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SOME PROPERTIES OF THE PRODUCT SEQUENCES INVOLVING FIBONACCI NUMBERS

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Abstract. Let F_n denote the n-th Fibonacci number for $n \in \mathbb{N}$. For every $m \in \mathbb{N}$, we define the P-Fibonacci sequence $P_{m,n}$ by the recurrence

$$P_{m,n} = F_m F_n (F_{n+1} - F_n) + F_{m+1} (F_n)^2.$$

In this paper, we investigate the structure and properties of the P-Fibonacci sequences, which arise from specific algebraic combinations of Fibonacci numbers. Through analytical exploration and pattern recognition, we uncover and prove several intriguing identities related to these sequences. We also present illustrative examples to highlight their recurring behaviors. This study not only contributes to the theoretical understanding of Fibonacci-related sequences but also lays groundwork for potential applications and further research in combinatorics, number theory, and algorithmic design.

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1. Introduction

The Fibonacci sequence is one of the most well-known and widely studied sequences in number theory. It was first introduced in 1202 by Leonardo of Pisa—better known as Fibonacci—in his book *Liber Abaci*, which presented the famous rabbit population problem. The name "Fibonacci," derived from *filius Bonacci* (son of Bonacci), was later popularized and has since become synonymous with the sequence itself [4] [6].

Beyond pure mathematics, the Fibonacci sequence is frequently observed in nature. In 2003, Posamentier and Lehmann in [5] highlighted various natural patterns linked to Fibonacci numbers – the number of spirals of seeds in a sunflower is the Fibonacci pairs: 13 (left-oriented spirals): 21 (right-oriented spirals), 21:34, 34:55, 55:89, and 89:144 which ensures an optimal packing of the seeds. Also, the leaf arrangement (*Phyllotaxis*) of some plants follows the Golden Ratio of the Fibonacci numbers $\phi \approx 1.618$, which enables them to receive the maximum amount of sunlight for photosynthesis.

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From a theoretical standpoint, Fibonacci numbers have long attracted attention for their rich structure and intriguing properties. In 1961, Vorob'ev [6] explored foundational identities of the sequence, while Burton [3] later expanded on these results in 2010, establishing additional algebraic and combinatorial relationships. More recently, Khaochim and Pongsriiam [2] examined the order of appearance of Fibonacci number products in modular arithmetic.

Building upon these foundational works, this paper introduces and investigates a new class of sequences derived from the products of Fibonacci numbers. Specifically, we define the P-Fibonacci sequence $P_{m,n}$, which generalizes the relationship between F_n and F_{n+m} using a two-parameter formula involving both multiplicative and additive components of the Fibonacci terms. Through this formulation, we uncover new identities, explore summation properties, and reveal recursive patterns embedded within this novel structure.

2. Preliminaries

In this section, we present the foundational definitions and theorems necessary for the development of our results. These include basic number-theoretic concepts such as the greatest common divisor and properties of the Fibonacci sequence.

Definition 1. [3] Let a and b be integers, with at least one of them nonzero. The greatest common divisor of a and b, denoted by $\gcd(a, b)$, is the positive integer d satisfying the following:

(a) $d \mid a$ and $d \mid b$.
(b) If $c \mid a$ and $c \mid b$, then $c \leq d$.

Theorem 2.1. [3] If $k > 0$, then $gcd(ka, kb) = kgcd(a, b)$.

Definition 2. [3] Two integers a and b, not both zero, are said to be relatively prime if $\gcd(a, b) = 1$.

Theorem 2.2. [3] Let a and b be integers, not both zero. Then a and b are relatively prime if and only if there exist integers x and y such that $1 = ax + by$.

Theorem 2.3. [1] If a and b are two consecutive integers such that $a = n$ and $b = n + 1$, where $n \in \mathbb{Z}$, then a and b are relatively prime.

Definition 3. [2] A Fibonacci sequence $(F_n)_{n\geq 1}$ is defined by $F_1 = F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. Each term of the F_n is called a Fibonacci number.

We now present some important identities related to the Fibonacci sequence obtained from [3,6,7].

Theorem 2.4. For all $n \in \mathbb{N}$, $gcd(F_n, F_{n+1}) = 1$.

Theorem 2.5. The greatest common divisor of two Fibonacci numbers is a Fibonacci number, that is, $gcd(F_m, F_n) = F_{gcd(m,n)}$.

