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CHARACTERIZATIONS OF INVERSELY S-INVARIANT ELEMENTS IN SEMIGROUPS OF FULL TRANSFORMATIONS AS MAGNIFYING ELEMENTS

SOR.NARTH HIRANSRI, JITTISAK RAKBUD, THANAKORN PRINYASART*

Department of Mathematics, Faculty of Science, Silpakorn University, Thailand

*Corresponding author: prinyasart_t@silpakorn.edu

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Abstract. In this paper, for any subsemigroup S of the full transformation semigroup $T(X)$, we introduce the notions of inversely left-S-invariant and inversely right-S-invariant elements of S. Our main aims are to give characterizations for inversely left-S-invariant and inversely right-S-invariant elements of S to be left and right magnifying, respectively. Moreover, we apply our results to characterize left and right magnifying elements in several well-known subsemigroups of $T(X)$.

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1. Introduction

In this paper, for any function f and an element x of the domain of f, the image of x under f is denoted by $f(x)$. For any sets X, Y, Z and functions $f : X \to Y$, $g : Y \to Z$, the composition of g and f, denoted by gf, is a function from X into Z defined by $gf(x) = g(f(x))$. Recall that a semigroup is a nonempty set S equipped with an associative binary operation on S. For any set X, let $T(X)$ denote the set of all functions from X into itself, also called full transformations on X. It is evident that $T(X)$ equipped with the composition forms a semigroup, called the full transformation semigroup on X. For any subset Y of X, let $\overline{T}(X,Y) := \{\alpha \in T(X) \mid \alpha(Y) \subseteq Y\}$. It is clear that $\overline{T}(X,Y)$ is a subsemigroup of $T(X)$.

The concept of left and right magnifying elements in semigroups was introduced by E. S. Ljapin [6] in 1978. An element α in a semigroup S is called *left* (or *right*) *magnifying* if there exists a proper subset M of S such that $S = \alpha M$ (or $S = M\alpha$). In 1994, K. D. Magill, Jr. [7] characterized left and right magnifying elements in any subsemigroup of $T(X)$ containing id_X. They also applied their results to some specific transformation semigroups.

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In recent years, magnifying elements of various subsemigroups of $\overline{T}(X,Y)$ have been studied. For example, in [4], [2], [5] and [8], the left and right magnifying elements of the following semigroups were studied respectively:

(1) $\overline{T}(X,Y)$,
(2) $T(X,Y) := \{\alpha \in T(X) \mid \mathrm{ran}(\alpha) \subseteq Y\}$,
(3) $B_T(X,Y) := \{\alpha \in T(X) \mid \alpha|_Y : Y \to Y$ is a bijection$\}$,
(4) $Fix_T(X,Y) := \{\alpha \in T(X) \mid \alpha|_Y = \mathrm{id}_Y\}$.

When Y is a subspace of a vector space X, some researchers also studied the magnifying elements in semigroups of linear transformations on X. For example, in [3] and [1], the left and right magnifying elements of the following semigroups of linear transformations were studied respectively:

(1) $\overline{L}(X,Y) := \{\alpha \in \overline{T}(X,Y) \mid \alpha$ is a linear transformation$\}$,
(2) $L(X,Y) := \overline{L}(X,Y) \cap T(X,Y)$.

In Section 2, we introduce the notion of inversely left-S-invariant elements in a subsemigroup S of $T(X)$ and establish some necessary and sufficient conditions for such an element of S to be left magnifying. Moreover, we apply the results to characterize left magnifying elements of the following well-known semigroups:

(1) $\overline{T}(X,Y)$,
(2) $T(X,Y)$,
(3) $B_T(X,Y)$,
(4) $Fix_T(X,Y)$,
(5) $\overline{F}(X,Y) := \{\alpha \in T(X,Y) \mid \mathrm{ran}(\alpha) = \alpha(Y)\}$,
(6) $\overline{L}(X,Y)$,
(7) $L(X,Y)$,
(8) $B_L(X,Y) := \overline{L}(X,Y) \cap B_T(X,Y)$,
(9) $Fix_L(X,Y) := \overline{L}(X,Y) \cap Fix_T(X,Y)$,
(10) $\overline{G}(X,Y) := \overline{L}(X,Y) \cap \overline{F}(X,Y)$.

Analogous to Section 2, in Section 3, the notion of inversely right-S-invariant elements is introduced and studied.

2. Left Magnifying Elements in Semigroups of Full Transformations

In this section, we study left magnifying elements in subsemigroups of $\overline{T}(X,Y)$. We begin with a lemma that gives a necessary and sufficient conditions for an element of a subsemigroup of $T(X)$ to be left magnifying.

