EXISTENCE OF PERIODIC SOLUTIONS FOR A NON-AUTONOMOUS DIFFERENTIAL SYSTEM OF EVEN DIMENSION n

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Abstract. In this paper, we study a non-autonomous differential system of even dimension n and aim to determine the conditions that ensure the existence of periodic solutions. Using the classical another first-order averaging theory, we establish sufficient conditions for the existence of these solutions.

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1. Introduction

One of the fundamental aspects of the qualitative theory of differential systems is the study of periodic orbits including their existence, number, and stability. For example, the study of periodic solutions in predator-prey models helps ecologists analyze the cyclical fluctuations in animal populations, providing insights for better wildlife management. In engineering, detecting periodic solutions in systems such as electrical circuits or mechanical vibrations is essential for maintaining stability and avoiding resonance effects that could cause system failure. It also plays a fundamental role in climate research, the medical field, and various other areas of study. In general, the study of phenomena with known periodic solutions facilitates the control and prediction of their outcomes.

Typically, it is difficult to find periodic solutions of differential systems using exact mathematical methods, and in many cases, it is not possible at all. Averaging theory is a useful method that helps researchers study periodic solutions in these systems. This method begins with the works of Lagrange and Laplace. For more explanations about averaging theory, see the books by Verhulst [1] and Sanders and Verhulst [2]. The main idea of the averaging theory is to reduce the complex problem of finding periodic solutions to finding the zeros of a system of nonlinear equations.

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Many researchers have studied problems related to the periodic behavior of solutions in higher-order differential equations and systems. Several papers, including [[3], [4], [5], [6], [7], [8], [9], [10]], provide useful examples and contributions in this area.

In [11], the authors studied the periodic solutions of a four-dimensional system using the averaging method of the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{u} \end{pmatrix} = \begin{matrix} -y + h_1(t) \\ x + h_2(t) \\ -u + h_3(t) \\ z + h_4(t) \end{matrix} + \varepsilon \begin{pmatrix} P_1(x, y, z, u) \\ P_2(x, y, z, u) \\ P_3(x, y, z, u) \\ P_4(x, y, z, u) \end{pmatrix}, \tag{1}$$

where P_i are polynomials in the variables x, y, z and u of degree n, $\mathrm{h}_i(t)$ are $2\pi-$periodic functions with $i = \overline{1, 4}$, and ε is a small parameter.

In [12], the authors provide sufficient conditions for the existence of periodic solutions for the differential system (2)

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{u} \end{pmatrix} = \begin{matrix} y \\ -x - \varepsilon G(t, x, y, z, u) \\ u \\ -z - \varepsilon H(t, x, y, z, u), \end{matrix} \tag{2}$$

where G and H are 2π-periodic functions in the variable t and ε is a small parameter.

In [13], the authors provide sufficient conditions for the existence of periodic solutions for the differential system (3)

$$\begin{aligned} \dot{x} &= y, \quad \dot{y} = -x - \varepsilon F(t, x, y, z, u, v, w) \\ \dot{z} &= u, \quad \dot{u} = -z - \varepsilon G(t, x, y, z, u, v, w) \\ \dot{v} &= w, \quad \dot{w} = -v - \varepsilon H(t, x, y, z, u, v, w), \end{aligned} \tag{3}$$

where F, G and H are 2π-periodic functions in the variable t and ε is a small parameter.

In this work, we investigate the existence of periodic solutions in system (4) using another first order averaging method

$$\begin{aligned} \dot{x}_1 &= -x_2 + f_1(t) + \varepsilon R_1(x_1, x_2, x_3, x_4, x_5, x_6, ..., x_{n-1}, x_n) \\ \dot{x}_2 &= x_1 + f_2(t) + \varepsilon R_2(x_1, x_2, x_3, x_4, x_5, x_6, ..., x_{n-1}, x_n) \\ \dot{x}_3 &= -x_4 + f_3(t) + \varepsilon R_3(x_1, x_2, x_3, x_4, x_5, x_6, ..., x_{n-1}, x_n) \\ \dot{x}_4 &= x_3 + f_4(t) + \varepsilon R_4(x_1, x_2, x_3, x_4, x_5, x_6, ..., x_{n-1}, x_n) \\ &\vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ \dot{x}_{n-1} &= -x_n + f_{n-1}(t) + \varepsilon R_{n-1}(x_1, x_2, x_3, x_4, x_5, x_6, ..., x_{n-1}, x_n) \\ \dot{x}_n &= x_{n-1} + f_n(t) + \varepsilon R_n(x_1, x_2, x_3, x_4, x_5, x_6, ..., x_{n-1}, x_n), \end{aligned} \tag{4}$$

where each R_i is a polynomial of degree m_i in the variables x_i, $f_i(t)$ are $2\pi-$periodic functions with $i=\overline{1,n}$, where n is even and ε denotes a small parameter.

2. Statement of the main result

We now present the main result in the form of the following theorem.