Theorem 2.6. Let $n \in \mathbb{N}$. Then the sum of the first n Fibonacci numbers is given by

$$F_1 + F_2 + F_3 + \cdots + F_{n-1} + F_n = F_{n+2} - 1.$$

Theorem 2.7. Let $n \in \mathbb{N}$. Then the sum of the squares of the first n Fibonacci numbers is given by

$$(F_1)^2 + (F_2)^2 + (F_3)^2 + \cdots + (F_n)^2 = F_n F_{n+1}.$$

Consider the table below that consists of the sequences of products of F_n and F_{n+m} for $1 \leq m \leq 5$.

Table 1. Sequences of Products of F_n and F_{n+m} for $1 \leq m \leq 5$

n	F_n	F_nF_{n+1}	F_nF_{n+2}	F_nF_{n+3}	F_nF_{n+4}	F_nF_{n+5}
1	1	1	2	3	5	8
2	1	2	3	5	8	13
3	2	6	10	16	26	42
4	3	15	24	39	63	102
5	5	40	65	105	170	275
6	8	104	168	272	440	712
7	13	273	442	715	1157	1872
8	21	714	1155	1869	3024	4893
9	34	1870	3026	4896	7922	12818
10	55	4895	7920	12815	20735	33550
11	89	12816	20737	33553	54290	87843
12	144	33552	54288	87840	142128	229968
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

The first pattern that most of us could possibly observe from Table 1 is that, for $3 \leq m \leq 5$, the nth terms of each sequences can be obtained by adding the two nth terms coming respectively from two preceding sequences. And, if you continue the process of multiplying F_n and F_{n+m}, the pattern repeats for $m > 5$ since

$$F_nF_{n+m} + F_nF_{n+m+1} = F_n(F_{n+m} + F_{n+m+1}) = F_nF_{n+m+2}.$$

Now, another pattern emerged within the terms of each sequence, that is, we can express them as a combination of Fibonacci numbers, as shown in Tables 2 to 6.

TABLE 2. Sequence of Products of F_n and F_{n+m} for $m = 1$

n	F_n	F_nF_{n+1}	
1	1	1	$= (1)(0) + (1)^2$
2	1	2	$= (1)(1) + (1)^2$
3	2	6	$= (2)(1) + (2)^2$
4	3	15	$= (3)(2) + (3)^2$
5	5	40	$= (5)(3) + (5)^2$
6	8	104	$= (8)(5) + (8)^2$
7	13	273	$= (13)(8) + (13)^2$
8	21	714	$= (21)(13) + (21)^2$
$\vdots$	$\vdots$		$\vdots$
n	F_n	F_nF_{n+1}	$= F_n(F_{n+1} - F_n) + (F_n)^2$

TABLE 3. Sequence of Products of F_n and F_{n+m} for $m = 2$

n	F_n	F_nF_{n+2}	
1	1	2	$= (1)(0) + (2)(1)^2$
2	1	3	$= (1)(1) + (2)(1)^2$
3	2	10	$= (2)(1) + (2)(2)^2$
4	3	24	$= (3)(2) + (2)(3)^2$
5	5	65	$= (5)(3) + (2)(5)^2$
6	8	168	$= (8)(5) + (2)(8)^2$
7	13	442	$= (13)(8) + (2)(13)^2$
8	21	1155	$= (21)(13) + (2)(21)^2$
$\vdots$	$\vdots$		$\vdots$
n	F_n	F_nF_{n+2}	$= F_n(F_{n+1} - F_n) + 2(F_n)^2$

TABLE 4. Sequence of Products of F_n and F_{n+m} for $m = 3$

n	F_n			F_nF_{n+3}
1	1	3	=	$(2)(1)(0) + (3)(1)^2$
2	1	5	=	$(2)(1)(1) + (3)(1)^2$
3	2	16	=	$(2)(2)(1) + (3)(2)^2$
4	3	39	=	$(2)(3)(2) + (3)(3)^2$
5	5	105	=	$(2)(5)(3) + (3)(5)^2$
6	8	272	=	$(2)(8)(5) + (3)(8)^2$
7	13	715	=	$(2)(13)(8) + (3)(13)^2$
8	21	1869	=	$(2)(21)(13) + (3)(21)^2$
⋮	⋮			⋮
n	F_n	F_nF_{n+3}	=	$2F_n(F_{n+1} - F_n) + 3(F_n)^2$

TABLE 5. Sequence of Products of F_n and F_{n+m} for $m = 4$

n	F_n			F_nF_{n+4}
1	1	5	=	$(3)(1)(0) + (5)(1)^2$
2	1	8	=	$(3)(1)(1) + (5)(1)^2$
3	2	26	=	$(3)(2)(1) + (5)(2)^2$
4	3	63	=	$(3)(3)(2) + (5)(3)^2$
5	5	170	=	$(3)(5)(3) + (5)(5)^2$
6	8	440	=	$(3)(8)(5) + (5)(8)^2$
7	13	1157	=	$(3)(13)(8) + (5)(13)^2$
8	21	3024	=	$(3)(21)(13) + (5)(21)^2$
⋮	⋮			⋮
n	F_n	F_nF_{n+4}	=	$3F_n(F_{n+1} - F_n) + 5(F_n)^2$