Lemma 2.1. *Let S be a subsemigroup of $T(X)$. Then the following statements hold:*

(1) *If α is a left magnifying element of S, then there exist distinct $\gamma_1, \gamma_2 \in S$ such that $\alpha\gamma_1 = \alpha\gamma_2$.*

(2) *If $\alpha \in S$ with $\alpha S = S$ and there exist distinct $\gamma_1, \gamma_2 \in S$ such that $\alpha\gamma_1 = \alpha\gamma_2$, then α is a left magnifying element.*

Proof. To show (1), let α be a left magnifying element of S. Then $S = \alpha M$ for some proper subset M of S. Let $\gamma_1 \in S \setminus M$. Since $\alpha\gamma_1 \in S = \alpha M$, it follows that $\alpha\gamma_1 = \alpha\gamma_2$ for some $\gamma_2 \in M$.

To prove (2), let $\alpha \in S$ such that $\alpha S = S$ and there are distinct $\gamma_1, \gamma_2 \in S$ such that $\alpha\gamma_1 = \alpha\gamma_2$. Choose $M = S \setminus \{\gamma_2\}$. Since $\alpha\gamma_2 = \alpha\gamma_1 \in \alpha M$, we have that $\alpha M = \alpha M \cup \{\alpha\gamma_2\} = \alpha S = S$. Therefore, α is a left magnifying element of S. □

For any α in a subsemigroup S of $T(X)$, by the regularity of $T(X)$, there exists $\gamma \in T(X)$ such that $\alpha\gamma\alpha = \alpha$. However, in this situation, such an element γ may not yield the inclusion $\gamma S \subseteq S$. For example, let $Y := \{-1, 0, 1\}$, and define $\alpha \in \overline{T}(\mathbb{Z}, Y)$ by $\alpha(x) := |x| - 1$ for all $x \in \mathbb{Z}$. If $\gamma \in T(\mathbb{Z})$ such that $\alpha\gamma\alpha = \alpha$, then $\gamma(1) \in \{-2, 2\}$, so $\gamma \notin \overline{T}(\mathbb{Z}, Y)$, which implies that $\gamma\overline{T}(\mathbb{Z}, Y) \nsubseteq \overline{T}(\mathbb{Z}, Y)$. This observation gives rise to the following definition.

Definition 1. Let S be a subsemigroup of $T(X)$ and $\alpha \in S$. We call α *inversely left-S-invariant* if there exists $\gamma \in T(X)$ such that $\alpha = \alpha\gamma\alpha$ and $\gamma S \subseteq S$.

The following remark follows directly from the definition:

Remark 1. Recall that for any semigroup S and $a \in S$, we call a a *regular element* of S if there exists $b \in S$ such that $aba = a$. This is equivalent to that there exists $c \in S$ such that $aca = a$ and $cac = c$. Such an element c is called an *inverse* of a. It is well-known that every element of $T(X)$ is regular. For any subsemigroup S of $T(X)$, the following statements clearly hold:

(1) α is inversely left-S-invariant if and only if there exists an inverse γ of α in $T(X)$ such that $\gamma S \subseteq S$.

(2) If α is a regular element of S, then α is inversely left-S-invariant.

We now proceed to state our main result, which provides a characterization of left magnifying elements.

Theorem 2.2. *Let S be a subsemigroup of $T(X)$, and let $\alpha \in S$ be inversely left-S-invariant. Then α is a left magnifying element of S if and only if the following two statements hold:*

(1) $\bigcup_{\beta \in S} \operatorname{ran}(\beta) \subseteq \operatorname{ran}(\alpha)$.

(2) *There exist distinct $\gamma_1, \gamma_2 \in S$ such that $\alpha\gamma_1 = \alpha\gamma_2$.*

Proof. Assume that α is a left magnifying element of S. By Lemma 2.1, we have that (2) holds. To show (1), let $\beta \in S$. Since $S = \alpha S$, we get $\beta = \alpha\gamma$ for some $\gamma \in S$. Therefore, $\operatorname{ran}(\beta) \subseteq \operatorname{ran}(\alpha)$. Hence, (1) holds.

Conversely, assume that (1) and (2) hold. By Lemma 2.1, it suffices to show that $\alpha S = S$. Let $\beta \in S$ be arbitrary. Since α is inversely left-S-invariant, $\alpha\gamma\alpha = \alpha$ for some $\gamma \in T(X)$ such that $\gamma S \subseteq S$. By (1), $\operatorname{ran}(\beta) \subseteq \operatorname{ran}(\alpha)$. Therefore, $\alpha\gamma\beta = \beta$. Since $\gamma\beta \in \gamma S \subseteq S$, we obtain that $\beta = \alpha(\gamma\beta) \in \alpha S$. Hence, $S = \alpha S$. $\square$

Next, we apply the main result to the semigroup S, where S is one of the following: $\overline{T}(X,Y)$, $T(X,Y)$, $B_T(X,Y)$, $Fix_T(X,Y)$, $\overline{F}(X,Y)$, $\overline{L}(X,Y)$, $L(X,Y)$, $B_L(X,Y)$, $Fix_L(X,Y)$, or $\overline{G}(X,Y)$. To do so, we first show that every left magnifying element in each of these semigroups is inversely left-S-invariant.