Theorem 1. *We consider the system defined by (4), and define the following set of equations*

$$\begin{aligned}
\mathcal{F}_1(x_{1,0},x_{2,0},x_{3,0},x_{4,0},...,x_{n-1,0},x_{n,0}) &= \tfrac{1}{2\pi}\int_0^{2\pi}(\cos(t)R_1(A(t))+\sin(t)R_2(A(t)))dt,\\
\mathcal{F}_2(x_{1,0},x_{2,0},x_{3,0},x_{4,0},...,x_{n-1,0},x_{n,0}) &= \tfrac{1}{2\pi}\int_0^{2\pi}(-\sin(t)R_1(A(t))+\cos(t)R_2(A(t)))dt,\\
\mathcal{F}_3(x_{1,0},x_{2,0},x_{3,0},x_{4,0},...,x_{n-1,0},x_{n,0}) &= \tfrac{1}{2\pi}\int_0^{2\pi}(\cos(t)R_3(A(t))+\sin(t)R_4(A(t)))dt,\\
\mathcal{F}_4(x_{1,0},x_{2,0},x_{3,0},x_{4,0},...,x_{n-1,0},x_{n,0}) &= \tfrac{1}{2\pi}\int_0^{2\pi}(-\sin(t)R_3(A(t))+\cos(t)R_4(A(t)))dt,\\
\vdots \qquad & \qquad \vdots \qquad\qquad \vdots\\
\mathcal{F}_{n-1}(x_{1,0},x_{2,0},x_{3,0},x_{4,0},...,x_{n-1,0},x_{n,0}) &= \tfrac{1}{2\pi}\int_0^{2\pi}(\cos(t)R_{n-1}(A(t))+\sin(t)R_n(A(t)))dt,\\
\mathcal{F}_n(x_{1,0},x_{2,0},x_{3,0},x_{4,0},...,x_{n-1,0},x_{n,0}) &= \tfrac{1}{2\pi}\int_0^{2\pi}(-\sin(t)R_{n-1}(A(t))+\cos(t)R_n(A(t)))dt,
\end{aligned}$$

where $A(t)=(a_1(t),a_2(t),a_3(t),a_4(t),...,a_{n-1}(t),a_n(t))$ which is represented by the following expression

$$\begin{aligned}
a_1(t) &= \cos(t)x_{1,0}-\sin(t)x_{2,0}+\int_0^t(\cos(t-s)f_1(s)-\sin(t-s)f_2(s))\,ds,\\
a_2(t) &= \sin(t)x_{1,0}+\cos(t)x_{2,0}+\int_0^t(\sin(t-s)f_1(s)+\cos(t-s)f_2(s))\,ds,\\
a_3(t) &= \cos(t)x_{3,0}-\sin(t)x_{4,0}+\int_0^t(\cos(t-s)f_3(s)-\sin(t-s)f_4(s))\,ds,\\
a_4(t) &= \sin(t)x_{3,0}+\cos(t)x_{4,0}+\int_0^t(\sin(t-s)f_3(s)+\cos(t-s)f_4(s))\,ds,\\
\vdots \qquad & \qquad \vdots \qquad\qquad \vdots\\
a_{n-1}(t) &= \cos(t)x_{n-1,0}-\sin(t)x_{n,0}+\int_0^t(\cos(t-s)f_{n-1}(s)-\sin(t-s)f_n(s))\,ds,\\
a_n(t) &= \sin(t)x_{n-1,0}+\cos(t)x_{n,0}+\int_0^t(\sin(t-s)f_{n-1}(s)+\cos(t-s)f_n(s))\,ds.
\end{aligned}$$

If the following conditions are satisfied

$$\begin{aligned}
&\int_0^{2\pi}(\cos(s)f_1(s)+\sin(s)f_2(s))ds=0,\\
&\int_0^{2\pi}(-\sin(s)f_1(s)+\cos(s)f_2(s))\,ds=0,\\
&\int_0^{2\pi}(\cos(s)f_3(s)+\sin(s)f_4(s))\,ds=0,\\
&\int_0^{2\pi}(-\sin(s)f_3(s)+\cos(s)f_4(s))\,ds=0,
\end{aligned} \tag{5}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\int_0^{2\pi} (\cos(s) f_{n-1}(s) + \sin(s) f_n(s)) ds = 0,$$

$$\int_0^{2\pi} (-\sin(s) f_{n-1}(s) + \cos(s) f_n(s)) ds = 0,$$

then for every $(x^*_{1,0}, x^*_{2,0}, x^*_{3,0}, x^*_{4,0}, ..., x^*_{n-1,0}, x^*_{n,0})$ *solution of the system*

$$\mathcal{F}_k (x_{1,0}, x_{2,0}, x_{3,0}, x_{4,0}, \ldots, x_{n-1,0}, x_{n,0}) = 0, \quad k = \overline{1, n},$$

and satisfying

$$\det \left(\frac{\partial (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4, ..., \mathcal{F}_{n-1}, \mathcal{F}_n)}{\partial (x_{1,0}, x_{2,0}, x_{3,0}, x_{4,0}, ..., x_{n-1,0}, x_{n,0})} \right) \Big|_{(x^*_{1,0}, x^*_{2,0}, x^*_{3,0}, x^*_{4,0}, ..., x^*_{n-1,0}, x^*_{n,0})} \neq 0.$$

