le t \le 2\pi \end{cases}$$

(3) $A(\cdot)_pB$: $(A(\cdot)_pB)^\alpha = \{(f_1^\alpha(t) \cdot g_1^\alpha(t), f_2^\alpha(t) \cdot g_2^\alpha(t)) \in \mathbb{R}^2 | 0 \le t \le 2\pi\}$

(4) $A(/)_pB$: $(A(/)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \in \mathbb{R}^2 | 0 \le t \le 2\pi\}$, where

$$x_\alpha(t) = \frac{f_1^\alpha(t)}{g_1^\alpha(t+\pi)} \quad (0 \le t \le \pi), \quad x_\alpha(t) = \frac{f_1^\alpha(t)}{g_1^\alpha(t-\pi)} \quad (\pi \le t \le 2\pi)$$

and

$$y_\alpha(t) = \frac{f_2^\alpha(t)}{g_2^\alpha(t+\pi)} \quad (0 \le t \le \pi), \quad y_\alpha(t) = \frac{f_2^\alpha(t)}{g_2^\alpha(t-\pi)} \quad (\pi \le t \le 2\pi)$$

For $\alpha = 0$ and $\alpha = 1$, $(A(*)_pB)^0 = \lim_{\alpha \to 0^+}(A(*)_pB)^\alpha$ and $(A(*)_pB)^1 = \lim_{\alpha \to 1^-}(A(*)_pB)^\alpha$, where $* = +,\ -,\ \cdot,\ /$.

Theorem 2.5. [16] *Let* $A = [[a_1,\ x_1,\ h_1,\ b_1,\ y_1]]^2$ *and* $B = [[a_2,\ x_2,\ h_2,\ b_2, y_2]]^2$ $(0 < h_1 < h_2 < 1)$ *be two generalized 2-dimensional quadratic fuzzy sets. For* $0 < \alpha < h_1$, *we have the followings.*

(1) *parametric addition* $A(+)_pB$:

$$(A(+)_pB)^\alpha = \Big\{(x,y) \in \mathbb{R}^2 \Big| \Big(\frac{x - x_1 - x_2}{a_1\sqrt{h_1 - \alpha} + a_2\sqrt{h_2 - \alpha}}\Big)^2 + \Big(\frac{y - y_1 - y_2}{b_1\sqrt{h_1 - \alpha} + b_2\sqrt{h_2 - \alpha}}\Big)^2 = 1\Big\}.$$

(2) *parametric subtraction* $A(-)_pB$:

$$(A(-)_pB)^\alpha = \Big\{(x,y) \in \mathbb{R}^2 \Big| \Big(\frac{x - x_1 + x_2}{a_1\sqrt{h_1 - \alpha} + a_2\sqrt{h_2 - \alpha}}\Big)^2 + \Big(\frac{y - y_1 + y_2}{b_1\sqrt{h_1 - \alpha} + b_2\sqrt{h_2 - \alpha}}\Big)^2 = 1\Big\}.$$

(3) *parametric multiplication* $A(\cdot)_pB$: $(A(\cdot)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$, *where*

$$x_\alpha(t) = x_1x_2 + (x_1a_2\sqrt{h_2 - \alpha} + x_2a_1\sqrt{h_1 - \alpha})\cos t + a_1a_2\sqrt{h_1 - \alpha}\sqrt{h_2 - \alpha}\cos^2 t$$

and

$$y_\alpha(t) = y_1y_2 + (y_1b_2\sqrt{h_2-\alpha} + y_2b_1\sqrt{h_1-\alpha})\sin t + b_1b_2\sqrt{h_1-\alpha}\sqrt{h_2-\alpha}\sin^2 t.$$

(4) *parametric division* $A(/)_pB$: $(A(/)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$, *where*

$$x_\alpha(t) = \frac{x_1 + a_1\sqrt{h_1-\alpha}\cos t}{x_2 - a_2\sqrt{h_2-\alpha}\cos t} \quad \text{and} \quad y_\alpha(t) = \frac{y_1 + b_1\sqrt{h_1-\alpha}\sin t}{y_2 - b_2\sqrt{h_2-\alpha}\sin t}.$$

If $\alpha = h_1$, *we have* $(A(*)_pB)^{h_1} = \lim_{\alpha\to h_1^-}(A(*)_pB)^\alpha, \quad * = +,\ -,\ \cdot,\ /$, *and for* $h_1 < \alpha \le h_2$, *by the Zadeh's max-min principle operations, we have to define*

$$(A(*)_pB)^\alpha = \emptyset, \quad * = +,\ -,\ \cdot,\ /$$

Example 2.6. [16] Let $A = [[6,\ 3,\ \frac{1}{2},\ 8,\ 5]]^2$ and $B = [[4,\ 2,\ \frac{2}{3},\ 5,\ 3]]^2$. Then by Theorem 2.5, we have the followings.

(1) For $0 < \alpha < \frac{1}{2}$, the α-set $(A(+)_pB)^\alpha$ of $A(+)_pB$ is

$$\Big\{(x,y)\in\mathbb{R}^2\Big|\Big(\frac{18(x-5)}{3\sqrt{3-6\alpha}+2\sqrt{4-6\alpha}}\Big)^2 + \Big(\frac{36(y-8)}{8\sqrt{3-6\alpha}+5\sqrt{4-6\alpha}}\Big)^2 = 1\Big\}.$$

(2) For $0 < \alpha < \frac{1}{2}$, the α-set $(A(-)_pB)^\alpha$ of $A(-)_pB$ is

$$\Big\{(x,y)\in\mathbb{R}^2\Big|\Big(\frac{18(x-1)}{3\sqrt{3-6\alpha}+2\sqrt{4-6\alpha}}\Big)^2 + \Big(\frac{36(y-2)}{8\sqrt{3-6\alpha}+5\sqrt{4-6\alpha}}\Big)^2 = 1\Big\}.$$

(3) For $0 < \alpha < \frac{1}{2}$, $(A(\cdot)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$, where

$$x_\alpha(t) = 6 + (\frac{4}{3}\sqrt{2-3\alpha} + 3\sqrt{1-2\alpha})\cos t + \frac{2}{3}\sqrt{(1-2\alpha)(2-3\alpha)}\cos^2 t,$$
$$y_\alpha(t) = 15 + (\frac{25}{9}\sqrt{2-3\alpha} + 6\sqrt{1-2\alpha})\sin t + \frac{10}{9}\sqrt{(1-2\alpha)(2-3\alpha)}\sin^2 t.$$