Lemma 2.3. *Let S be a subsemigroup of $\overline{T}(X,Y)$. If α is a left magnifying element of S, then $\beta(Y) \subseteq \alpha(Y)$ for all $\beta \in S$.*

Proof. Let α be a left magnifying element of S, and $\beta \in S$. Then $\beta = \alpha\gamma$ for some $\gamma \in S$. Since $\gamma(Y) \subseteq Y$, it follows that $\beta(Y) = \alpha(\gamma(Y)) \subseteq \alpha(Y)$. $\square$

Theorem 2.4. *Let S be $\overline{T}(X,Y)$, $T(X,Y)$, $B_T(X,Y)$, $Fix_T(X,Y)$, $\overline{F}(X,Y)$, $\overline{L}(X,Y)$, $L(X,Y)$, $B_L(X,Y)$, $Fix_L(X,Y)$ or $\overline{G}(X,Y)$. If α is a left magnifying element of S, then $\alpha(Y) = Y$.*

Proof. Let $y \in Y$. First, assume that S is $\overline{T}(X,Y)$, $T(X,Y)$, $B_T(X,Y)$, $Fix_T(X,Y)$ or $\overline{F}(X,Y)$. Define $\beta \in S$ by

$$\beta(x) := \begin{cases} x & \text{if } x \in Y, \\ y & \text{otherwise.} \end{cases}$$

Notice that $\beta(Y) = Y$. By Lemma 2.3, $Y = \beta(Y) \subseteq \alpha(Y)$.

Next, assume that S is $\overline{L}(X,Y)$, $L(X,Y)$, $B_L(X,Y)$, $Fix_L(X,Y)$ or $\overline{G}(X,Y)$. Let B be a basis for Y, and extend B to a basis $\tilde{B}$ for X. Let $\beta : X \to X$ be the linear transformation such that

$$\beta(x) := \begin{cases} x & \text{if } x \in B, \\ 0 & \text{if } x \in \tilde{B} \setminus B. \end{cases}$$

Then $\beta \in S$ and $\beta(Y) = Y$. By Lemma 2.3, $Y = \beta(Y) \subseteq \alpha(Y)$. $\square$

Theorem 2.5. *Let S be $\overline{T}(X,Y)$, $T(X,Y)$, $B_T(X,Y)$, $Fix_T(X,Y)$, $\overline{F}(X,Y)$, $\overline{L}(X,Y)$, $L(X,Y)$, $B_L(X,Y)$, $Fix_L(X,Y)$ or $\overline{G}(X,Y)$. For any $\alpha \in S$, if $\operatorname{ran}(\alpha) \cap Y = \alpha(Y)$, then α is regular in S.*

Proof. Let $\alpha \in S$ such that $\operatorname{ran}(\alpha) \cap Y = \alpha(Y)$. First, assume that S is $\overline{T}(X,Y)$, $T(X,Y)$, $B_T(X,Y)$, $Fix_T(X,Y)$ or $\overline{F}(X,Y)$. Since $Y \neq \varnothing$, there exists $y \in Y$. Since $\operatorname{ran}(\alpha) \cap Y = \alpha(Y)$, for each $b \in \operatorname{ran}(\alpha) \cap Y$, we can fix an element a_b in $Y \cap \alpha^{-1}(b)$. For each $b \in \operatorname{ran}(\alpha) \setminus Y$, we also fix an element a_b in $\alpha^{-1}(b)$. Define $\gamma : X \to X$ by

$$\gamma(x) := \begin{cases} a_x & \text{if } x \in \operatorname{ran}(\alpha), \\ a_{\alpha(y)} & \text{otherwise.} \end{cases}$$

One can show that $\gamma \in S$ and $\alpha\gamma\alpha = \alpha$.

Next, assume that S is $\overline{L}(X,Y)$, $L(X,Y)$, $B_L(X,Y)$, $Fix_L(X,Y)$ or $\overline{G}(X,Y)$. Let B be a basis for $\operatorname{ran}(\alpha) \cap Y$, and extend B to a basis for $\operatorname{ran}(\alpha)$ and Y, say B_α and B_Y, respectively. Then $B_\alpha \cup B_Y$ is a basis for $Y + \operatorname{ran}(\alpha)$. We extend $B_\alpha \cup B_Y$ to a basis $\tilde{B}$ for X. Since $\operatorname{ran}(\alpha) \cap Y = \alpha(Y)$, for each $b \in B$, we can fix an element a_b in $Y \cap \alpha^{-1}(b)$. For each $b \in B_\alpha \smallsetminus B$, we also fix an element a_b in $\alpha^{-1}(b)$. Let $\gamma : X \to X$ be the linear map such that

$$\gamma(x) := \begin{cases} a_x & \text{if } x \in B_\alpha, \\ 0 & \text{if } x \in \tilde{B} \smallsetminus B_\alpha. \end{cases}$$

One can show that $\gamma \in S$ and $\alpha\gamma\alpha = \alpha$. □

For any $\alpha \in T(X)$, if $\alpha(Y) = Y$, then $\operatorname{ran}(\alpha) \cap Y = \operatorname{ran}(\alpha) \cap \alpha(Y) = \alpha(Y)$. Then, by Theorem 2.4 and Theorem 2.5, we obtain the following corollary.