Through computational observation and analysis of the products F_nF_{n+m} for small values of m, a clear structure begins to emerge in the resulting expressions. As shown in Tables 2 to 6, for values $m = 1$ through $m = 5$, each product can be rewritten in the form

$$F_nF_{n+m} = a_mF_n(F_{n+1} - F_n) + b_m(F_n)^2,$$

TABLE 6. Sequence of Products of F_n and F_{n+m} for $m = 5$

n	F_n	F_nF_{n+5}	
1	1	8	$= (5)(1)(0) + (8)(1)^2$
2	1	13	$= (5)(1)(1) + (8)(1)^2$
3	2	42	$= (5)(2)(1) + (8)(2)^2$
4	3	102	$= (5)(3)(2) + (8)(3)^2$
5	5	275	$= (5)(5)(3) + (8)(5)^2$
6	8	712	$= (5)(8)(5) + (8)(8)^2$
7	13	1872	$= (5)(13)(8) + (8)(13)^2$
8	21	4893	$= (5)(21)(13) + (8)(21)^2$
$\vdots$	$\vdots$		$\vdots$
n	F_n	F_nF_{n+5}	$= 5F_n(F_{n+1} - F_n) + 8(F_n)^2$

where $a_m = F_m$ and $b_m = F_{m+1}$, the Fibonacci numbers themselves. This recurring structure is summarized in the Table 7, and it reveals a fascinating self-similarity: Fibonacci numbers not only define the sequence F_n, but also govern the coefficients in expressions involving their shifted products.

TABLE 7. General Equation of the Product Sequences of F_n and F_{n+m}

m	F_nF_{n+m}	
1	F_nF_{n+1}	$= F_n(F_{n+1} - F_n) + (F_n)^2$
2	F_nF_{n+2}	$= F_n(F_{n+1} - F_n) + 2(F_n)^2$
3	F_nF_{n+3}	$= 2F_n(F_{n+1} - F_n) + 3(F_n)^2$
4	F_nF_{n+4}	$= 3F_n(F_{n+1} - F_n) + 5(F_n)^2$
5	F_nF_{n+5}	$= 5F_n(F_{n+1} - F_n) + 8(F_n)^2$
$\vdots$		$\vdots$
m	F_nF_{n+m}	$= F_mF_n(F_{n+1} - F_n) + F_{m+1}(F_n)^2$

From this observation, we define a new sequence, termed the P-Fibonacci sequence, in the following section.

3. Main Results

Definition 4. *Let $m, n \in \mathbb{N}$. The P-Fibonacci sequence, denoted by $P_{m,n}$, of each pair F_n and F_{n+m} for every m is defined by*

$$P_{m,n} = F_mF_n(F_{n+1} - F_n) + F_{m+1}(F_n)^2.$$

A term of the P-Fibonacci sequence is called a P-Fibonacci number.

This formula captures the consistent structure observed in the initial tables and extends the product relationship between F_n and F_{n+m} into a two-parameter sequence. We then analyze the properties of this new sequence, starting with closed-form summation formulas for the case $m = 1$.

3.1. Summation Identities of *P*-Fibonacci Sequence.

Theorem 3.1. Let $n \in \mathbb{N}$. Then,

$$\sum_{i=1}^{n} P_{1,i} = \begin{cases} (F_{n+1})^2, & \text{if } n \text{ is odd,} \\ (F_{n+1})^2 - 1, & \text{if } n \text{ is even,} \end{cases}$$

where $P_{1,i} = F_i F_{i+1},$ and F_n denotes the nth Fibonacci number.

Proof. We consider two cases depending of the parity of n.

Case 1: If n is odd, then $n = 2k - 1$ for some $k \in \mathbb{N}$. We proceed by induction on k.

i) When $k = 1$, so $n = 1$:

$$\sum_{i=1}^{1} P_{1,i} = F_1 F_2 = (1)(1) = 1 = (F_2)^2.$$

ii) Assume that for some $s \in \mathbb{N}$,

$$\sum_{i=1}^{n} P_{1,i} = \sum_{i=1}^{2s-1} P_{1,i} = (F_{(2s-1)+1})^2 = (F_{2s})^2.$$

We show it holds for $n = 2(s+1) - 1 = 2s + 1$.