(4) For $0 < \alpha < \frac{1}{2}$, $(A(/)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$, where

$$x_\alpha(t) = \frac{54 + 27\sqrt{1-2\alpha}\cos t}{36 - 8\sqrt{2-3\alpha}\cos t}, \quad y_\alpha(t) = \frac{45 + 18\sqrt{1-2\alpha}\sin t}{27 - 5\sqrt{2-3\alpha}\sin t}.$$

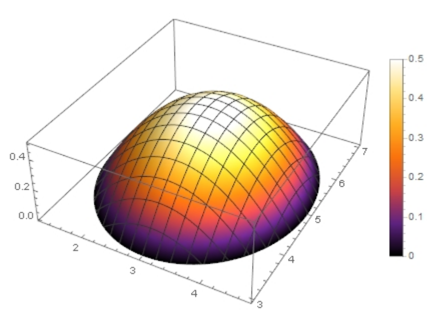

Figure 1. A

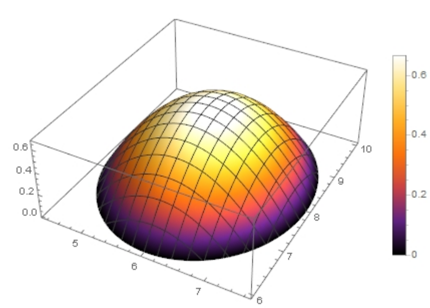

Figure 2. B

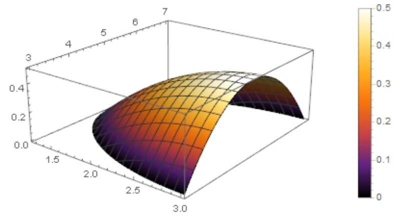

Figure 3. A_1

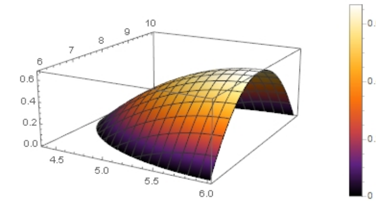

Figure 4. B_1

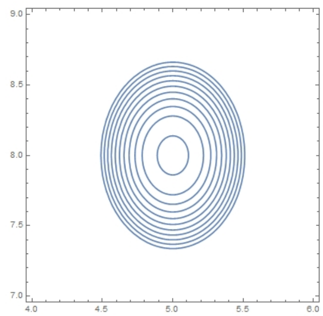

Figure 5. $A(+)B$ $(0 \leq \alpha \leq \frac{1}{2})$

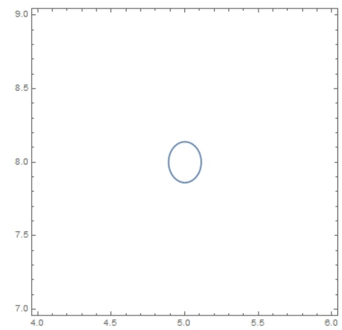

Figure 6. $A(+)B$ $(\frac{1}{2} \leq \alpha \leq 1)$

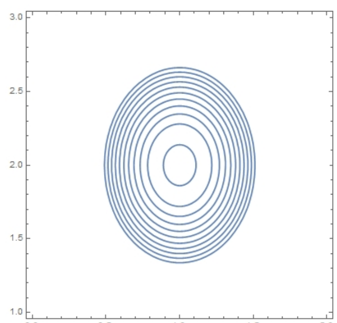

Figure 7. $A(-)B$ $(0 \leq \alpha \leq \frac{1}{2})$

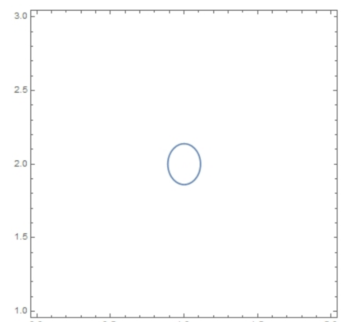

Figure 8. $A(-)B$ $(\frac{1}{2} \leq \alpha \leq 1)$

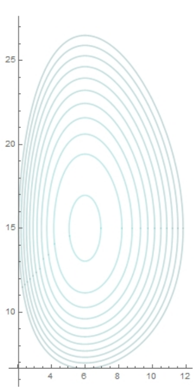

Figure 9. $A(\cdot)B$ $(0 \leq \alpha \leq \frac{1}{2})$

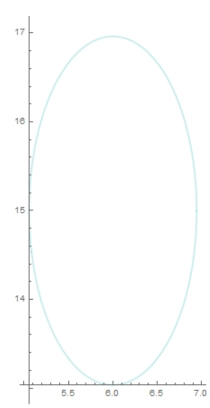

Figure 10. $A(\cdot)B$ $(\frac{1}{2} \leq \alpha \leq 1)$

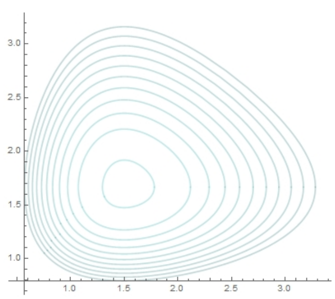

Figure 11. $A(/)B$ $(0 \le \alpha \le \frac{1}{2})$

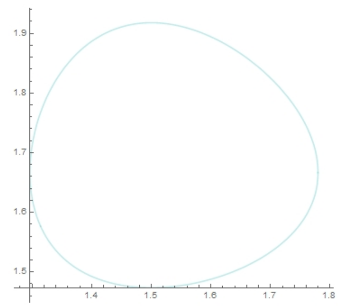

Figure 12. $A(/)B$ $(\frac{1}{2} \le \alpha \le 1)$

Fuzzy set A have a maximum membership value of $\frac{1}{2}$, and fuzzy set B a maximum of $\frac{2}{3}$. Figures 3 and 4 represent vertical cross-sections of the three-dimensional fuzzy sets shown in Figures 1 and 2, respectively. In classical cases where both fuzzy sets have a maximum value of 1, the result of the max-min composition operator also yields a fuzzy set with a maximum of 1. However, when the fuzzy sets have unequal and subunitary maximum values, as in this case, the resulting composition takes the form of a trapezoidal fuzzy set rather than a traditional quadratic fuzzy number. Notably, in each result, the maximum membership value of the composition is $\frac{1}{2}$, which is the lower of the two original maxima. This outcome illustrates the min behavior inherent in the max-min operator. Furthermore, we observe that the α-cut sets only exist for $\alpha \le \frac{1}{2}$. The absence of graphs for $\alpha > \frac{1}{2}$ confirms that no α-cut exists in that range, indicating that the support of the resulting fuzzy set is bounded above by $\frac{1}{2}$.