Corollary 2.6. *Let S be $\overline{T}(X,Y)$, $T(X,Y)$, $B_T(X,Y)$, $Fix_T(X,Y)$, $\overline{F}(X,Y)$, $\overline{L}(X,Y)$, $L(X,Y)$, $B_L(X,Y)$, $Fix_L(X,Y)$ or $\overline{G}(X,Y)$. If α is a left magnifying element of S, then α is regular, which also implies that α is inversely left-S-invariant.*

We are now ready to state the necessary and sufficient conditions for an element in the semigroups under consideration to be a left magnifying element.

Theorem 2.7. *Let S be $T(X,Y)$, $\overline{F}(X,Y)$, $L(X,Y)$ or $\overline{G}(X,Y)$. Then, for any $\alpha \in S$, α is a left magnifying element of S if and only if $\alpha(Y) = Y$, and $\alpha|_Y$ is not injective.*

Proof. Assume that α is a left magnifying element of S. By Theorem 2.4, we have that $\alpha(Y) = Y$. By Corollary 2.6, we have that α is inversely left-S-invariant. Since $\operatorname{ran}(\gamma) \subseteq Y$ for all $\gamma \in S$, by Theorem 2.2(2), we can conclude that $\alpha|_Y$ is not injective.

Coversely, assume that $\alpha(Y) = Y$ and $\alpha|_Y$ is not injective. Then $\operatorname{ran}(\alpha) \cap Y = \operatorname{ran}(\alpha) \cap \alpha(Y) = \alpha(Y)$. By Theorem 2.5, we have that α is inversely left-S-invariant. Notice that condition (1) in Theorem 2.2 holds since $\operatorname{ran}(\beta) \subseteq Y = \alpha(Y) \subseteq \operatorname{ran}(\alpha)$ for all $\beta \in S$. To show that condition (2) also holds, notice that since $\alpha|_Y$ is not injective, there exist distinct $y_1, y_2 \in Y$ such that $\alpha(y_1) = \alpha(y_2)$. For $i \in \{1, 2\}$, if S is $T(X,Y)$ or $\overline{F}(X,Y)$, define $\gamma_i \in S$ by $\gamma_i(x) := y_i$ for all $x \in X$. If S is $L(X,Y)$ or $\overline{G}(X,Y)$, let B be a basis for X, and for $i \in \{1, 2\}$, let $\gamma_i \in S$ such that $\gamma_i(x) = y_i$ for all $x \in B$. Thus, $\alpha\gamma_1 = \alpha\gamma_2$ but $\gamma_1 \neq \gamma_2$. By Theorem 2.2, α is a left magnifying element of S. □

Theorem 2.8. *Let S be $\overline{T}(X,Y)$, $B_T(X,Y)$, $Fix_T(X,Y)$, $\overline{L}(X,Y)$, $B_L(X,Y)$ or $Fix_L(X,Y)$. Then, for any $\alpha \in S$, α is a left magnifying element of S if and only if $\operatorname{ran}(\alpha) = X$, $\alpha(Y) = Y$, and α is not injective.*

Proof. Assume that α is a left magnifying element of S. By Theorem 2.4, we have that $\alpha(Y) = Y$. By Corollary 2.6, we have that α is inversely left-S-invariant. Since $\mathrm{id}_X \in S$, by Theorem 2.2(1), $X = \mathrm{ran}(\mathrm{id}_X) \subseteq \mathrm{ran}(\alpha)$. By Theorem 2.2(2), we also have that α is not injective.

Conversely, assume that $\mathrm{ran}(\alpha) = X$, $\alpha(Y) = Y$, and α is not injective. Then $\mathrm{ran}(\alpha) \cap Y = \mathrm{ran}(\alpha) \cap \alpha(Y) = \alpha(Y)$. By Theorem 2.5, we have that α is inversely left-S-invariant. Notice that condition (1) in Theorem 2.2 clearly holds. To show that condition (2) also holds, notice that since α is not injective, there exist distinct $x_1, x_2 \in X$ such that $\alpha(x_1) = \alpha(x_2)$. First, assume that S is $\overline{T}(X,Y)$, $B_T(X,Y)$ or $Fix_T(X,Y)$. If $Y \subsetneq X$, then for $i \in \{1,2\}$, we define $\gamma_i \in S$ by

$$\gamma_i(x) := \begin{cases} x_i & \text{if } x \in X \setminus Y, \\ x & \text{if } x \in Y. \end{cases}$$

