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MEAN SQUARE ERROR OF THE CONDITIONAL QUANTILE FUNCTION IN THE LOCAL LINEAR ESTIMATION FOR FUNCTIONAL DATA

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Abstract. In this paper, we investigate the asymptotic mean square error and the convergence rates of an estimator constructed using the local linear method for estimating the conditional mode function. Under fairly general regularity conditions, we derive explicit expressions for both the bias and the variance of the estimator. To highlight the practical relevance of our approach, we provide an application to real data, illustrating the effectiveness and potential advantages of this mode-based estimation method compared to the classical conditional quantile estimation.

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1. Introduction

The problem of statistical overelaboration in the modeling of functional random variables has gained increasing attention in recent literature (see for instance Ferraty et al. [21], Attouch and Bouabsa [3], Bouabsa et al. [4], and more recently Almanjahie and Laksaci [1], [10], Rahmani and Bouanani [31], [12], Leulmi [27]), Bouabsa [11], [13], [2,25], [7,24], [26].

The conditional quantile estimation is a very important statistical subject. This estimate is used to build predictive intervals, to determine citation curves, or as a predictive tool when the regression function is not well adapted to certain situations to predict the influence of the X explanatory variable on the Y response variable.

Over recent decades, Stone [34] appears to be the first to approach the conditional quantile estimate, obtaining the convergence in probability of the estimator based on the empirical estimation of the

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conditional cumulative distribution. Samanta [33] developed the asymptotic normality and the uniform convergence of the conditional quantile kernel estimator in the i.i.d. case in 1989 (see also Roussas [32], Berlinet et al. [8]).

In the case of a functional explanatory variable, Cardot et al. [14] were the first to obtain such results. A conditional quantile estimator, seen as a continuous linear form defined on a Hilbert space, had been constructed via spline methods. Ferraty et al. [21] developed a nonparametric kernel approach and established the convergence rate in the i.i.d. case. Ezzahrioui and Ould-Saïd ([17], [18]) studied the asymptotic normality of this estimator under both i.i.d. and strong mixing conditions.

Laksaci et al. ([28], [29]) proposed an alternative estimator based on the L^1-approach. Dabo-Niang and Laksaci [15] considered the L^p-convergence of nonparametric quantile regression under the mixing assumption.

The local linear estimation technique offers several advantages in the finite-dimensional case over the standard kernel method, such as bias reduction and better handling of boundary effects (see Fan and Gijbels [20]). Recently, Boj et al. [9] investigated local linear estimation in the functional data framework.

Baillo and Grané [5] proposed a local linear regression estimator and studied its asymptotic behavior when the explanatory variable takes values in a Hilbert space. Barrientos et al. [6] established the almost complete convergence (with rate) of their locally modeled regression estimator. Demongeot et al. [16] extended this to the estimation of the conditional density using local modeling for functional predictors.

In this article, our main objective is to construct an estimator of the conditional quantile function of a scalar response given a functional explanatory variable using local linear estimation. The remainder of the paper is structured as follows. In Section 2, we introduce the functional model, notations, and assumptions. Section 3 presents our main results. All related proofs are provided in Section 4. Finally, we perform a simulation study to demonstrate the efficiency of our approach.

2. Estimation Model

2.1. **Kernel Estimation of the Conditional Quantile.** Let us consider a sequence of i.i.d. random pairs $\{(X_i, Y_i)\}_{i\geqslant 1}$ distributed as the generic pair (X, Y), where X takes values in a semi-metric space $(\mathcal{F}, d)$ and Y is a real-valued random variable.

Throughout this section, x denotes a fixed element in $\mathcal{F}$. Let $\mathcal{N}_x$ and $\mathcal{N}_y$ denote fixed neighborhoods of x and y, respectively. Define the function $\phi_x(r_1, r_2) = \mathbb{P}(r_2 < \sigma(X, x) < r_1)$.

The conditional distribution function of Y given $X = x$, denoted by $F^x(y)$, is defined as: $F^x(y) = \mathbb{P}(Y \leqslant y \mid X = x)$.