3. A generalized 3-dimensional quadratic fuzzy set

In this section, we define the generalized 3-dimensional quadratic fuzzy sets on $\mathbb{R}^3$ as a generalization of a quadratic fuzzy numbers on $\mathbb{R}^3$. Then we want to define the parametric operations between two generalized 3-dimensional quadratic fuzzy sets. The α-cuts are regions in $\mathbb{R}^2$ but in $\mathbb{R}^3$ the α-cuts are cubics, which makes the existing method of calculations between α-cuts unusable. We interpret the existing method from a different perspective and apply the method to the cubic valued α-cuts on $\mathbb{R}^3$.

Definition 3.1. A fuzzy set A with a membership function

$$\mu_A(x,y,z) = \begin{cases} h - \left(\frac{(x-x_1)^2}{a^2} + \frac{(y-y_1)^2}{b^2} + \frac{(z-z_1)^2}{c^2}\right), & \text{if } \; b^2c^2(x-x_1)^2 \\ \qquad\qquad +c^2a^2(y-y_1)^2 + a^2b^2(z-z_1)^2 \le a^2b^2c^2h^2, \\ 0, \quad \text{otherwise}, \end{cases}$$

where $a, b, c > 0$ and $0 < h < 1$ is called *the generalized 3-dimensional quadratic fuzzy set* and denoted by $[[h, a, x_1, b, y_1, c, z_1]]^3$.

The α-cut A_α of a 3-dimensional quadratic fuzzy set $A = [h, a, x_1, b, y_1, c, z_1]^3$ is the following set

$$\begin{aligned}A_\alpha &= \left\{(x,y,z)\in\mathbb{R}^3 \;\middle|\; \frac{(x-x_1)^2}{a^2}+\frac{(y-y_1)^2}{b^2}+\frac{(z-z_1)^2}{c^2}\le h-\alpha\right\}\\ &= \left\{(x,y,z)\in\mathbb{R}^3 \;\middle|\; \frac{(x-x_1)^2}{a^2(h-\alpha)}+\frac{(y-y_1)^2}{b^2(h-\alpha)}+\frac{(z-z_1)^2}{c^2(h-\alpha)}\le 1\right\}.\end{aligned}$$

Definition 3.2. A 3-dimensional fuzzy number A defined on $\mathbb{R}^3$ is called *convex* fuzzy number if for all $\alpha \in (0,1)$, the α-cuts

$$A_\alpha = \{(x,y,z)\in\mathbb{R}^3 | \mu_A(x,y,z)\ge\alpha\}$$

are convex subsets in $\mathbb{R}^3$.

Theorem 3.3. [18] *Let A be a continuous convex fuzzy number defined on $\mathbb{R}^3$ and $A^\alpha = \{(x,y,z)\in\mathbb{R}^3|\mu_A(x,y,z)=\alpha\}$ be the α-set of A. Then for all $\alpha\in(0,1)$, there exist continuous functions $f_1^\alpha(s), f_2^\alpha(s,t)$ and $f_3^\alpha(s,t)(0\le s\le 2\pi, 0\le t\le\frac{\pi}{2})$ such that*

$$A^\alpha = \{(f_1^\alpha(s), f_2^\alpha(s,t), f_3^\alpha(s,t))\in\mathbb{R}^3|0\le s\le 2\pi, 0\le t\le\frac{\pi}{2}\}.$$

Definition 3.4. Let A and B are two continuous convex fuzzy numbers defined on $\mathbb{R}^3$ and

$$\begin{aligned}A^\alpha &= \{(x,y,z)\in\mathbb{R}^3|\mu_A(x,y,z)=\alpha\}\\ &= \{(f_1^\alpha(s), f_2^\alpha(s,t), f_3^\alpha(s,t))\in\mathbb{R}^3|0\le s\le 2\pi, 0\le t\le\frac{\pi}{2}\},\\ B^\alpha &= \{(x,y,z)\in\mathbb{R}^3|\mu_B(x,y,z)=\alpha\}\\ &= \{(g_1^\alpha(s), g_2^\alpha(s,t), g_3^\alpha(s,t))\in\mathbb{R}^3|0\le s\le 2\pi, 0\le t\le\frac{\pi}{2}\}\end{aligned}$$

be the α-sets of A and B, respectively. For $\alpha\in(0,1)$, we define that the parametric addition, parametric subtraction, parametric multiplication and parametric division of two fuzzy numbers A and B are fuzzy numbers that have their α-sets as follows.

(1) parametric addition $A(+)_pB$:

$$(A(+)_pB)^\alpha = \{(f_1^\alpha(s)+g_1^\alpha(s),\ f_2^\alpha(s,t)+g_2^\alpha(s,t), f_3^\alpha(s,t)+g_3^\alpha(s,t))\in\mathbb{R}^3|0\le s\le 2\pi, 0\le t\le\frac{\pi}{2}\}$$

(2) parametric subtraction $A(-)_pB$:

$$\begin{aligned}(A(-)_pB)^\alpha &= \{(f_1^\alpha(s)-g_1^\alpha(s+\pi),\ f_2^\alpha(s,t)-g_2^\alpha(s+\pi,t),\\ &\qquad f_3^\alpha(s,t)-g_3^\alpha(s+\pi,t))\in\mathbb{R}^3|0\le s\le \pi, 0\le t\le\frac{\pi}{2}\},\\ (A(-)_pB)^\alpha &= \{(f_1^\alpha(s)-g_1^\alpha(s-\pi),\ f_2^\alpha(s,t)-g_2^\alpha(s-\pi,t),\\ &\qquad f_3^\alpha(s,t)-g_3^\alpha(s-\pi,t))\in\mathbb{R}^3|\pi\le s\le 2\pi, 0\le t\le\frac{\pi}{2}\}\end{aligned}$$