If $Y = X$, then $S = \overline{T}(X,Y)$ since α is not injective, so for $i \in \{1,2\}$, we define $\gamma_i(x) := x_i$ for all $x \in X$. Then $\alpha\gamma_1 = \alpha\gamma_2$ but $\gamma_1 \neq \gamma_2$. By Theorem 2.2, α is a left magnifying element of S. Next, assume that S is $\overline{L}(X,Y)$, $B_L(X,Y)$ or $Fix_L(X,Y)$. Let B be a basis for Y, and extend B to a basis $\tilde{B}$ for X. If $B \subsetneq \tilde{B}$, then for $i \in \{1,2\}$, let $\gamma_i \in S$ such that

$$\gamma_i(x) := \begin{cases} x_i & \text{if } x \in \tilde{B} \setminus B, \\ x & \text{if } x \in B. \end{cases}$$

If $B = \tilde{B}$, then $S = \overline{L}(X,Y)$ since α is not injective, so for $i \in \{1,2\}$, we let $\gamma_i \in S$ such that $\gamma_i(x) := x_i$ for all $x \in \tilde{B}$. Then $\alpha\gamma_1 = \alpha\gamma_2$ but $\gamma_1 \neq \gamma_2$. By Theorem 2.2, α is a left magnifying element of S. □

3. Right Magnifying Elements in Semigroups of Full Transformations

In this section, we investigate right magnifying elements in subsemigroups of $\overline{T}(X,Y)$. We begin with a lemma that provides a necessary and sufficient conditions for an element of a subsemigroup of $T(X)$ to be a right magnifying element.

Lemma 3.1. *Let S be a subsemigroup of $T(X)$. Then the following statements hold:*

(1) *If α is a right magnifying element of S, then there exist distinct $\gamma_1, \gamma_2 \in S$ such that $\gamma_1\alpha = \gamma_2\alpha$.*

(2) *If $\alpha \in S$ with $S\alpha = S$ and there exist distinct $\gamma_1, \gamma_2 \in S$ such that $\gamma_1\alpha = \gamma_2\alpha$, then α is a right magnifying element.*

Proof. To show (1), let α be a right magnifying element of S. Then $S = M\alpha$ for some proper subset M of S. Let $\gamma_1 \in S \setminus M$. Since $\gamma_1\alpha \in S = M\alpha$, it follows that $\gamma_1\alpha = \gamma_2\alpha$ for some $\gamma_2 \in M$.

To prove (2), let $\alpha \in S$ such that $S\alpha = S$ and there are distinct $\gamma_1, \gamma_2 \in S$ such that $\gamma_1\alpha = \gamma_2\alpha$. Choose $M = S \setminus \{\gamma_2\}$. Since $\gamma_2\alpha = \gamma_1\alpha \in M\alpha$, we have that $M\alpha = M\alpha \cup \{\gamma_2\alpha\} = S\alpha = S$. Therefore, α is a right magnifying element of S. □

Similar to the previous section, we now define the concept of inversely right-S-invariant as follows.

Definition 2. Let S be a subsemigroup of $T(X)$ and $\alpha \in S$. We call α *inversely right-S-invariant* if there exists $\gamma \in T(X)$ such that $\alpha = \alpha\gamma\alpha$ and $S\gamma \subseteq S$.

The following remark follows directly from the definition:

Remark 2. For any subsemigroup S of $T(X)$, the following statements clearly hold:

(1) α is inversely right-S-invariant if and only if there exists an inverse γ of α in $T(X)$ such that $S\gamma \subseteq S$.

(2) If α is a regular element of S, then α is inversely right-S-invariant.

We now proceed to state our main result, which provides a characterization of right magnifying elements.

Theorem 3.2. *Let S be a subsemigroup of $T(X)$, and let $\alpha \in S$ be inversely right-S-invariant. Then α is a right magnifying element of S if and only if the following two statements hold:*

(1) $\alpha^{-1}(\alpha(x)) \subseteq \bigcap_{\beta \in S} \beta^{-1}(\beta(x))$ *for all* $x \in X$.

(2) *There exist distinct* $\gamma_1, \gamma_2 \in S$ *such that* $\gamma_1\alpha = \gamma_2\alpha$.

Proof. Assume that α is a right magnifying element of S. By Lemma 3.1, we have that (2) holds. To show (1), let $x \in X$, $\beta \in S$ and $a \in \alpha^{-1}(\alpha(x))$. Since $S = S\alpha$, we get $\beta = \gamma\alpha$ for some $\gamma \in S$. Since $\alpha(a) = \alpha(x)$, we have that $\beta(a) = \gamma\alpha(a) = \gamma\alpha(x) = \beta(x)$, which implies that $a \in \beta^{-1}(\beta(x))$. Hence, (1) holds.