The differential system (4) *possesses a periodic solution*

$$(x_1(t,\varepsilon), x_2(t,\varepsilon), x_3(t,\varepsilon), x_4(t,\varepsilon), ..., x_{n-1}(t,\varepsilon), x_n(t,\varepsilon))^t,$$

which tends toward the periodic solution given by

$$\begin{aligned}
x_1(t) &= \cos(t) x^*_{1,0} - \sin(t) x^*_{2,0} + \int_0^t (\cos(t-s) f_1(s) - \sin(t-s) f_2(s)) ds \\
x_2(t) &= \sin(t) x^*_{1,0} + \cos(t) x^*_{2,0} + \int_0^t (\sin(t-s) f_1(s) + \cos(t-s) f_2(s)) ds \\
x_3(t) &= \cos(t) x^*_{3,0} - \sin(t) x^*_{4,0} + \int_0^t (\cos(t-s) f_3(s) - \sin(t-s) f_4(s)) ds \\
x_4(t) &= \sin(t) x^*_{3,0} + \cos(t) x^*_{4,0} + \int_0^t (\sin(t-s) f_3(s) + \cos(t-s) f_4(s)) ds \\
&\vdots \qquad \vdots \qquad \vdots \\
x_{n-1}(t) &= \cos(t) x^*_{n-1,0} - \sin(t) x^*_{n,0} + \int_0^t (\cos(t-s) f_{n-1}(s) - \sin(t-s) f_n(s)) ds \\
x_n(t) &= \sin(t) x^*_{n-1,0} + \cos(t) x^*_{n,0} + \int_0^t (\sin(t-s) f_{n-1}(s) + \cos(t-s) f_n(s)) ds,
\end{aligned}$$

concerning the differential system

$$\begin{aligned}
\dot{x}_1 &= -x_2 + f_1(t) \\
\dot{x}_2 &= x_1 + f_2(t) \\
\dot{x}_3 &= -x_4 + f_3(t) \\
\dot{x}_4 &= x_3 + f_4(t) \\
&\vdots \qquad \vdots \\
\dot{x}_{n-1} &= -x_n + f_{n-1}(t) \\
\dot{x}_n &= x_{n-1} + f_n(t),
\end{aligned}$$

when $\varepsilon \to 0$*, it is important to note that this solution is periodic with a period of* 2π*.*

Theorem 1 is proved using the averaging theory for the study of periodic orbits, as explained in Section 3. A complete proof is given in Section 4. In Section 5, we provide two examples to illustrate how the theorem can be applied.

3. Another first order averaging theory

We consider the problem of the bifurcation of T- periodic solutions from differential systems of the form

$$\dot{x} = F_0\left(t, x\right) + \varepsilon F_1\left(t, x\right) + \varepsilon^2 F_2(t, x, \varepsilon), \tag{6}$$

with $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, for ε_0 sufficiently small. Here the functions F_0, $F_1 : \mathbb{R} \times \Omega \to \mathbb{R}^n$ and $F_2 : \mathbb{R} \times \Omega \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are C^2 functions, T-periodic in the first variable and Ω is an open subset of $\mathbb{R}^n$. One of the main assumptions is that the unperturbed system

$$\dot{x} = F_0\left(t, x\right), \tag{7}$$

has a submanifold of periodic solutions.

Let $x\left(t, z\right)$ be the solution of system (7) such that $x(0, z) = z$. We write the linearized system of the unperturbed system along the periodic solution $x(t, z)$ as

$$\dot{y} = D_x F_0(t, x(t, z, 0))y. \tag{8}$$

In what follows we denote by $M_z(t)$ some fundamental matrix of the linear differential system (8), and by $\xi : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k$ the projection of $\mathbb{R}^n$ onto its first k coordinates $\xi(x_1, ..., x_n) = (x_1, ..., x_k)$.

Theorem 2. *Let $V \in \mathbb{R}^k$ be open bounded with $Cl(V) \subset \Omega$, and let $\beta_0 : Cl(V) \to \mathbb{R}^{n-k}$ be a C^2 function. We assume*

(1) *$Z = \{z_\alpha = (\alpha, \beta_0\left(\alpha\right)), \ \alpha \in Cl\left(V\right)\} \subset \Omega$ and that for each $z_\alpha \in Z$ the solution $x(t, z_\alpha)$ of (7) is T-periodic.*
(2) *for each $z_\alpha \in Z$ there is a fundamental matrix $M_{z_\alpha}(t)$ of (8) such that the $M_{z_\alpha}^{-1}(0) - M_{z_\alpha}^{-1}(T)$ has in the right up corner the $k \times (n-k)$ zero matrix, and in the right down corner a $(n-k) \times (n-k)$ matrix Δ_α with $det(\Delta_\alpha) \neq 0$.*

Remark 3. *This proof was first given by Roseau [14]. A shorter version is available in [15].*

We consider the function $\mathcal{F} : Cl\left(V\right) \to \mathbb{R}^k$ defined by

$$\mathcal{F}\left(\alpha\right) = \xi\left(\int_0^T M_{z_\alpha}^{-1}\left(t\right) F_1\left(t, x\left(t, z_\alpha\right)\right) dt\right). \tag{9}$$

If there exists $a \in V$ with $\mathcal{F}\left(a\right) = 0$ and $\det\left(\left(\frac{d\mathcal{F}}{d\alpha}\right)\left(a\right)\right) \neq 0$, then there is a $T-$periodic solution $x\left(t, \varepsilon\right)$ of system (6) such that $x\left(0, \varepsilon\right) \to z_\alpha$ as $\varepsilon \to 0$.

