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ITERATIVE TECHNIQUES FOR COMMON SOLUTIONS OF MIXED EQUILIBRIUM PROBLEM AND FIXED POINTS OF μ-DEMICONTRACTIVE MAPPINGS

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Abstract. In this paper, we introduce and analyse a new algorithm for finding a common solution of the set of solutions of a Mixed Equilibrium Problem and the set of fixed points of a finite family of μ-demicontractive mappings. We prove a strong convergence result for the sequence generated by the algorithm and prove that the sequence converges strongly to a common solution of Mixed Equilibrium Problem and the set of fixed points of a finite family of μ-demicontractive mappings.

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1. Introduction

The theory of Equilibrium Problems (EP) has profoundly influenced the advancement of numerous scientific disciplines. This theoretical framework has emerged as a vital source of inspiration for addressing diverse problems in economics, optimization, and operations research. It provides a unified and generic approach to tackle challenges across these fields, encompassing variational inequalities, fixed point theory, Nash equilibrium, and game theory as special cases.

The concept of EP was first introduced in 1994 by Blum and Oettli [1] and Noor and Oettli [6]. Formally, the EP is defined as follows:

Let Σ be a real Hilbert space, $\mathcal{C} \neq \emptyset$ a closed and convex subset of Σ, and $F : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ an equilibrium function. The EP involves finding a point $\zeta^* \in \mathcal{E}$ such that

$$F(\zeta^*, \zeta) \geq 0, \quad \forall \zeta \in \mathcal{E}.$$

The solution to this problem is denoted as $EP(F)$. Over the years, the EP framework has been extended and generalized by numerous researchers, as seen in [8–10, 16, 17].

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Mixed Equilibrium Problem (MEP). One of the notable generalizations of EP is the Mixed Equilibrium Problem (MEP). The MEP aims to find a point $\zeta^* \in \mathcal{E}$ such that

$$F(\zeta^*, \eta) + \langle G(\zeta^*), \eta - \zeta^* \rangle + h(\eta, \zeta^*) - h(\zeta^*, \zeta^*) \geq 0, \quad \forall \eta \in \mathcal{E},$$

where $G : \mathcal{E} \to \Sigma$ is a nonlinear mapping, and $F, h : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ are two bifunctions. The solution set of MEP is denoted by $MEP(F, h)$. This formulation encapsulates a broad spectrum of problems, including fixed point problems, saddle point problems, variational inequality problems, optimization problems, and the equilibrium problem itself as special cases [1,4,6,15].

Iterative Methods for EP and MEP. In recent years, various iterative methods have been developed to approximate solutions for EP and MEP. These methods are often designed to find a common element between the solutions of equilibrium problems and the fixed points of finitely or infinitely many mappings. Yao et al. [15] introduced a novel hybrid iterative algorithm for identifying a common point between the fixed points of an infinite family of nonexpansive mappings and the solutions of MEP. Kuman and Jaiboon [5] analyzed a hybrid iterative scheme to find a common point among the solutions of MEP, fixed points of an infinite family of nonexpansive mappings, and solutions to variational inequalities for ξ-Lipschitz continuous and relaxed (m, v)-cocoercive mappings in Hilbert spaces. Yao et al. [14] proposed an extragradient method to approximate a common element between the fixed points of a demicontractive mapping and the solutions of MEP. Shehu [12] developed a hybrid iterative method to address a system of generalized mixed equilibrium problems, strict pseudocontractive mappings, and variational inequality problems in real Hilbert spaces.

This paper presents a new algorithm tailored for a finite family of μ-demicontractive mappings. The proposed algorithm is designed to strongly converge to a common solution of the Mixed Equilibrium Problem (MEP) and the set of fixed points of these mappings. By leveraging advanced analytical techniques, we demonstrate the algorithm's robustness and efficiency in finding common solutions within this framework.

2. Preliminaries

Definition 2.1. A mapping $\Phi : \Sigma \to \Sigma$ is said to be

(a) monotone, if

$$\langle \Phi(\zeta) - \Phi(\eta), \zeta - \eta \rangle \geq 0, \forall \zeta, \eta \in \Sigma;$$

(b) pseudomonotone, if

$$\langle \Phi(\zeta), \eta - \zeta \rangle \geq 0 \implies \langle \Phi(\eta), \eta - \zeta \rangle \geq 0, \forall \zeta, \eta \in \Sigma;$$

(c) contraction, if ∃ a constant $0 < \mu < 1$ such that

$$\|\Phi(\zeta) - \Phi(\eta)\| \leq \mu\|\zeta - \eta\|, \forall \zeta, \eta \in \Sigma;$$

(d) L-Lipschitz continuous, if

$$\|\Phi(\zeta) - \Phi(\eta)\| \leq L\|\zeta - \eta\|, \forall \zeta, \eta \in \Sigma;$$

(e) μ-demicontractive if $F(\Phi) \neq \emptyset$ and ∃ a constant $0 < \mu < 1$, with

$$\|\Phi(\zeta) - \zeta^{\dagger}\|^2 \leq \|\zeta - \zeta^{\dagger}\|^2 + \mu\|\zeta - \Phi(\zeta)\|^2, \forall \zeta \in \Sigma, \zeta^{\dagger} \in F(\Phi).$$

Definition 2.2. [7]. A Hilbert space Σ is said to satisfy the Opial property if, for every weakly convergent sequence (ζ_n) with weak limit $\zeta \in \Sigma$ it holds:

$$\liminf_{n\to\infty} \|\zeta_n - \zeta\| < \liminf_{n\to\infty} \|\zeta_n - \eta\|$$

for all $\eta \in \Sigma$ with $\zeta \neq \eta$.

Lemma 2.3. [13] *Suppose* $\zeta, \eta \in \Sigma$ *and* $\nu \in [0, 1]$. *Then following hold true:*

(a) $\|\zeta \pm \eta\|^2 = \|\zeta\|^2 \pm 2\langle\zeta, \eta\rangle + \|\eta\|^2$;
(b) $\|\zeta + \eta\|^2 \leq \|\zeta\|^2 + 2\langle\eta, \zeta + \eta\rangle$;
(c) $\nu\zeta + (1 - \nu)\eta\|^2 \leq \nu\|\zeta\|^2 + (1 - \nu)\|\eta\|^2 - \nu(1 - \nu)\|\zeta - \eta\|^2$.

Lemma 2.4. *Suppose* $\{\mu_n\} \subset \mathbb{R}^+$, $\{\nu_n\} \subset \mathbb{R}$ *and* $\{\lambda_n\} \subset (0, 1)$ *such that* $\sum\limits_{n=1}^{\infty} \lambda_n = \infty$,

$$\mu_{n+1} \leq (1 - \lambda_n)\mu_n + \lambda_n\nu_n, \forall n \in \mathbb{N}.$$

If $\limsup\limits_{n\to\infty} \nu_n \leq 0$ *and for any subsequence* $\{\mu_{n_i}\}$ *of* $\{\mu_n\}$ *satisfying* $\liminf\limits_{i\to\infty}(\mu_{n_i+1} - \mu_{n_i}) \geq 0$ *then* $\lim\limits_{n\to\infty} \mu_n = 0$.