(3) parametric multiplication $A(\cdot)_pB$:

$$(A(\cdot)_pB)^\alpha = \{(f_1^\alpha(s)\cdot g_1^\alpha(s),\ \ f_2^\alpha(s,t)\cdot g_2^\alpha(s,t),\ \ f_3^\alpha(s,t)\cdot g_3^\alpha(s,t))\in\mathbb{R}^3|0\le s\le 2\pi, 0\le t\le\frac{\pi}{2}\}$$

(4) parametric division $A(/)_pB$:

$$(A(/)_pB)^\alpha = \{(\frac{f_1^\alpha(s)}{g_1^\alpha(s+\pi)}, \frac{f_2^\alpha(s,t)}{g_2^\alpha(s+\pi,t)}, \frac{f_3^\alpha(s,t)}{g_3^\alpha(s+\pi,t)}) \in \mathbb{R}^3 | 0 \le s \le \pi, 0 \le t \le \frac{\pi}{2}\},$$

$$(A(/)_pB)^\alpha = \{(\frac{f_1^\alpha(s)}{g_1^\alpha(s-\pi)}, \frac{f_2^\alpha(s,t)}{g_2^\alpha(s-\pi,t)}, \frac{f_3^\alpha(s,t)}{g_3^\alpha(s-\pi,t)}) \in \mathbb{R}^3 | \pi \le s \le 2\pi, 0 \le t \le \frac{\pi}{2}\}$$

For $\alpha = 0$ and $\alpha = 1$, $(A(*)_pB)^0 = \lim_{\alpha\to 0^+}(A(*)_pB)^\alpha$ and $(A(*)_pB)^1 = \lim_{\alpha\to 1^-}(A(*)_pB)^\alpha$, where $* = +,\ -,\ \cdot,\ /$.

For $0 < h_1 < h_2 < 1$, let $A = [h_1,\ a_1,\ x_1,\ b_1,\ y_1,\ c_1,\ z_1]^3$ and $B = [h_2,\ a_2,\ x_2,\ b_2,\ y_2,\ c_2,\ z_2]^3$ be two generalized 3-dimensional quadratic fuzzy sets. If $0 \le \alpha < h_1$, $(A(*)_pB)^\alpha$ can be defined same as Definition 3.4. If $\alpha = h_1$,

$$(A(*)_pB)^{h_1} = \lim_{\alpha\to h_1^-}(A(*)_pB)^\alpha, \quad * = +,\ -,\ \cdot,\ /$$

If $h_1 < \alpha \le h_2$, by the Zadeh's max-min principle operations, we have to define

$$(A(*)_pB)^\alpha = \emptyset, \quad * = +,\ -,\ \cdot,\ /$$

Theorem 3.5. *Let $A = [[h_1, a_1, x_1, b_1, y_1, c_1, z_1]]^3$ and $B = [[h_2, a_2, x_2, b_2, y_2, c_2, z_2]]^3$ be two generalized 3-dimensional quadratic fuzzy sets. If $0 < h_1 < h_2 < 1$, then we have the followings.*
(1) For $0 < \alpha < h_1$, the α-set of $A(+)_pB$ is

$$(A(+)_pB)^\alpha = \Big\{(x,y,z) \in \mathbb{R}^3 \Big| \Big(\frac{x - x_1 - x_2}{a_1\sqrt{h_1-\alpha} + a_2\sqrt{h_2-\alpha}}\Big)^2 + \Big(\frac{y - y_1 - y_2}{b_1\sqrt{h_1-\alpha} + b_2\sqrt{h_2-\alpha}}\Big)^2 + \Big(\frac{z - z_1 - z_2}{c_1\sqrt{h_1-\alpha} + c_2\sqrt{h_2-\alpha}}\Big)^2 = 1\Big\}.$$

(2) For $0 < \alpha < h_1$, the α-set of $A(-)_pB$ is

$$(A(-)_pB)^\alpha = \Big\{(x,y,z) \in \mathbb{R}^3 \Big| \Big(\frac{x - x_1 + x_2}{a_1\sqrt{h_1-\alpha} + a_2\sqrt{h_2-\alpha}}\Big)^2 + \Big(\frac{y - y_1 + y_2}{b_1\sqrt{h_1-\alpha} + b_2\sqrt{h_2-\alpha}}\Big)^2 + \Big(\frac{z - z_1 + z_2}{c_1\sqrt{h_1-\alpha} + c_2\sqrt{h_2-\alpha}}\Big)^2 = 1\Big\}.$$

(3) For $0 < \alpha < h_1$, $(A(\cdot)_pB)^\alpha = \{(x_\alpha(s), y_\alpha(s,t), z_\alpha(s,t)) \in \mathbb{R}^3 \mid 0 \le s \le 2\pi, 0 \le t \le \frac{\pi}{2}\}$, where

$$x_\alpha(s) = x_1x_2 + (x_1a_2\sqrt{h_2-\alpha} + x_2a_1\sqrt{h_1-\alpha})\cos s + a_1a_2\sqrt{(h_1-\alpha)(h_2-\alpha)}\cos^2 s,$$

$$y_\alpha(s,t) = y_1y_2 + (y_1b_2\sqrt{h_2-\alpha} + y_2b_1\sqrt{h_1-\alpha})\sin s\cos t + b_1b_2\sqrt{(h_1-\alpha)(h_2-\alpha)}\sin^2 s\cos^2 t,$$

$$z_\alpha(s,t) = z_1z_2 + (z_1c_2\sqrt{h_2-\alpha} + z_2c_1\sqrt{h_1-\alpha})\sin s\sin t + c_1c_2\sqrt{(h_1-\alpha)(h_2-\alpha)}\sin^2 s\sin^2 t.$$