We assume that there exists an open set V with $Cl(V) \subset \Omega$ such that for each $z \in Cl\left(V\right), x\left(t, z, 0\right)$ is $T-$periodic, where $x\left(t, z, 0\right)$ denotes the solution of the unperturbed system (7) with $x\left(t, z, 0\right) = z$. The set $Cl\left(V\right)$ is isochronous for the system (6); i.e. it is a set formed only by periodic orbits, all of them having the same period. Then, an answer to the problem of the bifurcation of T-periodic solutions from the periodic solution $x\left(t, z, 0\right)$ contained in $Cl\left(V\right)$ is given in the following result.

Theorem 4. (*Perturbations of an isochronous set*)

We assume that there exists an open and bounded set V with $Cl\,(V) \subset \Omega$ such that for each $z \in Cl\,(V)$ the solution $x\,(t,z)$ is T-periodic, then we consider the function $\mathcal{F} : Cl(V) \to \mathbb{R}^n$ as

$$\mathcal{F}\,(z) = \frac{1}{T}\int_0^T M_z^{-1}(t,z)F_1\,(t, x(t,z))\,dt. \tag{10}$$

If there exists $a \in V$ such that $\mathcal{F}\,(a) = 0$ and

$$\det\left(\left(d\mathcal{F}/dz\right)(a)\right) \neq 0, \tag{11}$$

then there exists a T-periodic solution $x\,(t,\varepsilon)$ of system (6) such that $x(0,\varepsilon) \to a$ as $\varepsilon \to 0$.

Theorem 5. *Under the assumptions of Theorem* (4)*, for small* ε *ensures the existence and uniqueness of a T-periodic solution* $x(t,\varepsilon)$ *of system* (6) *such that* $x(0,\varepsilon) \to a$ *as* $\varepsilon \to 0$*, and if all eigenvalues of the matrix* $(d\mathcal{F}/dz)\,(a)$ *have negative real parts, then the periodic solution* $x(t,\varepsilon)$ *is stable. If some of the eigenvalue has positive real part the periodic solution* $x(t,\varepsilon)$ *is unstable.*

4. The proof of theorem

Using the results from Section 3, system (4) can be rewritten as system (6), with

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix}, \; F_0(t,x) = \begin{pmatrix} -x_2 + f_1\,(t) \\ x_1 + f_2\,(t) \\ -x_4 + f_3\,(t) \\ x_3 + f_4\,(t) \\ \vdots \\ -x_n + f_{n-1}\,(t) \\ x_{n-1} + f_n\,(t) \end{pmatrix},$$

and

$$F_1 = \begin{pmatrix} R_1\,(x_1, x_2, x_3, x_4, x_5, x_6, ..., x_{n-1}, x_n) \\ R_2\,(x_1, x_2, x_3, x_4, x_5, x_6, ..., x_{n-1}, x_n) \\ R_3\,(x_1, x_2, x_3, x_4, x_5, x_6, ..., x_{n-1}, x_n) \\ R_4\,(x_1, x_2, x_3, x_4, x_5, x_6, ..., x_{n-1}, x_n) \\ \vdots \\ R_{n-1}\,(x_1, x_2, x_3, x_4, x_5, x_6, ..., x_{n-1}, x_n) \\ R_n\,(x_1, x_2, x_3, x_4, x_5, x_6, ..., x_{n-1}, x_n) \end{pmatrix}.$$

We examine the periodic solutions of system (4) under n is even. By using

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = e^{At} \begin{pmatrix} x_{1,0} \\ x_{2,0} \\ x_{3,0} \\ x_{4,0} \\ \vdots \\ x_{n-1,0} \\ x_{n,0} \end{pmatrix} + \int_0^t e^{A(t-s)} \begin{pmatrix} f_1(s) \\ f_2(s) \\ f_3(s) \\ f_4(s) \\ \vdots \\ f_{n-1}(s) \\ f_n(s) \end{pmatrix} ds,$$

where

$$A = \begin{pmatrix} 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Hence, the following result is obtained

$$\begin{aligned} x_1(t) &= \cos(t)x_{1,0} - \sin(t)x_{2,0} + \int_0^t (\cos(t-s)f_1(s) - \sin(t-s)f_2(s))\,ds \\ x_2(t) &= \sin(t)x_{1,0} + \cos(t)x_{2,0} + \int_0^t (\sin(t-s)f_1(s) + \cos(t-s)f_2(s))\,ds \\ x_3(t) &= \cos(t)x_{3,0} - \sin(t)x_{4,0} + \int_0^t (\cos(t-s)f_3(s) - \sin(t-s)f_4(s))\,ds \\ x_4(t) &= \sin(t)x_{3,0} + \cos(t)x_{4,0} + \int_0^t (\sin(t-s)f_3(s) + \cos(t-s)f_4(s))\,ds \\ &\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ x_{n-1}(t) &= \cos(t)x_{n-1,0} - \sin(t)x_{n,0} + \int_0^t (\cos(t-s)f_{n-1}(s) - \sin(t-s)f_n(s))\,ds \\ x_n(t) &= \sin(t)x_{n-1,0} + \cos(t)x_{n,0} + \int_0^t (\sin(t-s)f_{n-1}(s) + \cos(t-s)f_n(s))\,ds. \end{aligned}$$