Note that

$$\sum_{i=1}^{n} P_{1,i} = \sum_{i=1}^{2s+1} P_{1,i} = P_{1,1} + P_{1,2} + P_{1,3} + \cdots + P_{1,2s-1} + P_{1,2s} + P_{1,2s+1}$$

Using the inductive hypothesis and Theorem 2.7,

$$\begin{aligned} \sum_{i=1}^{2s+1} P_{1,i} &= (F_{2s})^2 + P_{1,2s} + P_{1,2s+1} \\ &= (F_{2s})^2 + F_{2s}F_{2s+1} + F_{2s+1}F_{2s+2} \\ &= (F_{2s})^2 + F_{2s}F_{2s+1} + F_{2s}F_{2s+1} + (F_{2s+1})^2 \\ &= (F_{2s} + F_{2s+1})^2 \\ &= (F_{2s+2})^2. \end{aligned}$$

Hence, the identity holds for $k = s + 1 \in \mathbb{N}$.

Thus, for odd n,

$$\sum_{i=1}^{n} P_{1,i} = (F_{n+1})^2.$$

Case 2: If n is even, then $n = 2k$ for some $k \in \mathbb{N}$. We proceed by induction on k.

i) When $k = 1$, so $n = 2$:

$$\sum_{i=1}^{2} P_{1,i} = P_{1,1} + P_{1,2} = F_1F_2 + F_2F_3 = 3 = (F_{2+1})^2 - 1.$$

ii) Assume that for some $s \in \mathbb{N}$,

$$\sum_{i=1}^{n} P_{1,i} = \sum_{i=1}^{2s} P_{1,i} = (F_{2s+1})^2 - 1.$$

We show it holds for $n = 2(s+1) = 2s+2$.

Note that

$$\sum_{i=1}^{n} P_{1,i} = \sum_{i=1}^{2s+2} P_{1,i} = P_{1,1} + P_{1,2} + P_{1,3} + \ldots + P_{1,2s} + P_{1,2s+1} + P_{1,2s+2}$$

Using the inductive hypothesis and Theorem 2.7,

$$\begin{aligned}
\sum_{i=1}^{2s+2} P_{1,i} &= (F_{2s+1})^2 - 1 + P_{1,2s+1} + P_{1,2s+2} \\
&= (F_{2s+1})^2 - 1 + F_{2s+1}F_{2s+2} + F_{2s+2}F_{2s+3} \\
&= (F_{2s+1})^2 - 1 + F_{2s+1}F_{2s+2} + F_{2s+1}F_{2s+2} + (F_{2s+2})^2 \\
&= (F_{2s+1} + F_{2s+2})^2 - 1 \\
&= (F_{2s+3})^2 - 1.
\end{aligned}$$

Hence, the identity holds for $k = s+1 \in \mathbb{N}$.

Thus, for even n,

$$\sum_{i=1}^{n} P_{1,i} = (F_{n+1})^2 - 1.$$

□

Example 3.1. Let $n = 5$. Then,

$$\sum_{i=1}^{5} P_{1,i} = P_{1,1} + P_{1,2} + P_{1,3} + P_{1,4} + P_{1,5} = 1 + 2 + 6 + 15 + 40 = 64 = (F_{5+1})^2.$$

Example 3.2. Let $n = 4$. Then,

$$\sum_{i=1}^{4} P_{1,i} = P_{1,1} + P_{1,2} + P_{1,3} + P_{1,4} = 1 + 2 + 6 + 15 = 24 = (F_{4+1})^2 - 1.$$

Building on the result established in Theorem 3.1, which characterizes the sum

$$\sum_{i=1}^{n} F_i F_{i+1}$$

in terms of the square of a Fibonacci number, we now generalize this identity. Specifically, we consider sums of the form

$$\sum_{i=1}^{n} P_{m,i}.$$

The result, presented in Theorem 3.2, expresses this sum in terms of Fibonacci numbers and reflects a similar parity distinction based on whether n is odd or even.

Theorem 3.2. Let $m, n \in \mathbb{N}$. Then,

$$\sum_{i=1}^{n} P_{m,i} = \begin{cases} F_m(F_{n+1})^2 + F_n F_{n+1}(F_{m+1} - F_m), & \text{if } n \text{ is odd}, \\ F_m\left((F_{n+1})^2 - 1\right) + F_n F_{n+1}(F_{m+1} - F_m), & \text{if } n \text{ is even}. \end{cases}$$