Definition 2.5. A bifunction $F : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ is called as 2-monotone if

$$F(\zeta, \eta) + F(\eta, \vartheta) + F(\vartheta, \zeta) \leq, \forall \zeta, \eta, \vartheta \in \mathcal{E}.$$

Lemma 2.6. [2] *Suppose* Σ *is a real Hilbert space and* $\mathcal{E} \neq \emptyset$ *is a closed and convex subset of* Σ. *Suppose* $F, h : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ *are nonlinear mappings. Suppose for any* $\eta \in \Sigma$ *and for any* $\zeta \in \mathcal{E}$ ∃ *a bounded subset* $D_\zeta \subseteq \mathcal{E}$ *and* $\eta_\zeta \in \mathcal{E}$ *in such a way that for each* $\vartheta \in \mathcal{E}/D_\zeta$

$$F(\vartheta, \eta_\zeta) + h(\eta_\zeta, \vartheta) - f(\vartheta, \vartheta) + \frac{1}{r}\langle\eta_\zeta - \vartheta, \vartheta - \eta\rangle < 0.$$

Now define a mapping $T_r^{F,h} : \Sigma \to \mathcal{E}$ *as:*

$$T_r^{F,h}(\eta) = \left\{\zeta \in \mathcal{E} : F(\zeta, \vartheta) + h(\vartheta, \zeta) - h(\zeta, \zeta) + \frac{1}{r}\langle\vartheta - \zeta, \zeta - \eta\rangle \geq 0, \forall \vartheta \in \mathcal{E}\right\},$$

here $r \in \mathbb{R}^+$. *Then the following conclusions hold:*

(a) $T_r^{F,h}(\eta) \neq \emptyset, \forall \eta \in \Sigma$;

(b) $T_r^{F,h}$ *is a single valued mapping;*

(c) $T_r^{F,h}$ *is a firmly nonexpansive mapping.*

3. Main Results

Throughout this section, we assume that Σ is a real Hilbert space and $\mathcal{E}$ is a nonempty, closed and convex subset of Σ. $G : \Sigma \to \Sigma$ is a inverse strongly monotone mapping, $\digamma : \Sigma \to \Sigma$ is contraction with $0 < k < 1$ and $\Phi_i : \mathcal{E} \to \mathcal{E}$ is a finite family of μ-demicontractive mappings with $\bigcap_{i=1}^{n} F(\Phi_i) \neq \emptyset$. The solution set $\Xi = \{\zeta \in \text{ MEP}(F,h) \cap \bigcap_{i=1}^{n} F(\Phi_i)\}$ is nonempty.

Assumption 3.1. Suppose $F : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ and $h : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ are two bifunctions and satisfy below conditions:

(a) $F(\zeta,\zeta) = 0, \forall \zeta \in \mathcal{E}$;

(b) bifunction F is monotone

$$F(\zeta,\vartheta) + F(\vartheta,\zeta) \leq 0, \forall \zeta,\vartheta \in \mathcal{E};$$

(c) the bifunction h is generalized skew-symmetric

$$h(\zeta,\zeta) - h(\zeta,\vartheta) + h(\vartheta,\vartheta) - h(\vartheta,\eta) + h(\eta,\eta) - h(\eta,\zeta) \geq 0, \forall \zeta,\vartheta,\eta \in \mathcal{E}.$$

(d) for any $\zeta \in \mathcal{E}, \vartheta \to F(\zeta,\vartheta)$ is lower semi continuous and convex;

(e) bifunction $h(\cdot,\vartheta)$ is convex, and bifunction $h(\cdot,\cdot)$ is weakly continuous;

Algorithm 3.2. Given $\Upsilon > 0$. Let $\zeta_0, \zeta_1 \in \Sigma$. Define

$$\Upsilon_n = \begin{cases} \min\left\{\Upsilon, \frac{\varepsilon_n}{\|\zeta_n - \zeta_{n-1}\|}, \text{ if } \zeta_n \neq \zeta_{n-1}\right\} \\ \Upsilon, \quad \text{otherwise.} \end{cases}$$

$$\begin{aligned} \xi_n &= \zeta_n + \Upsilon_n(\zeta_n - \zeta_{n-1}), \\ \vartheta_n &= \Phi_{r_n}^{F,h}(\xi_n - r_n G(\xi_n)), \\ \eta_n &= \Phi_{r_n Q_n}^{F,h}(\vartheta_n - r_n G(\vartheta_n)), \\ \varpi_n &= \Gamma_n \eta_n + (1 - \Gamma_n)\Phi_i(\eta_n), \\ \zeta_{n+1} &= \chi_n \digamma(\zeta_n) + \omega_n \zeta_n + \tau_n \varpi_n. \end{aligned}$$

Here $Q_n = \{\xi \in \Sigma : \langle \xi_n - r_n G(\xi_n) - \vartheta_n, \vartheta_n - \xi\rangle \geq r_n G(\vartheta_n,\xi)\}$ and $\{\chi_n\}, \{\omega_n\}, \{\tau_n\} \subset [0,1]$ with $\chi_n + \omega_n + \tau_n = 1$, $r_n < L_1$, $\Upsilon_n \in [0,1)$. Let following conditions also satisfied:

(1) $\lim_{n\to\infty} \chi_n = 0$, $\sum_{n=0}^{\infty} \chi_n = \infty$;

(2) $0 < c < \omega_n, \tau_n \leq d < 1$;

(3) $\lim_{n\to\infty} \inf r_n > 0$;

(4) $\lim_{n\to\infty} \frac{\Upsilon_n}{\chi_n} = 0$;

(5) $\liminf_{n\to\infty}(\Gamma_n - \zeta) > 0$.