These solutions are periodic, having a period 2π if and only if

$$\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \\ \vdots \\ x_{n-1}(0) \\ x_n(0) \end{pmatrix} = \begin{pmatrix} x_1(2\pi) \\ x_2(2\pi) \\ x_3(2\pi) \\ x_4(2\pi) \\ \vdots \\ x_{n-1}(2\pi) \\ x_n(2\pi) \end{pmatrix}.$$

The conditions for the periodicity of these solutions are defined in Statement (5) of Theorem (1). It is evident that the set of periodic solutions has a dimension of n. Consequently, we aim to identify the periodic solutions of system (4) by determining the zeros $z = (x_{1,0}, x_{2,0}, x_{3,0}, x_{4,0}, ..., x_{n-1,0}, x_{n,0})$ of the system $\mathcal{F}(z) = 0$, where $\mathcal{F}(z)$ is defined in (10). The fundamental matrix $M(t)$ of the differential system described in equation (8) is given by

$$M\left(t\right) = M_z\left(t\right) = \begin{pmatrix} \cos(t) & -\sin(t) & 0 & 0 & \cdots & 0 & 0 \\ \sin(t) & \cos(t) & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cos(t) & -\sin(t) & \cdots & 0 & 0 \\ 0 & 0 & \sin(t) & \cos(t) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cos(t) & -\sin(t) \\ 0 & 0 & 0 & 0 & 0 & \sin(t) & \cos(t) \end{pmatrix}.$$

By computing the function $\mathcal{F}(z)$, we obtain the following system

$$\begin{aligned} \mathcal{F}_1(x_{1,0}, x_{2,0}, x_{3,0}, x_{4,0}, ..., x_{n-1,0}, x_{n,0}) &= 0, \\ \mathcal{F}_2(x_{1,0}, x_{2,0}, x_{3,0}, x_{4,0}, ..., x_{n-1,0}, x_{n,0}) &= 0, \\ \mathcal{F}_3(x_{1,0}, x_{2,0}, x_{3,0}, x_{4,0}, ..., x_{n-1,0}, x_{n,0}) &= 0, \\ \mathcal{F}_4(x_{1,0}, x_{2,0}, x_{3,0}, x_{4,0}, ..., x_{n-1,0}, x_{n,0}) &= 0, \\ \vdots \qquad & \vdots \\ \mathcal{F}_{n-1}(x_{1,0}, x_{2,0}, x_{3,0}, x_{4,0}, ..., x_{n-1,0}, x_{n,0}) &= 0, \\ \mathcal{F}_n(x_{1,0}, x_{2,0}, x_{3,0}, x_{4,0}, ..., x_{n-1,0}, x_{n,0}) &= 0, \end{aligned}$$

where $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$,$\mathcal{F}_4$,..., $\mathcal{F}_n$ are defined in Theorem (1).
Then for every $\left(x^*_{1,0}, x^*_{2,0}, x^*_{3,0}, x^*_{4,0}, ..., x^*_{n-1,0}, x^*_{n,0}\right)$ solution of the system

$$\mathcal{F}_k\left(x_{1,0}, x_{2,0}, x_{3,0}, x_{4,0}, ..., x_{n-1,0}, x_{n,0}\right) = 0, \tag{12}$$

$k = \overline{1, n}$, provide periodic orbits of the system (4) with $\varepsilon \neq 0$ being sufficiently small if they are simple, i.e. if

$$\det\left(\frac{\partial\left(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4, ..., \mathcal{F}_{n-1}, \mathcal{F}_n\right)}{\partial\left(x_{1,0}, x_{2,0}, x_{3,0}, x_{4,0}, ..., x_{n-1,0}, x_{n,0}\right)}\right)\Big|_{\left(x^*_{1,0}, x^*_{2,0}, x^*_{3,0}, x^*_{4,0}, ..., x^*_{n-1,0}, x^*_{n,0}\right)} \neq 0.$$

For every root $(x^*_{1,0}, x^*_{2,0}, x^*_{3,0}, x^*_{4,0}, ..., x^*_{n-1,0}, x^*_{n,0})$ of system (12), there exists a 2π-periodic solution

$$\left(x_1(t,\varepsilon), x_2(t,\varepsilon), x_3(t,\varepsilon), x_4(t,\varepsilon), ..., x_{n-1}(t,\varepsilon), x_n(t,\varepsilon)\right)^t,$$

of the differential system (4) with $\varepsilon \neq 0$ sufficiently small, which tends to the periodic solution given in the statement of Theorem (1) of the differential system

$$\begin{aligned}
\dot{x}_1 &= -x_2 + f_1(t) \\
\dot{x}_2 &= x_1 + f_2(t) \\
\dot{x}_3 &= -x_4 + f_3(t) \\
\dot{x}_4 &= x_3 + f_4(t) \\
&\vdots \quad \vdots \quad \vdots \\
\dot{x}_{n-1} &= -x_n + f_{n-1}(t) \\
\dot{x}_n &= x_{n-1} + f_n(t),
\end{aligned}$$

when $\varepsilon \to 0$. The demonstration of Theorem (1) has been proved.