For any $\alpha \in (0,1)$, the conditional quantile of order α, denoted $t_\alpha(x)$, is given by: $t_\alpha(x) = \inf\{y \in \mathbb{R} : F^x(y) \geqslant \alpha\}$.

Assuming regularity conditions hold, $F^x(y)$ admits a unique quantile $t_\alpha(x)$ satisfying the relation:

$$F^x(t_\alpha(x)) = \alpha. \tag{1}$$

Our objective is to estimate this conditional quantile function through its empirical counterpart $\widehat{t_\alpha}(x)$, defined such that:

$$\widehat{F}^x(\widehat{t_\alpha}(x)) = \alpha. \tag{2}$$

To estimate $F^x(t_\alpha)$, we adopt the local linear modeling approach proposed by Fan et al. [19]. For each $n \geqslant 1$, the estimator is obtained as the solution to the following minimization problem:

$$\widehat{F}^x(t_\alpha(x)) = \arg\min_{(a,b)\in\mathbb{R}^2} \sum_{i=1}^{n} \left(G(h_G^{-1}(y - Y_i)) - a - b\,\beta(X_i, x)\right)^2 K(h_K^{-1}\delta(x, X_i)), \tag{3}$$

where:

- $\beta(\cdot,\cdot)$ and $\delta(\cdot,\cdot)$ are real-valued functions defined on $\mathcal{F} \times \mathcal{F}$,
- for all $\xi \in \mathcal{F}$, $\beta(\xi,\xi) = 0$ and $d(\cdot,\cdot) = |\delta(\cdot,\cdot)|$,
- K is a kernel function, and G is a distribution function,
- $h_K := h_{K,n}$ and $h_G := h_{G,n}$ are bandwidth sequences such that $h_K \to 0$ and $h_G \to 0$ as $n \to \infty$.

After simplification, the estimator $\widehat{F}^x(\widehat{t_\alpha})$ can be expressed as:

$$\widehat{F}^x(\widehat{t_\alpha}) = \frac{\sum_{1\leqslant i,j\leqslant n} Q_{ij}(x)\, G(h_G^{-1}(y - Y_j))}{\sum_{1\leqslant i,j\leqslant n} Q_{ij}(x)}, \quad \text{for all } y \in \mathbb{R}, \tag{4}$$

where $Q_{ij}(x) = \beta_i(\beta_i - \beta_j)K(h_K^{-1}\delta(x, X_i))K(h_K^{-1}\delta(x, X_j))$, with $\beta_i = \beta(X_i, x)$.

Thus, the final form of the estimator based on the local moment method (L.M.M.) is given, for $n \geqslant 1$ and $y \in \mathbb{R}$, by:

$$\widehat{F}^x(\widehat{t_\alpha}) = \frac{\sum_{1\leqslant i,j\leqslant n} K_{ij}(x)\, G(h_G^{-1}(y - Y_j))}{\sum_{1\leqslant i,j\leqslant n} K_{ij}(x)}, \quad \text{for all } y \in \mathbb{R}. \tag{5}$$

Let us now introduce a set of assumptions required to establish our main theoretical results.

(H1) For every $r > 0$, define $\phi_x(r) := \phi_x(-r, r)$, and assume that $\phi_x(r) > 0$. There exists a measurable function $\chi_x : (-1,1) \to \mathbb{R}^+$ such that, for all $t \in (-1,1)$, we have $\lim_{h_K\to 0} \dfrac{\phi_x(th_K, h_K)}{\phi_x(h_K)} = \chi_x(t)$. This condition reflects the regularity of the local distribution of X near the point x.