Lemma 3.3. [3] *Suppose $\mathcal{E} \neq \emptyset$ is a closed and convex subset of Σ. Suppose $F : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ is a 2-monotone, $h : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ is a generalized skew symmetric bifunction satisfying the Assumption 3.1 and $G : \mathcal{E} \to \Sigma$ is a monotone and Lipschitz continuous on $\mathcal{E}$ with $L > 0$. If the sequence $\{\zeta_n\}$ is generated by (3.2) and $\zeta^\dagger \in MEP(F, h)$, then we have*

$$\|\eta_n - \zeta^\dagger\|^2 \leq \|\xi_n - \zeta^\dagger\|^2 - (1 - (r_n L)^2)\|\vartheta_n - \xi_n\|^2, \forall n \geq 1. \tag{3.1}$$

Lemma 3.4. *Suppose $\Sigma, \mathcal{E}, F, h, G, \Phi_i$ are defined as above with $\Xi \neq \emptyset$ and satisfying Assumptions 3.1. Then the sequence $\{\zeta_n\}$ generated by Algorithm 3.2 is bounded.*

Proof. Let $\zeta^\dagger \in MEP(F, h)$, then we have

$$\begin{aligned}\|\xi_n - \zeta^\dagger\| &= \|\zeta_n + \Upsilon_n(\zeta_n - \zeta_{n-1}) - \zeta^\dagger\| \\ &\leq \|\zeta_n - \zeta^\dagger\| + \Upsilon_n\|\zeta_n - \zeta_{n-1}\| \\ &\leq \|\zeta_n - \zeta^\dagger\| + \chi_n \cdot \frac{\Upsilon_n}{\chi_n}\|\zeta_n - \zeta_{n-1}\|.\end{aligned}$$

Since $\lim_{n\to\infty} \frac{\Upsilon_n}{\chi_n} = 0$ and hence $\lim_{n\to\infty} \frac{\Upsilon_n}{\chi_n}\|\zeta_n - \zeta_{n-1}\| = 0$, so $\exists\, M_1 > 0$ such that $\frac{\Upsilon_n}{\chi_n}\|\zeta_n - \zeta_{n-1}\| \leq M_1, \forall n \in \mathbb{N}$.

Now we get

$$\|\xi_n - \zeta^\dagger\| \leq \|\zeta_n - \zeta^\dagger\| + \chi_n M_1, \forall n \geq n_0. \tag{3.2}$$

$$\begin{aligned}\|\varpi_n - \zeta^\dagger\|^2 &= \|\Gamma_n\eta_n + (1 - \Gamma_n)\Phi_i(\eta_n) - \zeta^\dagger\|^2 \\ &= \|\Gamma_n(\eta_n - \zeta^\dagger) + (1 - \Gamma_n)(\Phi_i(\eta_n) - \zeta^\dagger)\|^2 \\ &\leq \Gamma_n\|\eta_n - \zeta^\dagger\|^2 + (1 - \Gamma_n)\|\Phi_i(\eta_n) - \zeta^\dagger\|^2 - \Gamma_n(1 - \Gamma_n)\|\eta_n - \Phi_i(\eta_n)\|^2 \\ &\leq \Gamma_n\|\eta_n - \zeta^\dagger\|^2 + (1 - \Gamma_n)(\|\eta_n - \zeta^\dagger\|^2 + \mu\|\eta_n - \Phi_i(\eta_n)\|^2) \\ &- \Gamma_n(1 - \Gamma_n)\|\eta_n - \Phi_i(\eta_n)\|^2 \\ &= \|\eta_n - \zeta^\dagger\|^2 - (1 - \Gamma_n)(\Gamma_n - \mu)\|\eta_n - \Phi_i(\eta_n)\|^2 \\ &\leq \|\eta_n - \zeta^\dagger\|^2.\end{aligned}$$

We get

$$\|\varpi_n - \zeta^\dagger\| \leq \|\eta_n - \zeta^\dagger\|. \tag{3.3}$$

From (3.1), we get

$$\|\eta_n - \zeta^\dagger\| \leq \|\xi_n - \zeta^\dagger\|. \tag{3.4}$$

Now again

$$\begin{aligned}
\|\zeta_{n+1} - \zeta^\dagger\| &= \|\chi_n F(\zeta_n) + \omega_n \zeta_n + \tau_n \varpi_n - \zeta^\dagger\| \\
&= \|\chi_n (F_n(\zeta_n) - \zeta^\dagger) + \omega_n(\zeta_n - \zeta^\dagger) + \tau_n(\varpi_n - \zeta^\dagger)\| \\
&\leq \chi_n \|F(\zeta_n) - F(\zeta^\dagger)\| + \chi_n \|F(\zeta^\dagger) - \zeta^\dagger\| + \omega_n \|\zeta_n - \zeta^\dagger\| + (1 - \chi_n - \omega_n)\|\varpi_n - \zeta^\dagger\| \\
&\leq \chi_n k \|\zeta_n - \zeta^\dagger\| + \chi_n \|F(\zeta^\dagger) - \zeta^\dagger\| + \omega_n \|\zeta_n - \zeta^\dagger\| + (1 - \chi_n - \omega_n)\|\varpi_n - \zeta^\dagger\|.
\end{aligned}$$

Now applying equations (3.2), (3.3) and (3.4), we get

$$\begin{aligned}
\|\zeta_{n+1} - \zeta^\dagger\| &\leq \chi_n k \|\zeta_n - \zeta^\dagger\| + \chi_n \|F(\zeta^\dagger) - \zeta^\dagger\| + \omega_n \|\zeta_n - \zeta^\dagger\| \\
&+ (1 - \chi_n - \omega_n)\|\zeta_n - \zeta^\dagger\| + \chi_n M_1 \\
&\leq \chi_n k \|\zeta_n - \zeta^\dagger\| + \chi_n \|F(\zeta^\dagger) - \zeta^\dagger\| + (1 - \chi_n)\|\zeta_n - \zeta^\dagger\| + \chi_n M_1 \\
&\leq (1 - \chi_n(1-k))\|\zeta_n - \zeta^\dagger\| + (1-k)\chi_n \left(\frac{M_1 + \|F(\zeta^\dagger) - \zeta^\dagger\|}{1-k} \right) \\
&\leq \max\left\{ \|\zeta_n - \zeta^\dagger\|, \frac{M_1 + \|F(\zeta^\dagger) - \zeta^\dagger\|}{1-k} \right\}.
\end{aligned}$$

Now using induction, we get

$$\|\zeta_n - \zeta^\dagger\| \leq \max\left\{ \|\zeta_0 - \zeta^\dagger\|, \frac{M_1 + \|F(\zeta^\dagger) - \zeta^\dagger\|}{1-k} \right\}. \tag{3.5}$$

Hence the sequence $\{\zeta_n\}$ is bounded, and thus the sequences $\{\vartheta_n\}, \{\eta_n\}, \{\varpi_n\}$ are also bounded. □

Theorem 3.5. *Suppose $\Sigma, \mathcal{E}, F, h, G, \Phi_i$ are defined as above with $\Xi \neq \emptyset$ and satisfying Assumptions 3.1. Then the sequence $\{\zeta_n\}$ generated by Algorithm 3.2 converges strongly to a point $\zeta^\dagger \in \Xi$, where $\zeta^\dagger = P_\Xi F(\zeta^\dagger)$.*

Proof. Suppose $\zeta^\dagger \in \Xi$, then we can easily see that mapping $P_\Xi F$ is contraction mapping. Using the Banach contraction mapping principle, $\exists$ a unique point $\zeta^\dagger \in \Xi$ such that $\zeta^\dagger = P_\Xi F(\zeta^\dagger)$. It gives

$$\langle F(\zeta^\dagger) - \zeta^\dagger, \zeta^* - \zeta^\dagger \rangle \leq 0, \forall \zeta^\dagger \in \Xi. \tag{3.6}$$