5. Applications

In this section, we present two examples to illustrate the results obtained in Theorem (1). The first system is in dimension 4, and the second one is in dimension 10.

Example 6. *This example illustrates the results obtained in Theorem (1), we consider the differential system (4) with $n = 4$ and*

$$F_0(t,x) = \begin{pmatrix} -x_2 + \sin(t) \\ x_1 + \cos(t) \\ -x_4 - \sin(t) \\ x_3 - \cos(t) \end{pmatrix}, \quad F_1(t,x) = \begin{pmatrix} 2x_1 - x_2 + x_1x_2 \\ 3x_1 + x_2 - x_1x_2 \\ x_3 - 2x_4 + x_3x_4 \\ x_3 + 2x_4 - x_3x_4 \end{pmatrix}.$$

The conditions (5) can be easily verified

$$\begin{aligned}
&2\int_0^{2\pi} \cos(s)\sin(s)ds = 0, && -2\int_0^{2\pi} \cos(s)\sin(s)ds = 0, \\
&\int_0^{2\pi} \left(-\sin^2(s) + \cos^2(s)\right) ds = 0, && \int_0^{2\pi} \left(\sin^2(s) - \cos^2(s)\right) ds = 0.
\end{aligned}$$

By calculating the functions $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ and $\mathcal{F}_4$, we obtain the following results

$$\begin{aligned}
\mathcal{F}_1(x_{1,0}, x_{2,0}, x_{3,0}, x_{4,0}) &= \frac{3}{2}x_{1,0} - 2x_{2,0} + \frac{1}{2}, \\
\mathcal{F}_2(x_{1,0}, x_{2,0}, x_{3,0}, x_{4,0}) &= 2x_{1,0} + \frac{3}{2}x_{2,0} + \frac{1}{2}, \\
\mathcal{F}_3(x_{1,0}, x_{2,0}, x_{3,0}, x_{4,0}) &= \frac{3}{2}x_{3,0} - \frac{3}{2}x_{4,0} - 1, \\
\mathcal{F}_4(x_{1,0}, x_{2,0}, x_{3,0}, x_{4,0}) &= \frac{3}{2}x_{3,0} + \frac{3}{2}x_{4,0} - 1.
\end{aligned}$$

The system $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3 = \mathcal{F}_4 = 0$, has a unique real solution, which can be given by $\left(-\frac{7}{25}, \frac{1}{25}, \frac{2}{3}, 0\right)$. The eigenvalues of the Jacobian matrix are $\left(\frac{3}{2} - 2i, \frac{3}{2} + 2i, \frac{3}{2} - \frac{3}{2}i, \frac{3}{2} + \frac{3}{2}i\right)^t$, which have four positive real parts. Since

$$\det\left(\frac{\partial(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4)}{\partial(x_{1,0}, x_{2,0}, x_{3,0}, x_{4,0})}\right)\Big|_{\left(-\frac{7}{25}, \frac{1}{25}, \frac{2}{3}, 0\right)} = \frac{225}{8} \neq 0,$$

then the differential system (4) *with* $n = 4$ *has an unstable periodic solution,* $\begin{pmatrix} x_1(t,\varepsilon) \\ x_2(t,\varepsilon) \\ x_3(t,\varepsilon) \\ x_4(t,\varepsilon) \end{pmatrix}$ *tending to the unstable periodic solution*

$$
\begin{aligned}
x_1(t) &= -\tfrac{7}{25}\cos(t) - \tfrac{1}{25}\sin(t),\\
x_2(t) &= \tfrac{1}{25}\cos(t) + \tfrac{18}{25}\sin(t),\\
x_3(t) &= \tfrac{2}{3}\cos(t),\\
x_4(t) &= -\tfrac{1}{3}\sin(t),
\end{aligned}
$$

of the differential system

$$
\begin{aligned}
\dot{x}_1 &= -x_2 + \sin(t),\\
\dot{x}_2 &= x_1 + \cos(t),\\
\dot{x}_3 &= -x_4 - \sin(t),\\
\dot{x}_4 &= x_3 - \cos(t),
\end{aligned}
$$

when $\varepsilon \to 0$.