Proof. Let $m, n \in \mathbb{N}$. By Definition 4, we have

$$\begin{aligned} \sum_{i=1}^{n} P_{m,i} &= \sum_{i=1}^{n} \left[F_m F_i(F_{i+1} - F_i) + F_{m+1}(F_i)^2\right] \\ &= \sum_{i=1}^{n} F_m F_i(F_{i+1} - F_i) + \sum_{i=1}^{n} F_{m+1}(F_i)^2 \\ &= F_m \sum_{i=1}^{n} F_i(F_{i+1} - F_i) + F_{m+1} \sum_{i=1}^{n} (F_i)^2 \\ &= F_m \left[\sum_{i=1}^{n} F_i F_{i+1} - \sum_{i=1}^{n} (F_i)^2\right] + F_{m+1} \sum_{i=1}^{n} (F_i)^2. \end{aligned}$$

Now, we will consider two cases:

Case 1: If n is odd, then by Theorem 3.1 and Theorem 2.7,

$$\begin{aligned} \sum_{i=1}^{n} P_{m,i} &= F_m \left[\sum_{i=1}^{n} F_i F_{i+1} - \sum_{i=1}^{n} (F_i)^2\right] + F_{m+1} \sum_{i=1}^{n} (F_i)^2 \\ &= F_m\left[(F_{n+1})^2 - F_n F_{n+1}\right] + F_{m+1}(F_n F_{n+1}) \\ &= F_m(F_{n+1})^2 - F_m(F_n F_{n+1}) + F_{m+1}(F_n F_{n+1}) \\ &= F_m(F_{n+1})^2 + F_n F_{n+1}(F_{m+1} - F_m). \end{aligned}$$

Case 2: If n is even, then by Theorem 3.1 and Theorem 2.7,

$$\begin{aligned}\sum_{i=1}^{n} P_{m,i} &= F_m\left[\sum_{i=1}^{n} F_iF_{i+1} - \sum_{i=1}^{n}(F_i)^2\right] + F_{m+1}\sum_{i=1}^{n}(F_i)^2 \\ &= F_m\big[\big((F_{n+1})^2 - 1\big) - F_nF_{n+1}\big] + F_{m+1}\big[F_nF_{n+1}\big] \\ &= F_m\big((F_{n+1})^2 - 1\big) - F_m(F_nF_{n+1}) + F_{m+1}(F_nF_{n+1}) \\ &= F_m\big((F_{n+1})^2 - 1\big) + F_nF_{n+1}(F_{m+1} - F_m).\end{aligned}$$

□

Example 3.3. Let $n = 5$ and $m = 4$. Then,

$$\sum_{i=1}^{5} P_{4,i} = P_{4,1} + P_{4,2} + P_{4,3} + P_{4,4} + P_{4,5} = 5 + 8 + 26 + 63 + 170 = 272.$$

On the other hand,

$$F_4(F_{5+1})^2 + F_5F_{5+1}(F_{4+1} - F_4) = 3\cdot 8^2 + 5\cdot 8\cdot(5-3) = 3\cdot 64 + 40\cdot 2 = 272.$$

Example 3.4. Let $n = 4$ and $m = 3$. Then,

$$\sum_{i=1}^{4} P_{3,i} = P_{3,1} + P_{3,2} + P_{3,3} + P_{3,4} = 3 + 5 + 16 + 39 = 63.$$

On the other hand,

$$F_3((F_{4+1})^2 - 1) + F_4F_{4+1}(F_{3+1} - F_3) = 2(5^2 - 1) + 3\cdot 5\cdot(3-2) = 2\cdot 24 + 15 = 63.$$

After generalizing the sum of products of consecutive Fibonacci numbers in Theorem 3.1 to shifted index sums in Theorem 3.2, we now turn our attention to horizontal and double-index summations involving these terms. To support the next major result, we first introduce a simple but essential identity regarding the sum of shifted Fibonacci numbers.

This identity, formalized in Lemma 3.3, allows us to express a sum of terms

$$\sum_{i=1}^{n} F_{i+1}$$

in a compact closed form. It plays a key role in simplifying the expressions that arise when summing rows or full rectangular blocks of Fibonacci product terms.

Lemma 3.3. Let $n \in \mathbb{N}$. Then

$$\sum_{i=1}^{n} F_{i+1} = F_{n+3} - 2.$$

Proof. Let $n \in \mathbb{N}$. We will now prove by induction on n.

i.) When $n = 1$,

$$\sum_{i=1}^{1} F_{i+1} = F_2 = 1.$$

Also,

$$F_{1+3} - 2 = F_4 - 2 = 3 - 2 = 1.$$

ii.) Assume that the identity holds for some $k \in \mathbb{N}$, that is,

$$\sum_{i=1}^{k} F_{i+1} = F_{k+3} - 2.$$

We aim to show that it also holds for $n = k + 1$.