Now,

$$\begin{aligned}
\|\xi_n - \zeta^\dagger\|^2 &= \|\zeta_n + \Upsilon_n(\zeta_n - \zeta_{n-1}) - \zeta^\dagger\|^2 \\
&= \|\zeta_n - \zeta^\dagger\|^2 + 2\Upsilon_n \langle \zeta_n - \zeta^\dagger, \zeta_n - \zeta_{n-1} \rangle + \Upsilon_n^2 \|\zeta_n - \zeta_{n-1}\|^2 \\
&\leq \|\zeta_n - \zeta^\dagger\|^2 + 2\Upsilon_n \|\zeta_n - \zeta_{n-1}\| \|\zeta_n - \zeta^\dagger\| + \Upsilon_n^2 \|\zeta_n - \zeta_{n-1}\|^2 \\
&= \|\zeta_n - \zeta^\dagger\|^2 + \Upsilon_n \|\zeta_n - \zeta_{n-1}\| [2\|\zeta_n - \zeta^\dagger\| + \Upsilon_n \|\zeta_n - \zeta_{n-1}\|] \\
&= \|\zeta_n - \zeta^\dagger\|^2 + \Upsilon_n \|\zeta_n - \zeta_{n-1}\| [2\|\zeta_n - \zeta^\dagger\| + \chi_n \cdot \frac{\Upsilon_n}{\chi_n} \|\zeta_n - \zeta_{n-1}\|]
\end{aligned}$$

$$\leq \|\zeta_n - \zeta^\dagger\|^2 + \Upsilon_n\|\zeta_n - \zeta_{n-1}\|[2\|\zeta_n - \zeta^\dagger\| + \chi_n M_1]$$
$$\leq \|\zeta_n - \zeta^\dagger\|^2 + \Upsilon_n\|\zeta_n - \zeta_{n-1}\|M_2, \tag{3.7}$$

here

$$M_2 = \sup_{n\in\mathbb{N}}[2\|\zeta_n - \zeta^\dagger\| + \chi_n M_1].$$

Again

$$\begin{aligned}
\|\zeta_{n+1} - \zeta^\dagger\|^2 &= \|\chi_n F(\zeta_n) + \omega_n\zeta_n + \tau_n\varpi_n - \zeta^\dagger\|^2 \\
&\leq \|\omega_n(\zeta_n - \zeta^\dagger) + \tau_n(\varpi_n - \zeta^\dagger)\|^2 + 2\chi_n\langle F(\zeta_n) - \zeta^\dagger, \zeta_{n+1} - \zeta^\dagger\rangle \\
&\leq \omega_n^2\|\zeta_n - \zeta^\dagger\|^2 + \tau_n^2\|\varpi_n - \zeta^\dagger\|^2 + 2\omega_n\tau_n(\|\zeta_n - \zeta^\dagger\|\|\varpi_n - \zeta^\dagger\|) \\
&+ 2\chi_n\langle F(\zeta_n) - \zeta^\dagger, \zeta_{n+1} - \zeta^\dagger\rangle \\
&\leq \omega_n^2\|\zeta_n - \zeta^\dagger\|^2 + \tau_n^2\|\varpi_n - \zeta^\dagger\|^2 + \omega_n\tau_n(\|\zeta_n - \zeta^\dagger\|^2 + \|\varpi_n - \zeta^\dagger\|^2) \\
&+ 2\chi_n\langle F(\zeta_n) - F(\zeta^\dagger), \zeta_{n+1} - \zeta^\dagger\rangle + 2\chi_n\langle F(\zeta^\dagger) - \zeta^\dagger, \zeta_{n+1} - \zeta^\dagger\rangle \\
&\leq \omega_n(\omega_n + \tau_n)\|\zeta_n - \zeta^\dagger\|^2 + \tau_n((\omega_n + \tau_n))\|\varpi_n - \zeta^\dagger\|^2 \\
&+ 2\chi_n\langle F(\zeta_n) - F(\zeta^\dagger), \zeta_{n+1} - \zeta^\dagger\rangle + 2\chi_n\langle F(\zeta^\dagger) - \zeta^\dagger, \zeta_{n+1} - \zeta^\dagger\rangle.
\end{aligned}$$

Now using equations (3.3) and (3.4) we get

$$\begin{aligned}
\|\zeta_{n+1} - \zeta^\dagger\|^2 \leq\ & \omega_n(\omega_n + \tau_n)\|\zeta_n - \zeta^\dagger\|^2 + \tau_n((\omega_n + \tau_n))\|\xi_n - \zeta^\dagger\|^2 \\
&+ 2\chi_n\langle F(\zeta_n) - F(\zeta^\dagger), \zeta_{n+1} - \zeta^\dagger\rangle + 2\chi_n\langle F(\zeta^\dagger) - \zeta^\dagger, \zeta_{n+1} - \zeta^\dagger\rangle.
\end{aligned}$$

Now applying (3.7) to the above equation, we get

$$\begin{aligned}
\|\zeta_{n+1} - \zeta^\dagger\|^2 &\leq \omega_n(\omega_n + \tau_n)\|\zeta_n - \zeta^\dagger\|^2 + \tau_n((\omega_n + \tau_n))\|\zeta_n - \zeta^\dagger\|^2 \\
&+ \tau_n((\omega_n + \tau_n))\Upsilon_n\|\zeta_n - \zeta_{n-1}\|M_2 \\
&+ 2\chi_n\langle F(\zeta_n) - F(\zeta^\dagger), \zeta_{n+1} - \zeta^\dagger\rangle + 2\chi_n\langle F(\zeta^\dagger) - \zeta^\dagger, \zeta_{n+1} - \zeta^\dagger\rangle \\
&\leq \omega_n(\omega_n + \tau_n)\|\zeta_n - \zeta^\dagger\|^2 + \tau_n((\omega_n + \tau_n))\|\zeta_n - \zeta^\dagger\|^2 \\
&+ \tau_n((\omega_n + \tau_n))\Upsilon_n\|\zeta_n - \zeta_{n-1}\|M_2 \\
&+ \chi_n k\|\zeta_n - \zeta^\dagger\|^2 + \chi_n k\|\zeta_{n+1} - \zeta^\dagger\|^2 + 2\chi_n\langle F(\zeta^\dagger) - \zeta^\dagger, \zeta_{n+1} - \zeta^\dagger\rangle \\
&\leq (\omega_n + \tau_n)^2\|\zeta_n - \zeta^\dagger\|^2 + \tau_n((\omega_n + \tau_n))\Upsilon_n\|\zeta_n - \zeta_{n-1}\|M_2 \\
&+ \chi_n k\|\zeta_n - \zeta^\dagger\|^2 + \chi_n k\|\zeta_{n+1} - \zeta^\dagger\|^2 + 2\chi_n\langle F(\zeta^\dagger) - \zeta^\dagger, \zeta_{n+1} - \zeta^\dagger\rangle \\
&\leq (1 - 2\chi_n + \chi_n k)\|\zeta_n - \zeta^\dagger\|^2 + \chi_n^2\|\zeta_n - \zeta^\dagger\|^2 + \tau_n((\omega_n + \tau_n))\Upsilon_n\|\zeta_n - \zeta_{n-1}\|M_2 \\
&+ \chi_n k\|\zeta_{n+1} - \zeta^\dagger\|^2 + 2\chi_n\langle F(\zeta^\dagger) - \zeta^\dagger, \zeta_{n+1} - \zeta^\dagger\rangle.
\end{aligned}$$