Example 7. *Consider the differential system* (4) *with* $n = 10$ *and*

$$
F_0(t,x) = \begin{pmatrix} -x_2 + \sin(t)\cos(t) \\ x_1 + \sin^2(t) \\ -x_4 + \sin(t)\cos(t) \\ x_3 + \cos^2(t) \\ -x_6 + \sin(t)\cos(t) \\ x_5 + \sin^2(t) \\ -x_8 + \sin(t)\cos(t) \\ x_7 + \sin^2(t) \\ -x_9 + \sin(t)\cos(t) \\ x_{10} + \cos^2(t) \end{pmatrix}, \quad F_1(t,x) = \begin{pmatrix} 2x_1x_2 + 2x_1 - 2x_2 \\ -2x_1x_2 + 2x_1 + 2x_2 \\ -x_3x_4 - x_3 + x_4 \\ x_3x_4 - x_3 - x_4 \\ x_5^2 - x_3 - x_4 - 3x_5 \\ x_6x_7 + x_6 - x_7 \\ -x_6x_7 + x_6 + x_7 \\ -x_8^2 + x_6 + x_7 + 3x_8 \\ x_9 + 4 \\ x_{10} + 1 \end{pmatrix}.
$$

Conditions (5) *can be readily verified*

$$
\begin{aligned}
&\int_0^{2\pi} \left(\sin(s)\cos^2(s) + \sin^3(s)\right) ds = 0, && \int_0^{2\pi} \left(-\sin^2(s)\cos(s) + \cos(s)\sin^2(s)\right) ds = 0,\\
&\int_0^{2\pi} \left(-\sin^2(s)\cos(s) + \cos(s)\sin^2(s)\right) ds = 0, && \int_0^{2\pi} \left(\sin(s)\cos^2(s) + \sin^3(s)\right) ds = 0,\\
&\int_0^{2\pi} \left(\sin(s)\cos^2(s) + \sin(s)\cos^2(s)\right) ds = 0, && \int_0^{2\pi} \left(-\sin^2(s)\cos(s) + \cos(s)\sin^2(s)\right) ds = 0,\\
&\int_0^{2\pi} \left(-\sin^2(s)\cos(s) + \cos^3(s)\right) ds = 0, && \int_0^{2\pi} \left(\sin(s)\cos^2(s) + \sin(s)\cos^2(s)\right) ds = 0,\\
&\int_0^{2\pi} \left(\sin(s)\cos^2(s) + \sin^3(s)\right) ds = 0, && \int_0^{2\pi} \left(-\sin^2(s)\cos(s) + \cos^3(s)\right) ds = 0\,.
\end{aligned}
$$

Computing the functions $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4, \mathcal{F}_5, \mathcal{F}_6, \mathcal{F}_7, \mathcal{F}_8, \mathcal{F}_9$ and $\mathcal{F}_{10}$ we find

$$\begin{aligned}
\mathcal{F}_1\left(x_{1,0},x_{2,0},x_{3,0},x_{4,0},x_{5,0},x_{6,0},x_{7,0},x_{8,0},x_{9,0},x_{10,0}\right) &= \frac{5}{2}x_{1,0}-\frac{5}{2}x_{2,0}+\frac{5}{2},\\
\mathcal{F}_2\left(x_{1,0},x_{2,0},x_{3,0},x_{4,0},x_{5,0},x_{6,0},x_{7,0},x_{8,0},x_{9,0},x_{10,0}\right) &= \frac{5}{2}x_{1,0}+\frac{5}{2}x_{2,0}+\frac{5}{2},\\
\mathcal{F}_3\left(x_{1,0},x_{2,0},x_{3,0},x_{4,0},x_{5,0},x_{6,0},x_{7,0},x_{8,0},x_{9,0},x_{10,0}\right) &= -\frac{7}{6}x_{3,0}+\frac{4}{3}x_{4,0}-\frac{7}{9},\\
\mathcal{F}_4\left(x_{1,0},x_{2,0},x_{3,0},x_{4,0},x_{5,0},x_{6,0},x_{7,0},x_{8,0},x_{9,0},x_{10,0}\right) &= -\frac{7}{6}x_{3,0}-\frac{4}{3}x_{4,0}-\frac{7}{9},\\
\mathcal{F}_5\left(x_{1,0},x_{2,0},x_{3,0},x_{4,0},x_{5,0},x_{6,0},x_{7,0},x_{8,0},x_{9,0},x_{10,0}\right) &= -\frac{1}{2}x_{3,0}-\frac{1}{2}x_{4,0}-\frac{15}{8}x_{5,0}\\
&- \frac{1}{8}x_{7,0}+\frac{1}{2}x_{8,0}-\frac{7}{3},\\
\mathcal{F}_6\left(x_{1,0},x_{2,0},x_{3,0},x_{4,0},x_{5,0},x_{6,0},x_{7,0},x_{8,0},x_{9,0},x_{10,0}\right) &= \frac{1}{2}x_{3,0}-\frac{1}{2}x_{4,0}-\frac{13}{8}x_{6,0}\\
&- \frac{1}{2}x_{7,0}+\frac{1}{8}x_{8,0}-\frac{1}{6},\\
\mathcal{F}_7\left(x_{1,0},x_{2,0},x_{3,0},x_{4,0},x_{5,0},x_{6,0},x_{7,0},x_{8,0},x_{9,0},x_{10,0}\right) &= \frac{1}{2}x_{5,0}+\frac{7}{8}x_{6,0}+2x_{7,0}\\
&- \frac{3}{8}x_{8,0}+\frac{5}{2},\\
\mathcal{F}_8\left(x_{1,0},x_{2,0},x_{3,0},x_{4,0},x_{5,0},x_{6,0},x_{7,0},x_{8,0},x_{9,0},x_{10,0}\right) &= -\frac{5}{8}x_{5,0}+\frac{1}{2}x_{6,0}+\frac{5}{8}x_{7,0}+2x_{8,0},\\
\mathcal{F}_9\left(x_{1,0},x_{2,0},x_{3,0},x_{4,0},x_{5,0},x_{6,0},x_{7,0},x_{8,0},x_{9,0},x_{10,0}\right) &= x_{9,0}+\frac{2}{3},\\
\mathcal{F}_{10}\left(x_{1,0},x_{2,0},x_{3,0},x_{4,0},x_{5,0},x_{6,0},x_{7,0},x_{8,0},x_{9,0},x_{10,0}\right) &= x_{10,0}.
\end{aligned}$$