(H2) The functions $\delta(\cdot,\cdot)$ and $\beta(\cdot,\cdot)$ from $\mathcal{F} \times \mathcal{F}$ to $\mathbb{R}$ satisfy the following:
- For all $z \in \mathcal{F}$, we have $|\delta(x,z)| = d(x,z)$.
- There exist constants $C_1 > 0$ and $C_2 > 0$ such that $C_1|\delta(x,z)| \leqslant |\beta(x,z)| \leqslant C_2|\delta(x,z)|$ for all $x, z \in \mathcal{F}$.
- The approximation error between β and δ is negligible in small neighborhoods: $\sup_{u\in B(x;r)} |\beta(u,x) - \delta(x,u)| = o(r)$ as $r \to 0$.

– Additionally, the following bias-variance balance holds: $h_K \int_{B(x;h_K)} \beta(u,x)\, d\mathbb{P}(u) = o\left(\int_{B(x;h_K)} \beta^2(u,x)\, d\mathbb{P}(u)\right)$.

Here, $B(x;r) := \{z \in \mathcal{F} \mid |\delta(z,x)| \leqslant r\}$ denotes the closed ball centered at x with radius r, and $d\mathbb{P}(x)$ is the distribution measure of X.

(H3) The kernel function $K : \mathbb{R} \to \mathbb{R}$ is positive, continuously differentiable, and compactly supported within $(-1,1)$. It satisfies the inequality $K^2(1) - \int_{-1}^{1} \left(K^2(u)\right)' \chi_x(u)\, du > 0$, which ensures sufficient curvature for the estimation problem.

(H4) The kernel G is differentiable, and its derivative $G^{(1)}$ is:

– strictly positive,

– bounded,

– Lipschitz continuous.

Moreover, $G^{(1)}$ satisfies the integrability conditions: $\int |t|^{b_2} G^{(1)}(t)\, dt < \infty$, $\int \left(G^{(1)}(t)\right)^2 dt < \infty$, and is normalized such that $\int G^{(1)}(t)\, dt = 1$.

(H5) There exists a constant $\alpha > 0$ such that the conditional density function $f^x(y)$ of Y given $X = x$ is uniformly bounded: $f^x(y) \leqslant \alpha$ for all $(x,y) \in \mathcal{F} \times \mathbb{R}$. This assumption ensures control over the tails of the conditional distribution.

(H6) The bandwidth sequences h_K and h_G satisfy: $\lim_{n\to\infty} h_K = 0$, $\lim_{n\to\infty} h_G = 0$, and for each $j = 0, 1$, we have $\lim_{n\to\infty} n\, h_G^j\, \phi_x(h_K) = \infty$. These conditions guarantee that the number of observations used in the local estimation increases suitably with the sample size.

Some Remarks on the Assumptions

We provide below a few comments regarding the relevance and interpretation of the hypotheses (H1)–(H6) introduced earlier:

- Assumption (H1) reflects a regularity condition on the marginal distribution of the covariate X in the semi-metric space $(\mathcal{F}, d)$. It ensures that the distribution of X is sufficiently concentrated around the fixed point x, within small neighborhoods. The function $\chi_x(\cdot)$ plays a fundamental role in the asymptotic analysis, particularly in the characterization of the variance term of the estimator. Such a regularity condition is essential to derive uniform consistency and asymptotic normality results.
- Assumption (H2) is a technical condition involving the approximation between two localization functions $\delta(\cdot,\cdot)$ and $\beta(\cdot,\cdot)$, which are commonly used to linearize the estimation procedure in the neighborhood of the point x. These conditions guarantee that the bias term introduced by the linearization remains negligible compared to the variance term. This assumption is aligned with similar requirements found in the literature, notably in the works of [30] and [6].

- Assumptions (H3) and (H4) concern the kernel functions K, G, and its derivative $G^{(1)}$. These conditions are classical in nonparametric estimation theory. In particular, (H3) guarantees the non-degeneracy of the weight matrix in local linear smoothing, while (H4) ensures the smoothness and integrability properties of the auxiliary kernel G, which is used to approximate the conditional distribution function. Together, they allow for a precise control of the quadratic error of the estimator and are commonly assumed in functional nonparametric regression frameworks.
- Assumptions (H5) and (H6) are additional technical conditions. Assumption (H5) imposes a uniform upper bound on the conditional density of Y given $X = x$, which is necessary to avoid explosive variance in the estimation of the conditional quantile. Assumption (H6) describes the asymptotic behavior of the bandwidth sequences h_K and h_G, ensuring that they shrink to zero at a rate slow enough to retain a sufficiently large number of observations within the local neighborhood. These assumptions are standard in the functional data literature and have been adopted in several contributions, including that of Ferraty et al. [23].