Furthermore, we have

$$x_0(s) = x_1x_2 + (x_1a_2\sqrt{h_2} + x_2a_1\sqrt{h_1})\cos s + a_1a_2\sqrt{h_1h_2}\cos^2 s,$$
$$y_0(s,t) = y_1y_2 + (y_1b_2\sqrt{h_2} + y_2b_1\sqrt{h_1})\sin s\cos t + b_1b_2\sqrt{h_1h_2}\sin^2 s\cos^2 t,$$
$$z_0(s,t) = z_1z_2 + (z_1c_2\sqrt{h_2} + z_2c_1\sqrt{h_1})\sin s\sin t + c_1c_2\sqrt{h_1h_2}\sin^2 s\sin^2 t,$$
$$x_{h_1}(s) = x_1x_2 + x_1a_2\sqrt{h_2 - h_1}\cos s,$$
$$y_{h_1}(s,t) = y_1y_2 + y_1b_2\sqrt{h_2 - h_1}\sin s\cos t,$$
$$z_{h_1}(s,t) = z_1z_2 + z_1c_2\sqrt{h_2 - h_1}\sin s\sin t,$$

and

$$(A(\cdot)_pB)^\alpha = \emptyset, \quad h_1 < \alpha \le h_2.$$

(4) *For* $0 < \alpha < h_1$, $(A(/)_pB)^\alpha = \{(x_\alpha(s), y_\alpha(s,t), z_\alpha(s,t)) \in \mathbb{R}^3 \mid 0 \le s \le 2\pi, 0 \le t \le \frac{\pi}{2}\}$, *where*

$$x_\alpha(s) = \frac{x_1 + a_1\sqrt{h_1-\alpha}\cos s}{x_2 - a_2\sqrt{h_2-\alpha}\cos s}, \quad y_\alpha(s,t) = \frac{y_1 + b_1\sqrt{h_1-\alpha}\sin s\cos t}{y_2 - b_2\sqrt{h_2-\alpha}\sin s\cos t}$$

and

$$z_\alpha(s,t) = \frac{z_1 + c_1\sqrt{h_1-\alpha}\sin s\sin t}{z_2 - c_2\sqrt{h_2-\alpha}\sin s\sin t}.$$

Furthermore, we have

$$x_0(s) = \frac{x_1 + a_1\sqrt{h_1}\cos s}{x_2 - a_2\sqrt{h_2}\cos s}, \quad y_0(s,t) = \frac{y_1 + b_1\sqrt{h_1}\sin s\cos t}{y_2 - b_2\sqrt{h_2}\sin s\cos t},$$
$$z_0(s,t) = \frac{z_1 + c_1\sqrt{h_1}\sin s\sin t}{z_2 - c_2\sqrt{h_2}\sin s\sin t}$$

and

$$(A(/)_pB)^\alpha = \emptyset, \quad h_1 < \alpha \le h_2.$$

Proof. Since A and B are continuous convex fuzzy sets defined on $\mathbb{R}^3$, by Theorem 3.3, there exists $f_1^\alpha(s), g_1^\alpha(s), f_i^\alpha(s,t), g_i^\alpha(s,t)$ $(i = 2,3)$ such that

$$A^\alpha = \{(f_1^\alpha(s), f_2^\alpha(s,t), f_3^\alpha(s,t)) \in \mathbb{R}^3 | 0 \le s \le 2\pi, 0 \le t \le \frac{\pi}{2}\},$$

and

$$B^\alpha = \{(g_1^\alpha(s), g_2^\alpha(s,t), g_3^\alpha(s,t)) \in \mathbb{R}^3 | 0 \le s \le 2\pi, 0 \le t \le \frac{\pi}{2}\}.$$

Since $A = [h_1, a_1, x_1, b_1, y_1, c_1, z_1]^3$ and $B = [h_2, a_2, x_2, b_2, y_2, c_2, z_2]^3$, we have

$$f_1^\alpha(s) = x_1 + a_1\sqrt{h_1-\alpha}\cos s, \quad f_2^\alpha(s,t) = y_1 + b_1\sqrt{h_1-\alpha}\sin s\cos t$$
$$f_3^\alpha(s,t) = z_1 + c_1\sqrt{h_1-\alpha}\sin s\sin t$$

and

$$g_1^\alpha(s) = x_2 + a_2\sqrt{h_2-\alpha}\cos s, \quad g_2^\alpha(s,t) = y_2 + b_2\sqrt{h_2-\alpha}\sin s\cos t$$
$$g_3^\alpha(s,t) = z_2 + c_2\sqrt{h_2-\alpha}\sin s\sin t.$$

(1) Since

$$f_1^\alpha(s)+g_1^\alpha(s)=x_1+x_2+(a_1\sqrt{h_1-\alpha}+a_2\sqrt{h_2-\alpha})\cos s,$$

$$f_2^\alpha(s,t)+g_2^\alpha(s,t)=y_1+y_2+(b_1\sqrt{h_1-\alpha}+b_2\sqrt{h_2-\alpha})\sin s\cos t$$

and

$$f_3^\alpha(s,t)+g_3^\alpha(s,t)=z_1+z_2+(c_1\sqrt{h_1-\alpha}+c_2\sqrt{h_2-\alpha})\sin s\sin t,$$

we have

$$(A(+)_pB)^\alpha=\Big\{(x,y,z)\in\mathbb{R}^3\Big|\Big(\frac{x-x_1-x_2}{a_1\sqrt{h_1-\alpha}+a_2\sqrt{h_2-\alpha}}\Big)^2+\Big(\frac{y-y_1-y_2}{b_1\sqrt{h_1-\alpha}+b_2\sqrt{h_2-\alpha}}\Big)^2+\Big(\frac{z-z_1-z_2}{c_1\sqrt{h_1-\alpha}+c_2\sqrt{h_2-\alpha}}\Big)^2=1\Big\}.$$

(2) If $0\le s\le\pi, 0<t<\frac{\pi}{2}$ and $0<\alpha<h_1$,

$$f_1^\alpha(s)-g_1^\alpha(s+\pi)=x_1-x_2+(a_1\sqrt{h_1-\alpha}+a_2\sqrt{h_2-\alpha})\cos s,$$

$$f_2^\alpha(s,t)-g_2^\alpha(s+\pi,t)=y_1-y_2+(b_1\sqrt{h_1-\alpha}+b_2\sqrt{h_2-\alpha})\sin s\cos t$$

and

$$f_3^\alpha(s,t)-g_3^\alpha(s+\pi,t)=z_1-z_2+(c_1\sqrt{h_1-\alpha}+c_2\sqrt{h_2-\alpha})\sin s\sin t.$$