The stability of the periodic solutions corresponding to a simple zero of $\mathcal{F}_i$, $i = \overline{1,10}$ is determined by the eigenvalues of the Jacobian matrix.

The system $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3 = \mathcal{F}_4 = \mathcal{F}_5 = \mathcal{F}_6 = \mathcal{F}_7 = \mathcal{F}_8 = \mathcal{F}_9 = \mathcal{F}_{10} = 0$, has one solution given by $(-1,0,-\frac{2}{3},0,-1,0,-1,0,-\frac{2}{3},0)^t$, the eigenvalues of the Jacobian matrix are

$$\begin{bmatrix}
2.5-2.5i\\
2.5+2.5i\\
-1.25-1.244432043i\\
-1.25+1.244432043i\\
1.88782929+0.4102156479i\\
1.88782929-0.4102156479i\\
-1.512758664\\
-1.762899916\\
1\\
1
\end{bmatrix},$$

which have six positive real parts. Since

$$\det\left(\frac{\partial\left(\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3,\mathcal{F}_4,\mathcal{F}_5,\mathcal{F}_6,\mathcal{F}_7,\mathcal{F}_8,\mathcal{F}_9,\mathcal{F}_{10}\right)}{\partial\left(x_{1,0},x_{2,0},x_{3,0},x_{4,0},x_{5,0},x_{6,0},x_{7,0},x_{8,0},x_{9,0},x_{10,0}\right)}\right)\Big|_{\left(-1,0,-\frac{2}{3},0,-1,0,-1,0,-\frac{2}{3},0\right)}=\frac{111475}{288},$$

then the differential system (4) *with* $n=10$ *has an unstable periodic solution*

$$\left(x_1(t,\varepsilon),x_2(t,\varepsilon),x_3(t,\varepsilon),x_4(t,\varepsilon),x_5(t,\varepsilon),x_6(t,\varepsilon),x_7(t,\varepsilon),x_8(t,\varepsilon),x_9(t,\varepsilon),x_{10}(t,\varepsilon)\right)^t,$$

tending to the unstable periodic solution

$$\begin{aligned}
x_1(t)&=-\tfrac{1}{2}\left(\cos(2t)+1\right),\\
x_2(t)&=-\tfrac{1}{2}\sin(2t),\\
x_3(t)&=-\tfrac{1}{6}\cos(2t)-\tfrac{1}{2},\\
x_4(t)&=\tfrac{1}{6}\sin(2t),\\
x_5(t)&=-\tfrac{1}{2}(\cos(2t)+1),\\
x_6(t)&=-\tfrac{1}{2}\sin(2t),\\
x_7(t)&=-\tfrac{1}{2}(\cos(2t)+1),\\
x_8(t)&=-\tfrac{1}{2}\sin(2t),\\
x_9(t)&=-\tfrac{1}{6}\cos(2t)-\tfrac{1}{2},\\
x_{10}(t)&=\tfrac{1}{6}\sin(2t),
\end{aligned}$$

of the differential system

$$\begin{aligned}
\dot{x}_1&=-x_2+\sin(t)\cos(t),\\
\dot{x}_2&=\ \ x_1+\sin^2(\,t),\\
\dot{x}_3&=-x_4+\sin(t)\cos(t),\\
\dot{x}_4&=x_3+\cos^2(t),\\
\dot{x}_5&=-x_6+\sin(t)\cos(t),\\
\dot{x}_6&=x_5+\sin^2(t),\\
\dot{x}_7&=-x_8+\sin(t)\cos(t),\\
\dot{x}_8&=x_7+\sin^2(t),\\
\dot{x}_9&=-x_9+\sin(t)\cos(t),\\
\dot{x}_{10}&=x_{10}+\cos^2(t),
\end{aligned}$$

when $\varepsilon\to0$.

Conclusion

In this work, we analyzed periodic solutions of non-autonomous polynomial differential system of even dimension n by applying the another first-order averaging theory, and we established sufficient criteria for the existence of periodic solutions of system (4).

Conflicts of Interest. The author declares that there are no conflicts of interest regarding the publication of this paper.

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