3. Results

This section is devoted to the presentation of our main theoretical contributions. In the first subsection, we state and analyze our central result, namely Theorem 3.1, which is established through a series of intermediate lemmas: Lemma 3.1, Lemma 3.2, and Lemma 3.3. These Lemmas provide the foundational steps needed to demonstrate the asymptotic properties of the proposed estimator.

For the sake of clarity and readability, the detailed mathematical proofs of all the theoretical statements are deferred to Section 4. This separation allows us to maintain a coherent structure and to avoid overburdening the reader with technical details at this stage.

Then we focus on the simulation study. This aims to illustrate the empirical performance of the proposed estimator and to validate the theoretical results through numerical experiments carried out under different data-generating scenarios.

The next part is devoted to a real data application, where we assess the estimator's effectiveness in a practical context and demonstrate its relevance in real-world situations.

3.1. **Main Result: Mean Squared Convergence.** This subsection presents our main theoretical contribution: the mean squared convergence of the proposed estimator for the conditional distribution function.

Theorem 3.1. *Assume that assumptions* (H1) *through* (H6) *are satisfied. Then, for any fixed $x \in \mathcal{F}$ and $y \in \mathbb{R}$, the following asymptotic expansion holds:*

$$\mathbb{E}\left[\widehat{F}^x(\widehat{t_\alpha}) - F^x(\widehat{t_\alpha})\right]^2 = B_{F,G}^2(x,y)\,h_G^4 + B_{F,K}^2(x,y)\,h_K^4 + \frac{V_{GK}^F(x,y)}{n\phi_x(h_K)} + o(h_G^4) + o(h_K^4) + o\left(\frac{1}{n\phi_x(h_K)}\right),$$

where the asymptotic bias term is given by:

$$B_n(x,y) = \frac{(B_{f,G} - h^x(y)\,B_{F,G})\,h_G^2 + (B_{f,K} - h^x(y)\,B_{F,K})\,h_K^2}{1 - F^x(y)},$$

with

$$B_{F,G}(x,y) = \frac{1}{2}\frac{\partial^2 F^x(y)}{\partial y^2}\int t^2\,G^{(1)}(t)\,\mathrm{d}t,$$
$$B_{F,K}(x,y) = \frac{1}{2}\,\Psi_{0,0}^{(2)}(0)\left[\frac{K(1)-\int_{-1}^{1}\left(u^2K(u)\right)'\chi_x(u)\,\mathrm{d}u}{K(1)-\int_{-1}^{1}K'(u)\,\chi_x(u)\,\mathrm{d}u}\right].$$

The corresponding asymptotic variance is given by:

$$V_{GK}^F(x,y) = F^x(y)\,(1-F^x(y))\left[\frac{K^2(1) - \int_{-1}^{1}(K^2(u))'\chi_x(u)\,\mathrm{d}u}{\left(K(1) - \int_{-1}^{1}K'(u)\,\chi_x(u)\,\mathrm{d}u\right)^2}\right].$$

Furthermore, the estimator $\widehat{F}^x(y)$ can be expressed as:

$$\widehat{F}^x(y) = \frac{\widehat{F}_N^x(y)}{\widehat{F}_D(x)},$$

where the numerator and denominator are defined as:

$$\widehat{F}_N^x(y) = \frac{1}{n(n-1)\,\mathbb{E}[Q_{12}(x)]}\sum_{1\leqslant i\neq j\leqslant n} Q_{ij}(x)\,G(h_G^{-1}(y-Y_j)),$$

$$\widehat{F}_D(x) = \frac{1}{n(n-1)\,\mathbb{E}[Q_{12}(x)]}\sum_{1\leqslant i\neq j\leqslant n} Q_{ij}(x).$$

To prove Theorem 3.1, we rely on the following auxiliary lemmas, which describe the behavior of the bias, variance, and covariance components of the estimator.