In the case of $\pi\le s\le 2\pi, 0<t<\frac{\pi}{2}$, we have

$$f_1^\alpha(s)-g_1^\alpha(s-\pi)=f_1^\alpha(s)-g_1^\alpha(s+\pi)$$

$$f_2^\alpha(s,t)-g_2^\alpha(s-\pi,t)=f_2^\alpha(s,t)-g_2^\alpha(s+\pi,t)$$

and

$$f_3^\alpha(s,t)-g_3^\alpha(s-\pi,t)=f_3^\alpha(s,t)-g_3^\alpha(s+\pi,t).$$

Thus

$$(A(-)_pB)^\alpha=\Big\{(x,y,z)\in\mathbb{R}^3\Big|\Big(\frac{x-x_1+x_2}{a_1\sqrt{h_1-\alpha}+a_2\sqrt{h_2-\alpha}}\Big)^2+\Big(\frac{y-y_1+y_2}{b_1\sqrt{h_1-\alpha}+b_2\sqrt{h_2-\alpha}}\Big)^2+\Big(\frac{z-z_1+z_2}{c_1\sqrt{h_1-\alpha}+c_2\sqrt{h_2-\alpha}}\Big)^2=1\Big\}.$$

(3) Let $(A(\cdot)_pB)^\alpha=\{(x_\alpha(s),y_\alpha(s,t),z_\alpha(s,t))\mid 0\le s\le 2\pi, 0\le t\le\frac{\pi}{2}\}$. From $f_1^\alpha(s), g_1^\alpha(s)$, $f_i^\alpha(s,t), g_i^\alpha(s,t)$ $(i=2,3)$, we have

$$x_\alpha(s)=x_1x_2+(x_1a_2\sqrt{h_2-\alpha}+x_2a_1\sqrt{h_1-\alpha})\cos s+a_1a_2\sqrt{(h_1-\alpha)(h_2-\alpha)}\cos^2 s,$$

$$y_\alpha(s,t)=y_1y_2+(y_1b_2\sqrt{h_2-\alpha}+y_2b_1\sqrt{h_1-\alpha})\sin s\cos t+b_1b_2\sqrt{(h_1-\alpha)(h_2-\alpha)}\sin^2 s\cos^2 t,$$

$$z_\alpha(s,t)=z_1z_2+(z_1c_2\sqrt{h_2-\alpha}+z_2c_1\sqrt{h_1-\alpha})\sin s\sin t+c_1c_2\sqrt{(h_1-\alpha)(h_2-\alpha)}\sin^2 s\sin^2 t.$$

Furthermore, we have

$$x_0(s) = x_1x_2 + (x_1a_2\sqrt{h_2} + x_2a_1\sqrt{h_1})\cos s + a_1a_2\sqrt{h_1h_2}\cos^2 s,$$
$$y_0(s,t) = y_1y_2 + (y_1b_2\sqrt{h_2} + y_2b_1\sqrt{h_1})\sin s\cos t + b_1b_2\sqrt{h_1h_2}\sin^2 s\cos^2 t,$$
$$z_0(s,t) = z_1z_2 + (z_1c_2\sqrt{h_2} + z_2c_1\sqrt{h_1})\sin s\sin t + c_1c_2\sqrt{h_1h_2}\sin^2 s\sin^2 t,$$
$$x_{h_1}(s) = x_1x_2 + x_1a_2\sqrt{h_2-h_1}\cos s,$$
$$y_{h_1}(s,t) = y_1y_2 + y_1b_2\sqrt{h_2-h_1}\sin s\cos t,$$
$$z_{h_1}(s,t) = z_1z_2 + z_1c_2\sqrt{h_2-h_1}\sin s\sin t$$

and

$$(A(\cdot)_pB)^\alpha = \emptyset, \quad h_1 < \alpha \le h_2.$$

(4) Let $(A(/)_pB)^\alpha = \{(x_\alpha(s), y_\alpha(s,t), z_\alpha(s,t)) \mid 0 \le s \le 2\pi, 0 \le t \le \frac{\pi}{2}\}$. Similarly, we have if $0 < \alpha < h_1$,

$$x_\alpha(s) = \frac{x_1 + a_1\sqrt{h_1-\alpha}\cos s}{x_2 - a_2\sqrt{h_2-\alpha}\cos s} \qquad y_\alpha(s,t) = \frac{y_1 + b_1\sqrt{h_1-\alpha}\sin s\cos t}{y_2 - b_2\sqrt{h_2-\alpha}\sin s\cos t}$$

and

$$z_\alpha(s,t) = \frac{z_1 + c_1\sqrt{h_1-\alpha}\sin s\sin t}{z_2 - c_2\sqrt{h_2-\alpha}\sin s\sin t}.$$

Furthermore, we have

$$x_0(s) = \frac{x_1 + a_1\sqrt{h_1}\cos s}{x_2 - a_2\sqrt{h_2}\cos s}, \qquad y_0(s,t) = \frac{y_1 + b_1\sqrt{h_1}\sin s\cos t}{y_2 - b_2\sqrt{h_2}\sin s\cos t},$$
$$z_0(s,t) = \frac{z_1 + c_1\sqrt{h_1}\sin s\sin t}{z_2 - c_2\sqrt{h_2}\sin s\sin t}$$

and

$$(A(/)_pB)^\alpha = \emptyset, \quad h_1 < \alpha \le h_2.$$

The proof is complete. □

Example 3.6. Let $A = [[\frac{1}{2},\ 6,\ 3,\ 8,\ 5,\ 4,\ 7]]^3$ and $B = [[\frac{2}{3},\ 4,\ 2,\ 5,\ 3,\ 6,\ 4]]^3$. Then by Theorem 3.5, we have the followings.