Simplifying above equation, we get

$$\begin{aligned}\|\zeta_{n+1}-\zeta^\dagger\|^2 &\le \left(1-\frac{2\chi_n(1-k)}{1-\chi_n k}\right)\|\zeta_n-\zeta^\dagger\|^2+\frac{2\chi_n(1-k)}{1-\chi_n k}\\ &\left[\frac{\tau_n(1-\chi_n)\Upsilon_n}{2\chi_n(1-k)}\|\zeta_n-\zeta_{n-1}\|M_2+\frac{\chi_n M_3}{2(1-k)}+\frac{1}{(1-k)}\langle F(\zeta^\dagger)-\zeta^\dagger,\zeta_{n+1}-\zeta^\dagger\rangle\right]\\ &=\left(1-\frac{2\chi_n(1-k)}{1-\chi_n k}\right)\|\zeta_n-\zeta^\dagger\|^2+\frac{2\chi_n(1-k)}{1-\chi_n k}\Pi_n\\ &\le\left(1-\frac{2\chi_n(1-k)}{1-\chi_n k}\right)\|\zeta_n-\zeta^\dagger\|^2+\frac{2\chi_n(1-k)}{1-\chi_n k}M',\end{aligned}$$

where

$$M_3=\sup_{n\in\mathbb{N}}\|\zeta_n-\zeta^\dagger\|^2,$$

$$M'=\sup_{n\in\mathbb{N}}\Pi_n,$$

$$\Pi_n=\left[\frac{\tau_n(1-\chi_n)\Upsilon_n}{2\chi_n(1-k)}\|\zeta_n-\zeta_{n-1}\|M_2+\frac{\chi_n M_3}{2(1-k)}+\frac{1}{(1-k)}\langle F(\zeta^\dagger)-\zeta^\dagger,\zeta_{n+1}-\zeta^\dagger\rangle\right].$$

To conclude our proof, we utilize Lemma 2.4. For that it is sufficient to prove that $\limsup_{n\to\infty}\Pi_n\le 0$ for any subsequence $\{\|\zeta_{n_k}-\zeta^\dagger\|\}$ of $\{\|\zeta_n-\zeta^\dagger\|\}$ with the condition

$$\liminf_{k\to\infty}\left\{\|\zeta_{n_k+1}-\zeta^\dagger\|-\|\zeta_{n_k}-\zeta^\dagger\|\right\}\ge 0. \tag{3.8}$$

to establish this assume that there exists a subsequence $\{\|\zeta_{n_k}-\zeta^\dagger\|\}$ of $\{\|\zeta_n-\zeta^\dagger\|\}$ in such a way that (3.8) holds.

$$\begin{aligned}&\liminf_{k\to\infty}\left\{\|\zeta_{n_k+1}-\zeta^\dagger\|^2-\|\zeta_{n_k}-\zeta^\dagger\|^2\right\}\\ &=\liminf_{k\to\infty}\left\{\left(\|\zeta_{n_k+1}-\zeta^\dagger\|-\|\zeta_{n_k}-\zeta^\dagger\|\right)\left(\|\zeta_{n_k+1}-\zeta^\dagger\|-\|\zeta_{n_k}-\zeta^\dagger\|\right)\right\}\ge 0.\end{aligned} \tag{3.9}$$

Now,

$$\begin{aligned}\|\zeta_{n_k+1}-\zeta^\dagger\|^2 &\le \omega_{n_k}(\omega_{n_k}+\tau_{n_k})\|\zeta_{n_k}-\zeta^\dagger\|^2+\tau_{n_k}(\omega_{n_k}+\tau_{n_k})\|\varpi_{n_k}-\zeta^\dagger\|^2\\ &+2\chi_{n_k}\langle F(\zeta_{n_k})-F(\zeta^\dagger),\zeta_{n_k+1}-\zeta^\dagger\rangle+2\chi_{n_k}\langle F(\zeta^\dagger)-\zeta^\dagger,\zeta_{n_k+1}-\zeta^\dagger\rangle\\ &\le \omega_{n_k}(\omega_{n_k}+\tau_{n_k})\|\zeta_{n_k}-\zeta^\dagger\|^2+\tau_{n_k}(\omega_{n_k}+\tau_{n_k})\|\eta_{n_k}-\zeta^\dagger\|^2\\ &-\tau_{n_k}(\omega_{n_k}+\tau_{n_k})(1-\Gamma_{n_k})(\Gamma_{n_k}-\mu)\|\eta_{n_k}-\Phi_i(\eta_{n_k})\|^2\\ &+2\chi_{n_k}\langle F(\zeta_{n_k})-F(\zeta^\dagger),\zeta_{n_k+1}-\zeta^\dagger\rangle+2\chi_{n_k}\langle F(\zeta^\dagger)-\zeta^\dagger,\zeta_{n_k+1}-\zeta^\dagger\rangle\\ &\le \omega_{n_k}(\omega_{n_k}+\tau_{n_k})\|\zeta_{n_k}-\zeta^\dagger\|^2+\tau_{n_k}(\omega_{n_k}+\tau_{n_k})\|\zeta_{n_k}-\zeta^\dagger\|^2\\ &+\Upsilon_{n_k}\tau_{n_k}(\omega_{n_k}+\tau_{n_k})\|\zeta_{n_k}-\zeta_{n_k-1}\|M_2\\ &-\tau_{n_k}(\omega_{n_k}+\tau_{n_k})(1-\Gamma_{n_k})(\Gamma_{n_k}-\mu)\|\eta_{n_k}-\Phi_i(\eta_{n_k})\|^2\\ &+2\chi_{n_k}\langle F(\zeta_{n_k})-F(\zeta^\dagger),\zeta_{n_k+1}-\zeta^\dagger\rangle+2\chi_{n_k}\langle F(\zeta^\dagger)-\zeta^\dagger,\zeta_{n_k+1}-\zeta^\dagger\rangle\end{aligned}$$