Conversely, assume that (1) and (2) hold. By Lemma 3.1, it suffices to show that $S\alpha = S$. Let $\beta \in S$ be arbitrary. Since α is inversely right-S-invariant, $\alpha\gamma\alpha = \alpha$ for some $\gamma \in T(X)$ such that $S\gamma \subseteq S$. By (1), we have that $\beta\gamma\alpha = \beta$. Since $\beta\gamma \in S\gamma \subseteq S$, we obtain that $\beta = (\beta\gamma)\alpha \in S\alpha$. Hence, $S = S\alpha$. □

Next, we apply the main result to the semigroup S, where S is one of the following: $\overline{T}(X,Y)$, $T(X,Y)$, $B_T(X,Y)$, $Fix_T(X,Y)$, $\overline{F}(X,Y)$, $\overline{L}(X,Y)$, $L(X,Y)$, $B_L(X,Y)$, $Fix_L(X,Y)$, or $\overline{G}(X,Y)$. To do so, we first show that every right magnifying element in each of these semigroups is inversely right-S-invariant.

Lemma 3.3. *Let S be a subsemigroup of $\overline{T}(X,Y)$. If α is a right magnifying element of S, then $\alpha^{-1}(Y) \subseteq \beta^{-1}(Y)$ for all $\beta \in S$.*

Proof. Let α be a right magnifying element of S and $\beta \in S$. Then $\beta = \gamma\alpha$ for some $\gamma \in S$. Thus, $\alpha^{-1}(Y) \subseteq \alpha^{-1}\left(\gamma^{-1}(Y)\right) = \beta^{-1}(Y)$. □

Theorem 3.4. *Let S be $\overline{T}(X,Y)$, $B_T(X,Y)$, $Fix_T(X,Y)$, $\overline{F}(X,Y)$, $\overline{L}(X,Y)$, $B_L(X,Y)$, $Fix_L(X,Y)$ or $\overline{G}(X,Y)$. If α is a right magnifying element of S, then $\mathrm{ran}(\alpha) \cap Y = \alpha(Y)$.*

Proof. This theorem is trivial when S is $B_T(X,Y)$, $Fix_T(X,Y)$, $\overline{F}(X,Y)$, $B_L(X,Y)$, $Fix_L(X,Y)$ or $\overline{G}(X,Y)$ since for every $\alpha \in S$, $\operatorname{ran}(\alpha) \cap Y = \alpha(Y)$. Now, assume that S is $\overline{T}(X,Y)$ or $\overline{L}(X,Y)$, and let α be a right magnifying element of S. Since $\operatorname{id}_X \in S$, by Lemma 3.3, we have that $\alpha^{-1}(Y) \subseteq \operatorname{id}_X^{-1}(Y) = Y$. Thus, $\operatorname{ran}(\alpha) \cap Y = \alpha(\alpha^{-1}(Y)) \subseteq \alpha(Y)$. Moreover, since $\alpha(Y) \subseteq Y$, we can conclude that $\operatorname{ran}(\alpha) \cap Y = \alpha(Y)$. $\square$

Theorem 3.5. *Let S be $\overline{T}(X,Y)$, $T(X,Y)$, $B_T(X,Y)$, $Fix_T(X,Y)$, $\overline{F}(X,Y)$, $\overline{L}(X,Y)$, $L(X,Y)$, $B_L(X,Y)$, $Fix_L(X,Y)$ or $\overline{G}(X,Y)$. If α is a right magnifying element of S, then α is inversely right-S-invariant.*

Proof. By Theorem 2.5 and Theorem 3.4, this theorem holds when S is $\overline{T}(X,Y)$, $B_T(X,Y)$, $Fix_T(X,Y)$, $\overline{F}(X,Y)$, $\overline{L}(X,Y)$, $B_L(X,Y)$, $Fix_L(X,Y)$ or $\overline{G}(X,Y)$. Now, let α be a right magnifying element of S, and assume that S is $T(X,Y)$. Since $Y \neq \varnothing$, there exists $y \in Y$. For each $b \in \operatorname{ran}(\alpha)$, fix an element a_b in $\alpha^{-1}(b)$. Define $\gamma : X \to X$ by

$$\gamma(x) := \begin{cases} a_x & \text{if } x \in \operatorname{ran}(\alpha), \\ y & \text{otherwise.} \end{cases}$$

It is clear that $\alpha\gamma\alpha = \alpha$. Since $T(X,Y)$ is a right ideal of $T(X)$, it is also obvious that $T(X,Y)\gamma \subseteq T(X,Y)$.

Next, assume that S is $L(X,Y)$. Let B be a basis for $\operatorname{ran}(\alpha)$, and extend B to a basis for X, say $\tilde{B}$. For each $b \in \operatorname{ran}(\alpha)$, fix an element a_b in $\alpha^{-1}(b)$. Let $\gamma : X \to X$ be the linear transformation such that

$$\gamma(x) = \begin{cases} a_x & \text{if } x \in B, \\ 0 & \text{if } x \in \tilde{B} \smallsetminus B. \end{cases}$$

It is clear that $\alpha\gamma\alpha = \alpha$. Since $L(X,Y)$ is a right ideal of $L(X)$, it is obvious that $L(X,Y)\gamma \subseteq L(X,Y)$. $\square$

For convenience, we also need the following two lemmas.