Now,

$$\sum_{i=1}^{k+1} F_{i+1} = \sum_{i=1}^{k} F_{i+1} + F_{k+2}.$$

By inductive hypothesis,

$$\sum_{i=1}^{k+1} F_{i+1} = F_{k+3} - 2 + F_{k+2} = F_{k+4} - 2.$$

This completes the inductive step. □

Using this lemma, we establish Theorem 3.4, which focuses on a horizontal summation—that is, summing across the row index i while keeping the column index n fixed.

Theorem 3.4. Let $m, n \in \mathbb{N}$. Then,

$$\sum_{i=1}^{m} P_{i,n} = F_n F_{n+1}(F_{m+2} - 1) + (F_n)^2(F_{m+1} - 1).$$

Proof. Let $m, n \in \mathbb{N}$. By Definition 4,

$$\begin{aligned}\sum_{i=1}^{m} P_{i,n} &= \sum_{i=1}^{m} \left[F_i F_n\big(F_{n+1} - F_n\big) + F_{i+1}(F_n)^2\right] \\ &= \big(F_n F_{n+1} - (F_n)^2\big) \sum_{i=1}^{m} F_i + (F_n)^2 \sum_{i=1}^{m} F_{i+1}.\end{aligned}$$

Using Theorem 2.6 and Lemma 3.3, we have

$$\begin{aligned}\sum_{i=1}^{m} P_{i,n} &= \big(F_n F_{n+1} - (F_n)^2\big)\big(F_{m+2} - 1\big) + (F_n)^2(F_{m+3} - 2) \\ &= F_n F_{n+1}(F_{m+2} - 1) + (F_n)^2(F_{m+3} - F_{m+2} - 1) \\ &= F_n F_{n+1}(F_{m+2} - 1) + (F_n)^2(F_{m+1} - 1).\end{aligned}$$

□

Example 3.5. Let $m = 5$ and $n = 4$. Then,

$$\sum_{i=1}^{5} P_{i,4} = P_{1,4} + P_{2,4} + P_{3,4} + P_{4,4} + P_{5,4} = 15 + 24 + 39 + 63 + 102 = 243.$$

Also,

$$F_4 F_{4+1}(F_{5+2} - 1) + (F_4)^2 (F_{5+1} - 1) = 3 \cdot 5 \cdot (13 - 1) + 3^2 \cdot (8 - 1) = 180 + 63 = 243.$$

Building upon Theorem 3.4, we arrive at Theorem 3.5, which presents the main result: a formula for the double sum

$$\sum_{i=1}^{m} \sum_{j=1}^{n} P_{i,j},$$

the total sum over an $m \times n$ rectangle of Fibonacci products. As in Theorem 3.1, the final form depends on whether n is odd or even, capturing a subtle pattern in the interplay of Fibonacci sequences.

Theorem 3.5. Let $m, n \in \mathbb{N}$. Define $P_{i,j} = F_i F_{j+1}$. Then,

$$\sum_{i=1}^{m} \sum_{j=1}^{n} P_{i,j} = \begin{cases} (F_{n+1})^2 (F_{m+2} - 1) + F_n F_{n+1}(F_{m+1} - 1), & \text{if } n \text{ is odd,} \\ \left((F_{n+1})^2 - 1\right)(F_{m+2} - 1) + F_n F_{n+1}(F_{m+1} - 1), & \text{if } n \text{ is even.} \end{cases}$$

Proof. Let $m, n \in \mathbb{N}$.

Case 1: Suppose n is odd. Using Theorem 3.2,

$$\begin{aligned} \sum_{i=1}^{m} \sum_{j=1}^{n} P_{i,j} &= \sum_{i=1}^{m} \left[F_i (F_{n+1})^2 + F_n F_{n+1}(F_{i+1} - F_i)\right] \\ &= (F_{n+1})^2 \sum_{i=1}^{m} F_i + F_n F_{n+1} \left[\sum_{i=1}^{m} F_{i+1} - \sum_{i=1}^{m} F_i\right]. \end{aligned}$$

By Theorem 2.6 and Lemma 3.3,

$$\begin{aligned} \sum_{i=1}^{m} \sum_{j=1}^{n} P_{i,j} &= (F_{n+1})^2 (F_{m+2} - 1) + F_n F_{n+1}\left[(F_{m+3} - 2) - (F_{m+2} - 1)\right] \\ &= (F_{n+1})^2 (F_{m+2} - 1) + F_n F_{n+1}(F_{m+1} - 1). \end{aligned}$$