$$\begin{aligned}
&\leq \omega_{n_k}(\omega_{n_k}+\tau_{n_k})\|\zeta_{n_k}-\zeta^\dagger\|^2+\tau_{n_k}(\omega_{n_k}+\tau_{n_k})\|\zeta_{n_k}-\zeta^\dagger\|^2\\
&+\Upsilon_{n_k}\tau_{n_k}(\omega_{n_k}+\tau_{n_k})\|\zeta_{n_k}-\zeta_{n_k-1}\|M_2\\
&-\tau_{n_k}(\omega_{n_k}+\tau_{n_k})(1-\Gamma_{n_k})(\Gamma_{n_k}-\mu)\|\eta_{n_k}-\Phi_i(\eta_{n_k})\|^2\\
&+\chi_{n_k}k\|\zeta_{n_k}-\zeta^\dagger\|^2+\chi_{n_k}k\|\zeta_{n_k+1}-\zeta^\dagger\|^2+2\chi_{n_k}\langle F(\zeta^\dagger)-\zeta^\dagger,\zeta_{n_k+1}-\zeta^\dagger\rangle\\
&\leq(\omega_{n_k}+\tau_{n_k})^2\|\zeta_{n_k}-\zeta^\dagger\|^2+\Upsilon_{n_k}\tau_{n_k}(\omega_{n_k}+\tau_{n_k})\|\zeta_{n_k}-\zeta_{n_k-1}\|M_2\\
&-\tau_{n_k}(\omega_{n_k}+\tau_{n_k})(1-\Gamma_{n_k})(\Gamma_{n_k}-\mu)\|\eta_{n_k}-\Phi_i(\eta_{n_k})\|^2\\
&+\chi_{n_k}k\|\zeta_{n_k}-\zeta^\dagger\|^2+\chi_{n_k}k\|\zeta_{n_k+1}-\zeta^\dagger\|^2+2\chi_{n_k}\langle F(\zeta^\dagger)-\zeta^\dagger,\zeta_{n_k+1}-\zeta^\dagger\rangle\\
&\leq(1-2\chi_{n_k}+\chi_{n_k}k)\|\zeta_{n_k}-\zeta^\dagger\|^2+\chi_{n_k}^2\|\zeta_{n_k}-\zeta^\dagger\|^2\\
&+\Upsilon_{n_k}\tau_{n_k}(\omega_{n_k}+\tau_{n_k})\|\zeta_{n_k}-\zeta_{n_k-1}\|M_2\\
&-\tau_{n_k}(\omega_{n_k}+\tau_{n_k})(1-\Gamma_{n_k})(\Gamma_{n_k}-\mu)\|\eta_{n_k}-\Phi_i(\eta_{n_k})\|^2\\
&+\chi_{n_k}k\|\zeta_{n_k}-\zeta^\dagger\|^2+\chi_{n_k}k\|\zeta_{n_k+1}-\zeta^\dagger\|^2+2\chi_{n_k}\langle F(\zeta^\dagger)-\zeta^\dagger,\zeta_{n_k+1}-\zeta^\dagger\rangle.
\end{aligned}$$

Simplifying the above equation, we get

$$\begin{aligned}
\|\zeta_{n_k+1}-\zeta^\dagger\|^2 &\leq \left(1-\frac{2\chi_{n_k}(1-k)}{1-\chi_{n_k}k}\right)\|\zeta_{n_k}-\zeta^\dagger\|^2+\frac{2\chi_{n_k}(1-k)}{1-\chi_{n_k}k}\\
&\left[\frac{\tau_{n_k}(1-\chi_{n_k})\Upsilon_{n_k}}{2\chi_{n_k}(1-k)}\|\zeta_{n_k}-\zeta_{n_k-1}\|M_2+\frac{\chi_{n_k}M_3}{2(1-k)}+\frac{1}{(1-k)}\langle F(\zeta^\dagger)-\zeta^\dagger,\zeta_{n_k+1}-\zeta^\dagger\rangle\right]\\
&-\frac{\tau_{n_k}(\omega_{n_k}+\tau_{n_k})}{1-\chi_{n_k}k}(1-\Gamma_{n_k})(\Gamma_{n_k}-\mu)\|\eta_{n_k}-\Phi_i(\eta_{n_k})\|^2\\
&\leq \left(1-\frac{2\chi_{n_k}(1-k)}{1-\chi_{n_k}k}\right)\|\zeta_{n_k}-\zeta^\dagger\|^2+\frac{2\chi_{n_k}(1-k)}{1-\chi_{n_k}k}\Pi_{n_k}\\
&-\frac{\tau_{n_k}(\omega_{n_k}+\tau_{n_k})}{1-\chi_{n_k}k}(1-\Gamma_{n_k})(\Gamma_{n_k}-\mu)\|\eta_{n_k}-\Phi_i(\eta_{n_k})\|^2\\
&\leq \|\zeta_{n_k}-\zeta^\dagger\|^2+\frac{2\chi_{n_k}(1-k)}{1-\chi_{n_k}k}M'\\
&-\frac{\tau_{n_k}(\omega_{n_k}+\tau_{n_k})}{1-\chi_{n_k}k}(1-\Gamma_{n_k})(\Gamma_{n_k}-\mu)\|\eta_{n_k}-\Phi_i(\eta_{n_k})\|^2.
\end{aligned}$$

It gives us

$$\begin{aligned}
&\limsup_{k\to\infty}\left(\frac{\tau_{n_k}(\omega_{n_k}+\tau_{n_k})}{1-\chi_{n_k}k}(1-\Gamma_{n_k})(\Gamma_{n_k}-\mu)\|\eta_{n_k}-\Phi_i(\eta_{n_k})\|^2\right)\\
&\qquad\leq \limsup_{k\to\infty}\left(\|\zeta_{n_k}-\zeta^\dagger\|^2+\frac{2\chi_{n_k}(1-k)}{1-\chi_{n_k}k}M'-\|\zeta_{n_k+1}-\zeta^\dagger\|^2\right)\\
&\qquad\leq -\liminf_{k\to\infty}\left(\|\zeta_{n_k+1}-\zeta^\dagger\|^2-\|\zeta_{n_k}-\zeta^\dagger\|^2\right)\leq 0.
\end{aligned}$$