Lemma 3.6. *Let S be $T(X,Y)$, $B_T(X,Y)$, $Fix_T(X,Y)$, $\overline{F}(X,Y)$, $L(X,Y)$, $B_L(X,Y)$, $Fix_L(X,Y)$ or $\overline{G}(X,Y)$. If $\alpha \in S$ is injective, then α is inversely right-S-invariant.*

Proof. Let $\alpha \in S$ be injective. First, assume that S is $T(X,Y)$, $B_T(X,Y)$, $Fix_T(X,Y)$ or $\overline{F}(X,Y)$. Since $Y \neq \varnothing$, there exists $y \in Y$. For each $b \in \operatorname{ran}(\alpha)$, let a_b be the unique element in $\alpha^{-1}(b)$. Define $\gamma : X \to X$ by

$$\gamma(x) := \begin{cases} a_x & \text{if } x \in \operatorname{ran}(\alpha), \\ y & \text{otherwise.} \end{cases}$$

It is clear that $\alpha\gamma\alpha = \alpha$ and $S\gamma \subseteq S$.

Next, assume that S is $L(X,Y)$, $B_L(X,Y)$, $Fix_L(X,Y)$ or $\overline{G}(X,Y)$. Let B be a basis for $\operatorname{ran}(\alpha)$, and extend B to a basis for X, say $\tilde{B}$. For each $b \in \operatorname{ran}(\alpha)$, let a_b be the unique element in $\alpha^{-1}(b)$. Let

$\gamma : X \to X$ be the linear transformation such that

$$\gamma(x) = \begin{cases} a_x & \text{if } x \in B, \\ 0 & \text{if } x \in \tilde{B} \setminus B. \end{cases}$$

It is clear that $\alpha\gamma\alpha = \alpha$ and $S\gamma \subseteq S$. □

Lemma 3.7. *Let S be $\overline{T}(X,Y)$, $T(X,Y)$, $B_T(X,Y)$, $Fix_T(X,Y)$, $\overline{F}(X,Y)$, $\overline{L}(X,Y)$, $L(X,Y)$, $B_L(X,Y)$, $Fix_L(X,Y)$ or $\overline{G}(X,Y)$. For any $\alpha \in S$, if α is injective and $\operatorname{ran}(\alpha) \neq X$, then there are $\gamma_1, \gamma_2 \in S$ such that $\gamma_1\alpha = \gamma_2\alpha$ but $\gamma_1 \neq \gamma_2$.*

Proof. Let $\alpha \in S$ such that α is injective and $\operatorname{ran}(\alpha) \neq X$. First, assume that S is $\overline{T}(X,Y)$, $T(X,Y)$, $B_T(X,Y)$, $Fix_T(X,Y)$ or $\overline{F}(X,Y)$. Let $a \in X \setminus \operatorname{ran}(\alpha)$ and $b \in Y \cap \operatorname{ran}(\alpha)$. Let $\gamma_1 := \alpha \in S$ and define $\gamma_2 \in S$ by

$$\gamma_2(x) := \begin{cases} \alpha(b) & \text{if } x = a, \\ \alpha(x) & \text{otherwise.} \end{cases}$$

Then $\gamma_1\alpha = \gamma_2\alpha$. Since α is injective and $a \neq b$, we have that $\gamma_1(a) = \alpha(a) \neq \alpha(b) = \gamma_2(a)$.

Next, assume that S is $\overline{L}(X,Y)$, $L(X,Y)$, $B_L(X,Y)$, $Fix_L(X,Y)$ or $\overline{G}(X,Y)$. Let B be a basis for $Y \cap \operatorname{ran}(\alpha)$. Then extend B to bases for Y and $\operatorname{ran}(\alpha)$, say B_Y and B_α, respectively. Then $B_Y \cup B_\alpha$ is a basis for $Y + \operatorname{ran}(\alpha)$. We extend $B_Y \cup B_\alpha$ to a basis $\tilde{B}$ for X. Let $\gamma_1 := \alpha \in S$ and let $\gamma_2 \in S$ such that

$$\gamma_2(x) := \begin{cases} 0 & \text{if } x \in \tilde{B} \setminus B_\alpha, \\ \alpha(x) & \text{if } x \in B_\alpha. \end{cases}$$

Then $\gamma_1\alpha = \gamma_2\alpha$. Since α is injective and $B_\alpha \subsetneq \tilde{B}$, we have that $\gamma_1(\tilde{B} \setminus B_\alpha) = \alpha(\tilde{B} \setminus B_\alpha) \neq \{0\} = \gamma_2(\tilde{B} \setminus B_\alpha)$, so $\gamma_1 \neq \gamma_2$. □

We are now ready to state the necessary and sufficient conditions for an element in the semigroups under consideration to be a right magnifying element.