Case 2: Suppose n is even. Using Theorem 3.2,

$$\begin{aligned} \sum_{i=1}^{m} \sum_{j=1}^{n} P_{i,j} &= \sum_{i=1}^{m} \left[F_i \left((F_{n+1})^2 - 1\right) + F_n F_{n+1}(F_{i+1} - F_i)\right] \\ &= \left((F_{n+1})^2 - 1\right) \sum_{i=1}^{m} F_i + F_n F_{n+1} \left[\sum_{i=1}^{m} F_{i+1} - \sum_{i=1}^{m} F_i\right]. \end{aligned}$$

By Theorem 2.6 and Lemma 3.3,

$$\sum_{i=1}^{m} \sum_{j=1}^{n} P_{i,j} = \left((F_{n+1})^2 - 1\right)(F_{m+2} - 1) + F_n F_{n+1}\left[(F_{m+3} - 2) - (F_{m+2} - 1)\right]$$

$$= \big((F_{n+1})^2 - 1\big)(F_{m+2} - 1) + F_n F_{n+1}(F_{m+1} - 1).$$

□

Example 3.6. Let $m = 4$ and $n = 3$. Then,

$$\sum_{i=1}^{4}\sum_{j=1}^{3} P_{i,j} = P_{1,1} + P_{1,2} + P_{1,3} + P_{2,1} + P_{2,2} + P_{2,3} + P_{3,1} + P_{3,2} + P_{3,3} + P_{4,1} + P_{4,2} + P_{4,3}$$

$$= 1 + 2 + 6 + 2 + 3 + 10 + 3 + 5 + 16 + 5 + 8 + 26 = 87.$$

Also,

$$(F_{3+1})^2(F_{4+2} - 1) + F_3 F_{3+1}(F_{4+1} - 1) = 3^2 \cdot (8 - 1) + 2 \cdot 3 \cdot (5 - 1) = 87.$$

Example 3.7. Let $m = 4$ and $n = 3$. Then,

$$\sum_{i=1}^{3}\sum_{j=1}^{4} P_{i,j} = P_{1,1} + P_{1,2} + P_{1,3} + P_{1,4} + P_{2,1} + P_{2,2} + P_{2,3} + P_{2,4} + P_{3,1} + P_{3,2} + P_{3,3} + P_{3,4}$$

$$= 1 + 2 + 6 + 15 + 2 + 3 + 10 + 24 + 3 + 5 + 16 + 39 = 126.$$

Also,

$$((F_{4+1})^2 - 1)(F_{3+2} - 1) + F_4 F_{4+1}(F_{3+1} - 1) = (25 - 1)(5 - 1) + 3 \cdot 5 \cdot (3 - 1) = 126.$$

3.2. **GCD of *P*-Fibonacci Sequence.** To explore the greatest common divisors within the P-Fibonacci sequence, we begin by establishing a fundamental number-theoretic result. Lemma 3.6 states that every two consecutive odd integers are relatively prime. Building on this idea, we derive two significant results that link the structure of the P-Fibonacci sequence to the classical Fibonacci numbers.

Lemma 3.6. Every two consecutive odd integers are said to be relatively prime.

Proof. Let $a = 2k + 1$ and $b = 2k + 3$ be two consecutive arbitrary odd integers for some $k \in \mathbb{Z}$. Note that if we let two integers $x = k + 1$ and $y = -k$, then

$$\begin{aligned} ax + by &= (2k + 1)(k + 1) + (2k + 3)(-k) \\ &= 2k^2 + 3k + 1 - 2k^2 - 3k \\ &= 1. \end{aligned}$$

Thus, by Theorem 2.2, $2k+1$ and $2k+3$ are relatively prime. Since $2k+1$ and $2k+3$ are two consecutive arbitrary odd integers, therefore, every two consecutive odd integers are relatively prime. □

Example 3.8. Let $a = 5$ and $b = 7$. Note that if we let $x = 3$ and $y = -2$, then

$$ax + by = (5)(3) + (7)(-2) = 15 - 14 = 1.$$

Theorem 3.7. Let $n \in \mathbb{N}$. Then $\gcd(P_{1,n}, P_{1,n+1}) = F_{n+1}$.

Proof. Let $n \in \mathbb{N}$. By Definition 4,

$$P_{1,n} = F_1 F_n (F_{n+1} - F_n) + F_2 (F_n)^2 = F_n F_{n+1}.$$

and

$$P_{1,n+1} = F_1 F_{n+1} (F_{n+2} - F_{n+1}) + F_2 (F_{n+1})^2 = F_{n+1} F_{n+2}.$$

Now, using Theorem 2.1 and Theorem 2.5, we have

$$\begin{aligned} gcd(P_{m,n}, P_{m,n+1}) &= gcd(P_{1,n}, P_{1,n+1}) \\ &= gcd(F_n F_{n+1}, F_{n+1} F_{n+2}) \\ &= F_{n+1} \big[gcd(F_n, F_{n+2}) \big] \\ &= F_{n+1} \big[F_{gcd(n,n+2)} \big]. \qquad (1) \end{aligned}$$