And we get

$$\lim_{k\to\infty}\|\eta_{n_k}-\Phi_i(\eta_{n_k})\|=0. \tag{3.10}$$

If we follow the same approach as above, we can easily get

$$\lim_{k\to\infty} \|\vartheta_{n_k} - \xi_{n_k}\| = 0. \tag{3.11}$$

We also have

$$\begin{aligned}
\|\vartheta_{n_k} - \eta_{n_k}\|^2 &= \left\|\Phi^{F,h}_{r_{n_k}}(\xi_{n_k} - r_{n_k}G(\xi_{n_k})) - \Phi^{F,h}_{r_{n_k}Q_{n_k}}(\vartheta_{n_k} - r_{n_k}G(\vartheta_{n_k}))\right\|^2 \\
&\le \|(\xi_{n_k} - r_{n_k}G(\xi_{n_k})) - (\vartheta_{n_k} - r_{n_k}G(\vartheta_{n_k}))\|^2 \\
&\le \|\xi_{n_k} - \vartheta_{n_k}\|^2 + (r_{n_k})^2\|G(\xi_{n_k}) - G(\vartheta_{n_k})\|^2 \\
&\le \|\xi_{n_k} - \vartheta_{n_k}\|^2 + (r_{n_k}L)^2\|\xi_{n_k} - \vartheta_{n_k}\|^2 \\
&\le (1 + (r_{n_k}L)^2)\|\xi_{n_k} - \vartheta_{n_k}\|^2.
\end{aligned}$$

Applying the above equation (3.11), we get

$$\lim_{k\to\infty} \|\vartheta_{n_k} - \eta_{n_k}\| = 0. \tag{3.12}$$

Again we have

$$\|\varpi_{n_k} - \eta_{n_k}\| = \|\Gamma_{n_k}\eta_{n_k} + (1 - \Gamma_{n_k})\Phi_i(\eta_{n_k}) - \eta_{n_k}\| = (1 - \Gamma_{n_k})\|\Phi_i(\eta_{n_k}) - \eta_{n_k}\|.$$

Again applying (3.10) to the above equation, we get

$$\lim_{k\to\infty} \|\varpi_{n_k} - \eta_{n_k}\| = 0. \tag{3.13}$$

Now

$$\|\xi_{n_k} - \zeta_{n_k}\| \le \Upsilon_{n_k}\|\zeta_{n_k} - \zeta_{n_k-1}\| = \chi_{n_k} \cdot \frac{\Upsilon_{n_k}}{\chi_{n_k}}\|\zeta_{n_k} - \zeta_{n_k-1}\|$$

and we can easily get from the above equation that

$$\lim_{k\to\infty} \|\xi_{n_k} - \zeta_{n_k}\| = 0. \tag{3.14}$$

Using above limits, we can also get

$$\begin{aligned}
\lim_{k\to\infty} \|\vartheta_{n_k} - \xi_{n_k}\| &\le \lim_{k\to\infty} \|\vartheta_{n_k} - \zeta_{n_k}\| + \lim_{k\to\infty} \|\zeta_{n_k} - \xi_{n_k}\| = 0, \\
\lim_{k\to\infty} \|\vartheta_{n_k} - \zeta_{n_k}\| &\le \lim_{k\to\infty} \|\vartheta_{n_k} - \xi_{n_k}\| + \lim_{k\to\infty} \|\zeta_{n_k} - \xi_{n_k}\| = 0, \\
\lim_{k\to\infty} \|\varpi_{n_k} - \zeta_{n_k}\| &\le \lim_{k\to\infty} \|\varpi_{n_k} - \eta_{n_k}\| + \lim_{k\to\infty} \|\eta_{n_k} - \zeta_{n_k}\| = 0, \\
\lim_{k\to\infty} \|\eta_{n_k} - \zeta_{n_k}\| &\le \lim_{k\to\infty} \|\eta_{n_k} - \vartheta_{n_k}\| + \lim_{k\to\infty} \|\vartheta_{n_k} - \zeta_{n_k}\| = 0.
\end{aligned} \tag{3.15}$$

Now we prove that $\limsup_{k\to\infty} \Pi_{n_k} \le 0$. To prove this it is sufficient to prove that

$$\limsup_{k\to\infty} \langle F(\zeta^\dagger) - \zeta^\dagger, \zeta_{n_k+1} - \zeta^\dagger\rangle \le 0.$$

Suppose $\left\{\zeta_{n_{k_j}}\right\}$ is a subsequence of $\{\zeta_{n_k}\}$ such that

$$\limsup_{j\to\infty}\langle F(\zeta^\dagger)-\zeta^\dagger,\zeta_{n_{k_j}+1}-\zeta^\dagger\rangle=\limsup_{k\to\infty}\langle F(\zeta^\dagger)-\zeta^\dagger,\zeta_{n_k+1}-\zeta^\dagger\rangle. \tag{3.16}$$

Now we prove that $\lim\limits_{k\to\infty}\|\zeta_{n_k+1}-\zeta_{n_k}\|=0$. So we have

$$\begin{aligned}\|\zeta_{n_k+1}-\zeta_{n_k}\| &= \|\chi_{n_k}F(\zeta_{n_k})+\omega_{n_k}\zeta_{n_k}+\tau_{n_k}\varpi_{n_k}-\zeta_{n_k}\|\\ &= \|\chi_{n_k}(F(\zeta_{n_k})-\zeta_{n_k})+\omega_{n_k}\zeta_{n_k}+\tau_{n_k}(\varpi_{n_k}-\zeta_{n_k})-(1-\chi_{n_k}-\tau_{n_k})\zeta_{n_k}\|\\ &= \|\chi_{n_k}(F(\zeta_{n_k})-\zeta_{n_k})+\tau_{n_k}(\varpi_{n_k}-\zeta_{n_k})\|\\ &\le \chi_{n_k}\|F(\zeta_{n_k})-\zeta_{n_k}\|+\tau_{n_k}\|\varpi_{n_k}-\zeta_{n_k}\|.\end{aligned}$$

Applying condition (1) and (3.13), we get

$$\lim_{k\to\infty}\|\zeta_{n_k+1}-\zeta_{n_k}\|=0. \tag{3.17}$$

Since the sequence $\{\zeta_n\}$ is bounded in Σ there exists a subsequence $\{\zeta_{n_k}\}$ of $\{\zeta_n\}$ which converges weakly in Σ. Suppose $\zeta_{n_k}\rightharpoonup\zeta^*\in\Sigma$. From (3.15), we can get that $\{\eta_{n_k}\}$ and $\{\vartheta_{n_k}\}$ also converges weakly to ζ^*. SInce $\vartheta_n=\Phi_{r_n}^{F,h}(\xi_n-r_nG(\xi_n))$, we get

$$F(\vartheta_n,\vartheta)+h(\vartheta,\vartheta_n)-h(\vartheta_n,\vartheta_n)+\frac{1}{r_n}\langle\vartheta-\vartheta_n,\vartheta_n-(\xi_n-r_nG(\xi_n))\rangle\ge 0,\forall\vartheta\in\mathcal{E}.$$

using the monotonicty of F, we get

$$h(\vartheta,\vartheta_n)-h(\vartheta_n,\vartheta_n)+\frac{1}{r_n}\langle\vartheta-\vartheta_n,\vartheta_n-\xi_n\rangle\ge F(\vartheta,\vartheta_n)+\langle G(\xi_n),\vartheta_n-\vartheta\rangle,\forall\vartheta\in\mathcal{E}.$$