Theorem 3.8. *Let S be $T(X,Y)$, $B_T(X,Y)$, $Fix_T(X,Y)$, $\overline{F}(X,Y)$, $L(X,Y)$, $B_L(X,Y)$, $Fix_L(X,Y)$ or $\overline{G}(X,Y)$. Then, for any $\alpha \in S$, α is a right magnifying element of S if and only if α is injective and $\operatorname{ran}(\alpha) \neq X$.*

Proof. Assume that α is a right magnifying element of S. By Theorem 3.5, we have that α is inversely right-S-invariant. By Theorem 3.2(2), we have that $\operatorname{ran}(\alpha) \neq X$. Next, we will show that α is injective. First, assume that S is $B_T(X,Y)$, $Fix_T(X,Y)$, $B_L(X,Y)$ or $Fix_L(X,Y)$. Then $\operatorname{id}_X \in S$. By Theorem 3.2(1), $\alpha^{-1}(\alpha(x)) \subseteq \operatorname{id}_X^{-1}(\operatorname{id}_X(x)) = \{x\}$ for all $x \in X$. Thus, α is injective. Now, assume that S is $T(X,Y)$, $\overline{F}(X,Y)$, $L(X,Y)$ or $\overline{G}(X,Y)$. If $|Y| = 1$, then $|S| = 1$, so S does not contain any right magnifying element, which contradicts to the assumption that α is a right magnifying element of S. Thus, $|Y| > 1$. Then, for any distinct elements $x_1, x_2 \in X$, there is $\beta \in S$ such that $\beta(x_1)$ and $\beta(x_2)$ are distinct elements in Y. It follows that $\alpha^{-1}(\alpha(x)) = \{x\}$ for all $x \in X$ by Theorem 3.2(1). Hence, α is injective.

Conversely, assume that α is injective and $\operatorname{ran}(\alpha) \neq X$. By Lemma 3.6, α is inversely right-S-invariant. Since α is injective, condition (1) in Theorem 3.2 holds. By Lemma 3.7, condition (2) in Theorem 3.2 holds. Hence, α is a right magnifying element of S. □

Theorem 3.9. *Let S be $\overline{T}(X, Y)$ or $\overline{L}(X, Y)$. Then, for any $\alpha \in S$, α is a right magnifying element of S if and only if α is injective, $\operatorname{ran}(\alpha) \neq X$ and $\operatorname{ran}(\alpha) \cap Y = \alpha(Y)$.*

Proof. Assume that α is a right magnifying element of S. By Theorem 3.4, we have that $\operatorname{ran}(\alpha) \cap Y = \alpha(Y)$. By Theorem 3.5, α is inversely right-S-invariant. By Theorem 3.2(2), we have that $\operatorname{ran}(\alpha) \neq X$. Since $\mathrm{id}_X \in S$, we have that $\alpha^{-1}(\alpha(x)) \subseteq \mathrm{id}_X^{-1}(\mathrm{id}_X(x)) = \{x\}$ for all $x \in X$ by Theorem 3.2(1). Hence, α is injective.

Conversely, assume that α is injective, $\operatorname{ran}(\alpha) \neq X$ and $\operatorname{ran}(\alpha) \cap Y = \alpha(Y)$. By Theorem 3.5, α is inversely right-S-invariant. Since α is injective, condition (1) in Theorem 3.2 holds. By Lemma 3.7, condition (2) in Theorem 3.2 holds. Hence, α is a right magnifying element of S. □

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] S. Baupradist, T. Panityakul, R. Chinram, Left and Right Magnifying Elements in Semigroups of Linear Transformations With Restricted Range, Int. J. Math. Comput. Sci. 14 (2019), 1–8. https://future-in-tech.net/14.1/R-Baupradist.pdf.

[2] R. Chinram, S. Baupradist, Magnifying Elements in a Semigroup of Transformations with Restricted Range, Missouri J. Math. Sci. 30 (2018), 54–58. https://doi.org/10.35834/mjms/1534384954.

[3] R. Chinram, S. Baupradist, Magnifying Elements in Semigroups of Linear Transformations with Invariant Subspaces, J. Interdiscip. Math. 21 (2018), 1457–1462. https://doi.org/10.1080/09720502.2018.1507709.

[4] R. Chinram, S. Baupradist, Magnifying Elements of Semigroups of Transformations with Invariant Set, Asian-Eur. J. Math. 12 (2019), 1950056. https://doi.org/10.1142/s1793557119500566.

[5] S. Khirabdhi, S.N. Singh, Magnifiers in Semigroups of Transformations Whose Restrictions Belong to a Given Semigroup, Asian-Eur. J. Math. 16 (2023), 2350227. https://doi.org/10.1142/s1793557123502273.

[6] E.S. Ljapin, Semigroups, Translations of Mathematical Monographs, Vol.3, American Mathematical Society, Providence, 1978. https://doi.org/10.1090/mmono/003.

[7] K.D. Magill Jr., Magnifying Elements of Transformation Semigroups, Semigroup Forum 48 (1994), 119–126. http://eudml.org/doc/135302.

[8] M. Petapirak, T. Kaewnoi, R. Chinram, On Magnifying Elements in Transformation Semigroups with a Fixed Point Set, Thai J. Math. 18 (2020), 1041–1049. https://thaijmath2.in.cmu.ac.th/index.php/thaijmath/article/view/1054.