Note that if n is odd, that is, $n = 2k - 1$ for some $k \in \mathbb{Z}^+$,

$$\begin{aligned} gcd(n, n+2) &= gcd\big(2k-1, (2k-1)+2\big) \\ &= gcd(2k-1, 2k+1) \\ &= 1 \qquad (2) \end{aligned}$$

since $2k - 1$ and $2k + 1$ are two consecutive odd integers, in which by Lemma 3.6, they are relatively prime. Using equations (1) and (2),

$$gcd(P_{m,n}, P_{m,n+1}) = F_{n+1} F_1 = F_{n+1}.$$

Note also that if n is even, that is, $n = 2k$ for some $k \in \mathbb{Z}^+$,

$$\begin{aligned} gcd(n, n+2) &= gcd(2k, 2k+2) \\ &= 2gcd(k, k+1) \\ &= 2(1) = 2 \qquad (3) \end{aligned}$$

since k and $k + 1$ are two consecutive integers, in which by Theorem 2.3, they are relatively prime. Using equations (1) and (3),

$$gcd(P_{m,n}, P_{m,n+1}) = F_{n+1} F_2 = F_{n+1}.$$

Therefore, $gcd(P_{1,n}, P_{1,n+1}) = F_{n+1}$. □

Example 3.9. Let $n = 7$ and $m = 1$. Now,

$$\gcd(P_{1,7}, P_{1,8}) = \gcd(F_7 F_8, F_8 F_9) = F_8 \gcd(F_7, F_9) = F_8 \gcd(13, 34) = F_8 \cdot 1 = F_{7+1}.$$

Theorem 3.8. Let $m, n \in \mathbb{N}$. Then $\gcd(P_{m,n}, P_{m+1,n}) = F_n$.

Proof. Let $m, n \in \mathbb{N}$. By Definition 4 and Theorem 2.1,

$$\begin{aligned} gcd(P_{m,n}, P_{m+1,n}) &= gcd(F_n F_{n+m}, F_n F_{n+m+1}) \\ &= F_n \left[gcd(F_{n+m}, F_{n+m+1}) \right]. \end{aligned}$$

Note that F_{n+m} and F_{n+m+1} are two consecutive Fibonacci numbers. So, by Theorem 2.4, $gcd(F_{n+m}, F_{n+m+1}) = 1$. Thus,

$$\begin{aligned} gcd(P_{m,n}, P_{m+1,n}) &= gcd(F_n F_{n+m}, F_n F_{n+m+1}) \\ &= F_n \left[gcd(F_{n+m}, F_{n+m+1}) \right] \\ &= F_n(1) \\ &= F_n. \end{aligned}$$

□

Example 3.10. Let $m = 4$ and $n = 7$. Now,

$$\gcd(P_{4,7}, P_{4+1,7}) = \gcd(P_{4,7}, P_{5,7}) = \gcd(1157, 1872) = 13 = F_7.$$

In Theorem 3.7, we show that for any $n \in \mathbb{N}$, the greatest common divisor of two consecutive terms in the first order P-Fibonacci sequence satisfies

$$\gcd(P_{1,n}, P_{1,n+1}) = F_{n+1}.$$

Further generalizing, Theorem 3.8 asserts that for any $m, n \in \mathbb{N}$, the greatest common divisor of two terms from successive orders at the same index satisfies

$$\gcd(P_{m,n}, P_{m+1,n}) = F_n.$$

These results highlight the deep interconnection between P-Fibonacci sequences and classical Fibonacci numbers through their gcd properties.

4. Conclusion

The results presented in this paper open several avenues for further exploration in the field of Fibonacci number theory and its applications. The formulation of the P-Fibonacci sequence reveals a novel structure embedded within the classical Fibonacci sequence, particularly in the interaction between terms F_n and F_{n+m}. Future researchers are encouraged to investigate generalizations of this sequence under different recurrence relations, such as those involving Lucas numbers or Tribonacci-type sequences, to determine whether similar patterns or identities emerge.

In addition, the closed-form summation identities and recursive properties derived for $P_{m,n}$ suggest potential applications in discrete mathematics, coding theory, and cryptographic algorithms that utilize number-theoretic properties. Further study may focus on combinatorial interpretations, matrix representations, or connections to modular arithmetic and prime factorizations within the sequence. Finally, as this work primarily considers integer-valued Fibonacci sequences, future research could also examine analogous formulations in generalized Fibonacci sequences defined over rings or fields.

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