(1) For $0 < \alpha < \frac{1}{2}$, the α-set of $A(+)_pB$ is

$$(A(+)_pB)^\alpha = \Big\{(x,y,z) \in \mathbb{R}^3 \Big| \Big(\frac{3(x-5)}{9\sqrt{2-4\alpha} + 4\sqrt{6-9\alpha}}\Big)^2 + \Big(\frac{3(y-8)}{12\sqrt{2-4\alpha} + 5\sqrt{6-9\alpha}}\Big)^2 + \Big(\frac{z-11}{2\sqrt{2-4\alpha} + 2\sqrt{6-9\alpha}}\Big)^2 = 1\Big\}.$$

(2) For $0 < \alpha < \frac{1}{2}$, the α-set of $A(-)_pB$ is

$$(A(-)_pB)^\alpha = \Big\{(x,y,z)\in\mathbb{R}^3 \Big| \Big(\frac{3(x-1)}{9\sqrt{2-4\alpha}+4\sqrt{6-9\alpha}}\Big)^2 + \Big(\frac{3(y-2)}{12\sqrt{2-4\alpha}+5\sqrt{6-9\alpha}}\Big)^2 + \Big(\frac{z-3}{2\sqrt{2-4\alpha}+2\sqrt{6-9\alpha}}\Big)^2 = 1\Big\}.$$

(3) For $0<\alpha<\frac{1}{2}$, $(A(\cdot)_pB)^\alpha = \{(x_\alpha(s), y_\alpha(s,t), z_\alpha(s,t)) \mid 0\le s\le 2\pi, 0\le t\le \frac{\pi}{2}\}$, where

$$x_\alpha(s) = 6 + (4\sqrt{6-9\alpha}+6\sqrt{2-4\alpha})\cos s + 4\sqrt{(2-4\alpha)(6-9\alpha)}\cos^2 s,$$

$$y_\alpha(s,t) = 15 + (\frac{25}{3}\sqrt{6-9\alpha}+12\sqrt{2-4\alpha})\sin s\cos t + \frac{20}{3}\sqrt{(2-4\alpha)(6-9\alpha)}\sin^2 s\cos^2 t$$

and

$$z_\alpha(s,t) = 28 + (14\sqrt{6-9\alpha}+8\sqrt{2-4\alpha})\sin s\sin t + 4\sqrt{(2-4\alpha)(6-9\alpha)}\sin^2 s\sin^2 t.$$

(4) For $0<\alpha<\frac{1}{2}$, $(A(/)_pB)^\alpha = \{(x_\alpha(s), y_\alpha(s,t), z_\alpha(s,t)) \mid 0\le s\le 2\pi, 0\le t\le \frac{\pi}{2}\}$, where

$$x_\alpha(s) = \frac{9+9\sqrt{2-4\alpha}\cos s}{6-4\sqrt{6-9\alpha}\cos s} \quad y_\alpha(s,t) = \frac{15+12\sqrt{2-4\alpha}\sin s\cos t}{9-5\sqrt{6-9\alpha}\sin s\cos t}$$

and

$$z_\alpha(s,t) = \frac{7+2\sqrt{2-4\alpha}\sin s\sin t}{4-2\sqrt{6-9\alpha}\sin s\sin t}.$$