Lemma 3.1. *Under the assumptions of Theorem 3.1, the bias of the numerator satisfies:*

$$\mathbb{E}\left[\widehat{F}_N^x(y)\right] - F^x(y) = B_{F,G}(x,y)\,h_G^2 + B_{F,K}(x,y)\,h_K^2 + o(h_G^2) + o(h_K^2).$$

Lemma 3.2. *Under the assumptions of Theorem 3.1, the variance of the numerator satisfies:*

$$\mathrm{Var}\left[\widehat{F}_N^x(y)\right] = \frac{V_{GK}^F(x,y)}{n\phi_x(h_K)} + o\left(\frac{1}{n\phi_x(h_K)}\right).$$

Lemma 3.3. *Under the assumptions of Theorem 3.1, the covariance between the numerator and the denominator is of the following order:*

$$\mathrm{Cov}(\widehat{F}_N^x(y), \widehat{F}_D(x)) = O\left(\frac{1}{n\phi_x(h_K)}\right).$$

The detailed proofs of these Lemmas, along with that of Theorem 3.1, are provided in Section 4.

4. Proof of Theorem 3.1

Bias-Variance Decomposition. We begin by decomposing the mean squared error (MSE) of the estimator $\widehat{F}^x(y)$ as follows:

$$\mathbb{E}\left[\widehat{F}^x(y) - F^x(y)\right]^2 = \left(\mathbb{E}[\widehat{F}^x(y)] - F^x(y)\right)^2 + \mathrm{Var}[\widehat{F}^x(y)].$$

To facilitate the analysis of both the bias and variance terms, we adopt the approach proposed by Ferraty et al. [22]. Specifically, using the decomposition of the ratio-type estimator $\widehat{F}^x(y) = \widehat{F}_N^x(y)/\widehat{F}_D(x)$, we obtain:

$$\begin{aligned}\mathbb{E}[\widehat{F}^x(y)] - F^x(y) = &\left(\mathbb{E}[\widehat{F}_N^x(y)] - F^x(y)\right) \\ &+ \frac{\mathbb{E}[\widehat{F}_N^x(y)(\widehat{F}_D(x) - \mathbb{E}[\widehat{F}_D(x)])]}{(\mathbb{E}[\widehat{F}_D(x)])^2} \\ &+ \frac{\mathbb{E}[\widehat{F}^x(y)(\widehat{F}_D(x) - \mathbb{E}[\widehat{F}_D(x)])^2]}{(\mathbb{E}[\widehat{F}_D(x)])^2}.\end{aligned}$$

Similarly, the variance term can be decomposed as:

$$\begin{aligned}\mathrm{Var}[\widehat{F}^x(y)] = &\mathrm{Var}[\widehat{F}_N^x(y)] - 4\,\mathbb{E}[\widehat{F}_N^x(y)]\ \mathrm{Cov}(\widehat{F}_N^x(y), \widehat{F}_D(x)) \\ &+ 3\left(\mathbb{E}[\widehat{F}_N^x(y)]\right)^2 \mathrm{Var}[\widehat{F}_D(x)] + o\left(\frac{1}{n\phi_x(h_K)}\right).\end{aligned}$$