Hence $\forall\vartheta\in\mathcal{E}$,

$$h(\vartheta,\vartheta_{n_k})-h(\vartheta_{n_k},\vartheta_{n_k})+\left\langle\vartheta-\vartheta_{n_k},\frac{\vartheta_{n_k}-\xi_{n_k}}{r_{n_k}}\right\rangle\ge F(\vartheta,\vartheta_{n_k})+\langle G(\xi_{n_k}),\vartheta_{n_k}-\vartheta\rangle. \tag{3.18}$$

For any $\nu\in(0,1)$, we can see that $\vartheta_\nu=\nu\vartheta+(1-\nu)\zeta^*\in\mathcal{E}$. Then from (3.18), we get

$$\begin{aligned}\langle G(\vartheta_\nu),\vartheta_\nu-\vartheta_{n_k})\rangle\ge{}&\langle G(\vartheta_\nu)-G(\vartheta_{n_k}),\vartheta_\nu-\vartheta_{n_k}\rangle-h(\vartheta_\nu,\vartheta_{n_k})+h(\vartheta_{n_k})-h(\vartheta_\nu,\vartheta_{n_k})\\ &+h(\vartheta_{n_k},\vartheta_{n_k})-\left\langle\vartheta-\vartheta_{n_k},\frac{\vartheta_{n_k}-\xi_{n_k}}{r_{n_k}}\right\rangle+F(\vartheta_\nu,\vartheta_{n_k})\\ &+\langle G(\vartheta_{n_k})-G(\xi_{n_k}),\vartheta_\nu-\vartheta_{n_k}\rangle.\end{aligned} \tag{3.19}$$

Since the mapping G is monotone and Lipschitz continuous, we also have

$$\lim_{k\to\infty}\|G(\vartheta_{n_k})-G(\xi_{n_k})\|\le\lim_{k\to\infty}L\|\vartheta_{n_k}-\xi_{n_k}\|=0.$$

and hence

$$\lim_{k\to\infty}\|G(\vartheta_{n_k})-G(\xi_{n_k})\|=0.$$

Using the assumption 3.1 and the above inequality (3.19), we have

$$h(\vartheta_\nu, \zeta^*) - h(\zeta^*, \zeta^*) \geq F(\vartheta_\nu, \zeta^*) - \langle G(\vartheta_\nu), \vartheta_\nu - \zeta^* \rangle.$$

Now,

$$\begin{aligned} 0 &= F(\vartheta_\nu, \vartheta_\nu) \\ &\leq \nu F(\vartheta_\nu, \vartheta) + (1-\nu) F(\vartheta_\nu, \zeta^*) \\ &\leq \nu F(\vartheta_\nu, \vartheta) + (1-\nu)[h(\vartheta_\nu, \zeta^*) - h(\zeta^*, \zeta^*) + \langle G(\vartheta_\nu), \vartheta_\nu - \zeta^* \rangle] \\ &= \nu F(\vartheta_\nu, \vartheta) + (1-\nu)[h(\vartheta_\nu, \zeta^*) - h(\zeta^*, \zeta^*)] + (1-\nu)\langle G(\vartheta_\nu), \nu\vartheta + (1-\nu)\zeta^* - \zeta^* \rangle \\ &= F(\vartheta_\nu, \vartheta) + (1-\nu)[h(\vartheta, \zeta^*) - h(\zeta^*, \zeta^*) + \langle G(\vartheta_\nu), \vartheta - \zeta^* \rangle] \end{aligned} \tag{3.20}$$

For $\nu \to 0$, we get

$$F(\zeta^*, \vartheta) + h(\vartheta, \zeta^*) - h(\zeta^*, \zeta^*) + \langle F(\zeta^*), \vartheta - \zeta^* \rangle \geq 0, \forall \vartheta \in \mathcal{E},$$

and it gives us $\zeta^* \in MEP(F, h)$. Further using the demiclosedness of Φ_i and (3.10) and (3.15), we get $\zeta^* \in \bigcap_{i=1}^{n} F(\Phi_i)$. Hence $\zeta^* \in \Xi$. Now finally, using (3.6), (3.16) and (3.17), we get

$$\begin{aligned} \limsup_{k\to\infty} \langle F(\zeta^\dagger) - \zeta^\dagger, \zeta_{n_k+1} - \zeta^\dagger \rangle &= \limsup_{k\to\infty} \langle F(\zeta^\dagger) - \zeta^\dagger, \zeta_{n_k+1} - \zeta_{n_k} \rangle \\ &\quad + \limsup_{k\to\infty} \langle F(\zeta^\dagger) - \zeta^\dagger, \zeta_{n_k} - \zeta^\dagger \rangle \\ &= \lim_{j\to\infty} \langle F(\zeta^\dagger) - \zeta^\dagger, \zeta_{n_{k_j}} - \zeta^\dagger \rangle \\ &= \langle F(\zeta^\dagger) - \zeta^\dagger, \zeta^* - \zeta^\dagger \rangle \\ &\leq 0. \end{aligned} \tag{3.21}$$

Applying Lemma 2.4, we get that the sequence $\{\zeta_n\}$ generated by Algorithm 3.2 converges to $\zeta^\dagger \in \Xi$. □

4. Examples

Example 4.1. Let $\Sigma = \mathbb{R}$ and $\mathcal{E} = [0, 1]$, define $G(\zeta) = 3\zeta$, $F : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ as $F(\zeta, \eta) = (\zeta - 2)(\eta - \zeta)$, for all $\zeta, \eta \in \mathcal{E}$ and $h : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ as $h(\zeta, \eta) = \eta - \zeta$ for all $\zeta, \eta \in \mathcal{E}$. Here $MEP(F, h) = \{\frac{3}{4}\}$. Define mappings $F : \mathcal{E} \to \mathcal{E}$ by $F(\zeta) = \frac{\zeta}{3}$, $\Phi_1 : \mathcal{E} \to \mathcal{E}$ by

$$\Phi_1(\zeta) = \begin{cases} \frac{3}{4}, \text{ if } 0 \leq \zeta < 1, \\ \frac{1}{2}, \text{ if } \zeta = 1, \end{cases}$$

and $\Phi_2 : \mathcal{E} \to \mathcal{E}$ by

$$\Phi_2(\zeta) = \begin{cases} \frac{3}{4}, \text{ if } 0 \le \zeta \le \frac{3}{4}, \\ \frac{1}{9}, \text{ if } \frac{3}{4} < \zeta \le 1. \end{cases}$$

Here $\Xi = \{\zeta \in \text{ MEP}(F,h) \cap \bigcap_{i=1}^{2} F(\Phi_i)\} = \{\frac{3}{4}\}$ and all the conditions of Theorem 3.5 are satisfied. Hence the sequence $\{\zeta_n\}$ generated by Algorithm (3.2) converges strongly to $\frac{3}{4}$.

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