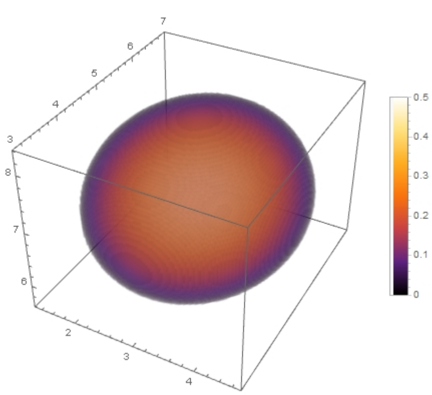

Figure 13. A

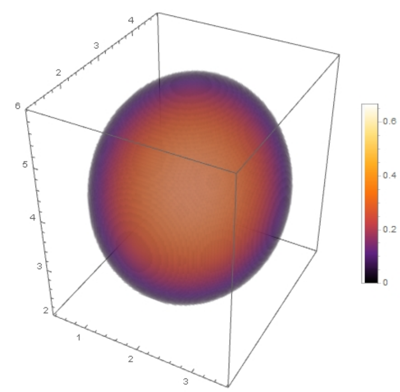

Figure 14. B

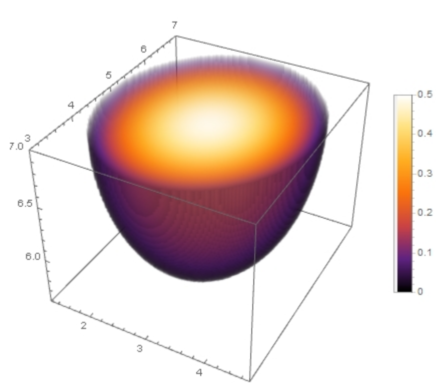

Figure 15. $A/2$

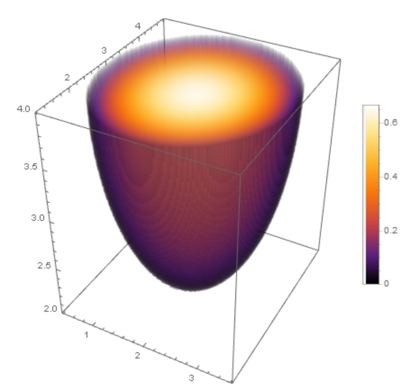

Figure 16. $B/2$

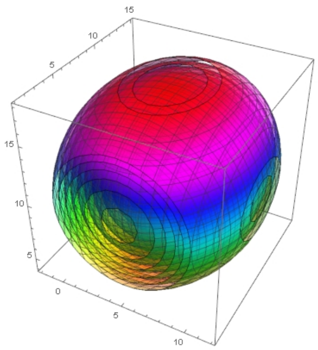

Figure 17. $A(+)B$ $(0 \leq \alpha \leq \frac{1}{2})$

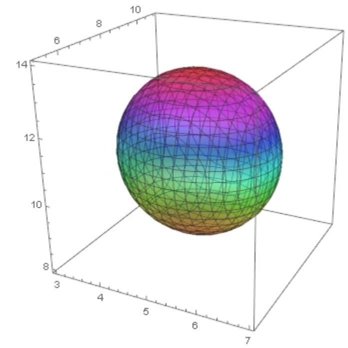

Figure 18. $A(+)B$ $(\frac{1}{2} \leq \alpha \leq 1)$

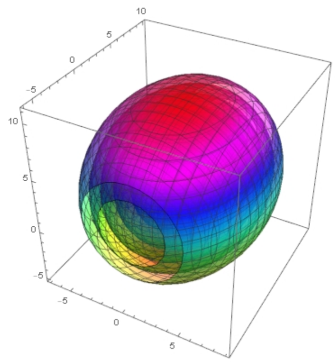

Figure 19. $A(-)B$ $(0 \leq \alpha \leq \frac{1}{2})$

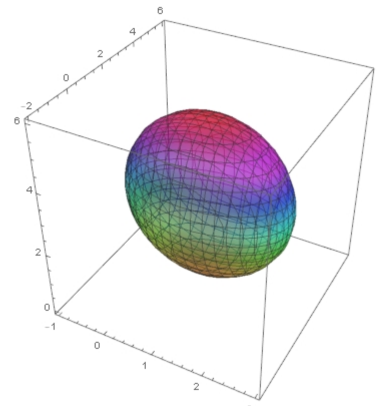

Figure 20. $A(-)B$ $(\frac{1}{2} \leq \alpha \leq 1)$

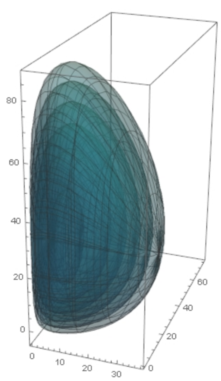

Figure 21. $A(\cdot)B$ $(0 \leq \alpha \leq \frac{1}{2})$

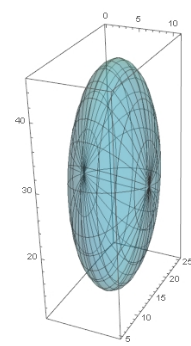

Figure 22. $A(\cdot)B$ $(\frac{1}{2} \leq \alpha \leq 1)$

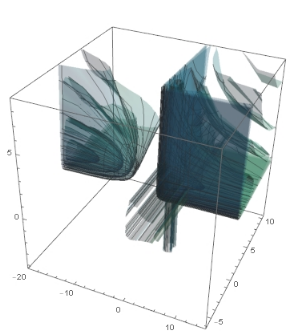

Figure 23. $A(/)B$ $(0 \leq \alpha \leq \frac{1}{2})$

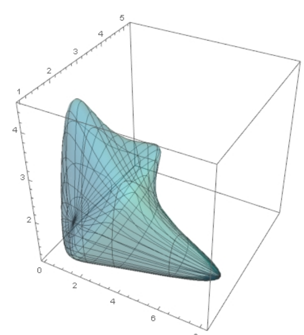

Figure 24. $A(/)B$ $(\frac{1}{2} \leq \alpha \leq 1)$

Since the domain under consideration is three-dimensional, the membership function values are visualized using color density. As illustrated in Figures 15 and 16, cross-sectional cuts of the fuzzy sets allow us to observe the internal membership values through color gradation. By examining the color bar on the right side of each figure, it can be confirmed that fuzzy set A has a maximum membership value of $\frac{1}{2}$, while fuzzy set B has a maximum of $\frac{2}{3}$. For each result of the max-min composition operator, six α-cut surfaces are visualized for the range $0 \leq \alpha \leq \frac{1}{2}$. These α-cuts are clearly distinguishable, as several overlapping 3D surfaces are rendered in each composite result. Although the system was designed to visualize six α-cut surfaces in the range $\frac{1}{2} \leq \alpha \leq 1$ as well, only a single surface is actually displayed for each case. Moreover, each of these matches the graph corresponding to $\alpha = \frac{1}{2}$, providing clear evidence that no α-cut sets exist for $\alpha > \frac{1}{2}$. Consistent with the two-dimensional case, the results of the max-min composition in three dimensions take the form of trapezoidal fuzzy sets, rather than traditional quadratic fuzzy numbers. The maximum membership value of each composite result is $\frac{1}{2}$, which corresponds to the minimum of the two original maxima. Furthermore, we observe that α-cuts exist only for $0 \leq \alpha \leq \frac{1}{2}$. The absence of any visualized α-cuts for $\alpha > \frac{1}{2}$ confirms that the support of the resulting fuzzy set is bounded above by $\frac{1}{2}$. This visual evidence provides a strong graphical validation of Theorem 3.5, which states that when applying the max-min composition operator to fuzzy sets with maximum values h_1 and h_2(where $0 < h_1 < h_2 < 1$), the resulting fuzzy set will have a maximum value of h_1, and α-cut sets will exist only for $0 \leq \alpha \leq h_1$.

4. Conclusion

In Chapter 2, we analyzed the max-min composite operator applied to general two-dimensional quadratic fuzzy sets. Given two such sets A and B, with respective maximum membership values h_1 and h_2 (where $0 < h_1 < h_2 < 1$) , we found that the resulting α-cuts from the operation exist only for $0 < \alpha < h_1$; that is, no α-cuts exist for values greater than h_1. We illustrated this behavior through a graphical example. Upon interpreting the resulting graphs, we observed that operations such as $A(+)B$ and $A(-)B$ largely preserve the quadratic fuzzy form in the outer sections of the domain. However, the overall shape more closely resembles that of a trapezoidal fuzzy set, where the graph is composed of quadratic curves on both ends and a flatter central section. In contrast, the remaining two operations yielded more complex structures, emphasizing the utility of graphical representations in enhancing interpretability and application.

In Chapter 3, we extended our computations to general three-dimensional quadratic fuzzy sets. As with the two-dimensional case, two primary structural results emerged: one resembling a trapezoidal shape, and the other displaying complex, non-quadratic structures. While a 3D graph was presented for illustrative purposes, it is more challenging to directly perceive similarities with the 2D case. This is because the membership values in 3D are represented as color intensities, revealing only the values along the outer surface of the fuzzy set. To fully understand the membership distribution, it is necessary to analyze cross-sectional views. From such cross-sectional analysis, we observed that the color intensity remains relatively uniform in the central region, suggesting that the overall structure maintains a trapezoidal-like profile, similar to the 2D case.

These findings further reinforce the value of studying general quadratic fuzzy sets, which extend beyond the limitations of classical quadratic fuzzy numbers. The ability to model non-normal fuzzy sets and capture a broader range of uncertainty structures makes these results especially applicable in domains demanding high precision and flexibility, including fuzzy decision-making systems [19], high-dimensional fuzzy control systems [20], information fusion and sensor data integration [21], and fuzzy image processing and segmentation [22].

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