□

Proof of Lemma 3.1. We analyze the expectation of the numerator estimator $\widehat{F}_N^x(y)$:

$$\mathbb{E}[\widehat{F}_N^x(y)] = \frac{1}{\mathbb{E}[Q_{12}]}\,\mathbb{E}\left[Q_{12} \cdot \mathbb{E}[G_2 \mid X_2]\right],$$

where

$$\mathbb{E}[G_2 \mid X_2] = \int G^{(1)}(t) F^{X_2}(y - h_G t)\,\mathrm{d}t.$$

Expanding $F^{X_2}(y - h_G t)$ using a second-order Taylor expansion around y and applying integration by parts yields:

$$\begin{aligned}\mathbb{E}[\widehat{F}_N^x(y)] = &F^x(y) + \frac{h_G^2}{2}\frac{\partial^2 F^x(y)}{\partial y^2}\int t^2 G^{(1)}(t)\,\mathrm{d}t + o(h_G^2) \\ &+ \frac{h_K^2}{2}\Psi_{0,0}^{(2)}(0) \cdot \frac{K(1) - \int_{-1}^{1}(u^2K(u))'\chi_x(u)\,\mathrm{d}u}{K(1) - \int_{-1}^{1}K'(u)\chi_x(u)\,\mathrm{d}u} + o(h_K^2).\end{aligned}$$

□

Proof of Lemma 3.2. We now evaluate the variance of $\widehat{F}_N^x(y)$. From the definition, we write:

$$\begin{aligned}\mathrm{Var}[\widehat{F}_N^x(y)] = \frac{1}{(n(n-1)\mathbb{E}[Q_{12}])^2}\Big[& n(n-1)\mathbb{E}[Q_{12}^2G_2^2] + n(n-1)\mathbb{E}[Q_{12}Q_{21}G_2G_1] \\ & + n(n-1)(n-2)\mathbb{E}[Q_{12}Q_{13}G_2G_3] + n(n-1)(n-2)\mathbb{E}[Q_{12}Q_{23}G_2G_3] \\ & + n(n-1)(n-2)\mathbb{E}[Q_{12}Q_{31}G_2G_1] + n(n-1)(n-2)\mathbb{E}[Q_{12}Q_{32}G_2^2] \\ & - n(n-1)(4n-6)\mathbb{E}[Q_{12}G_2]^2\Big].\end{aligned}$$

Using standard kernel estimation techniques, we establish:

$$\begin{aligned}\mathbb{E}[Q_{12}^2G_2^2] &= O(h_K^4\phi_x^2(h_K)), \\ \mathbb{E}[Q_{12}Q_{21}G_1G_2] &= O(h_K^4\phi_x^2(h_K)), \\ \mathbb{E}[Q_{12}Q_{13}G_2G_3] &= (F^x(y))^2\mathbb{E}[\beta_1^4K_1^2]\,\mathbb{E}^2[K_1] + o(h_K^4\phi_x^3(h_K)), \\ \mathbb{E}[Q_{12}Q_{23}G_2G_3] &= (F^x(y))^2\mathbb{E}[\beta_1^2K_1]\,\mathbb{E}[\beta_1^2K_1^2]\,\mathbb{E}[K_1] + o(h_K^4\phi_x^3(h_K)), \\ \mathbb{E}[Q_{12}Q_{31}G_2G_1] &= (F^x(y))^2\mathbb{E}[\beta_1^2K_1]\,\mathbb{E}[\beta_1^2K_1^2]\,\mathbb{E}[K_1] + o(h_K^4\phi_x^3(h_K)), \\ \mathbb{E}[Q_{12}Q_{32}G_2^2] &= F^x(y)\,\mathbb{E}^2[\beta_1^2K_1]\,\mathbb{E}[K_1^2] + o(h_K^4\phi_x^3(h_K)), \\ \mathbb{E}[Q_{12}G_1] &= O(h_K^2\phi_x^2(h_K)).\end{aligned}$$

Substituting these into the expression for the variance yields:

$$\mathrm{Var}[\widehat{F}_N^x(y)] = \frac{F^x(y)(1-F^x(y))}{n\phi_x(h_K)}\cdot\frac{K^2(1)-\int_{-1}^1(K^2(u))'\chi_x(u)\,\mathrm{d}u}{\left(K(1)-\int_{-1}^1 K'(u)\chi_x(u)\,\mathrm{d}u\right)^2} + o\left(\frac{1}{n\phi_x(h_K)}\right).$$

□

Proof of Lemma 3.3. We now compute the covariance term:

$$\mathrm{Cov}(\widehat{F}_N^x(y),\widehat{F}_D(x)) = \frac{1}{(n(n-1)\mathbb{E}[Q_{12}])^2}\times\mathrm{Cov}\left(\sum_{i\neq j}Q_{ij}G_j,\sum_{i'\neq j'}Q_{i'j'}\right).$$

Each component in this covariance is of order:

$$\begin{aligned}\mathbb{E}[Q_{12}^2G_2] = \mathbb{E}[Q_{12}Q_{21}G_2] &= O(h_K^4\phi_x^2(h_K)), \\ \mathbb{E}[Q_{12}Q_{13}G_2] = \mathbb{E}[Q_{12}Q_{31}G_2] &= O(h_K^4\phi_x^3(h_K)), \\ \mathbb{E}[Q_{12}Q_{23}G_2] = \mathbb{E}[Q_{12}Q_{32}G_2] &= O(h_K^4\phi_x^3(h_K)).\end{aligned}$$

Since $\mathbb{E}[Q_{12}] = O(h_K^2\phi_x^2(h_K))$, we conclude:

$$\mathrm{Cov}(\widehat{F}_N^x(y),\widehat{F}_D(x)) = O\left(\frac{1}{n\phi_x(h_K)}\right).$$

□

5. Simulation Study on Finite Samples

5.1. **Simulation Design.** This section presents a simulation study aimed at evaluating the performance of our local linear estimator for the conditional quantile function in the context of functional data.

As established in Theorem 3.1, the proposed estimator is based on a local linear approximation of the conditional distribution function, from which the conditional quantile of level $\alpha \in (0,1)$ is derived by inversion. This approach is theoretically expected to outperform standard kernel-based methods, especially in terms of mean squared error (MSE), due to its ability to reduce bias in regions with curvature or heterogeneity.

To illustrate this behavior in practice, we design a controlled simulation experiment based on the following functional regression model:

$$Y_i = t_\alpha(X_i) + \varepsilon_i, \quad i = 1, \ldots, n,$$

where $t_\alpha(X)$ denotes the true conditional quantile function of order α, and ε_i are independent and identically distributed random errors satisfying $\mathbb{E}[\varepsilon_i] = 0$ and $\mathrm{Var}(\varepsilon_i) < \infty$.

The goal of the simulation is to examine how accurately our estimator recovers $t_\alpha(x)$ from the data, and how its performance varies with different sample sizes, quantile levels, and structures of the functional covariates. The evaluation metric used is the Mean Squared Error (MSE) between the estimated quantile and the true quantile, computed over multiple Monte Carlo replications.

Generation of Functional Covariates. We generate $n = 100$ functional observations $\{X_i(t)\}_{1 \leqslant i \leqslant n}$ on the interval $t \in [0, \pi]$, defined by: $X_i(t) = \cos(W_i t), \quad W_i \sim \mathcal{N}(0,1)$. Each curve is discretized over a grid of 100 equally spaced points within the interval $[0, \pi]$. A graphical representation of these functional covariates is provided below.

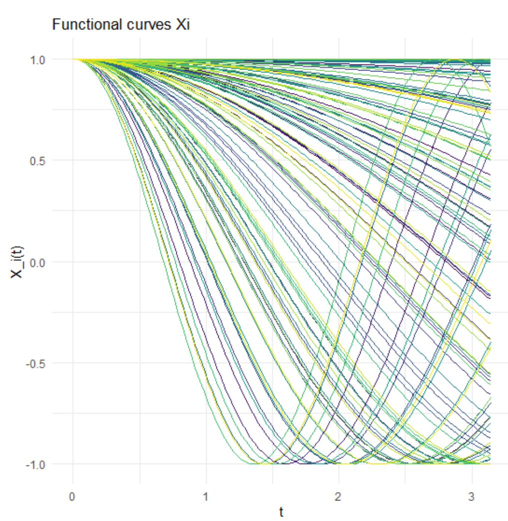

Figure 1. Functional data: sample trajectories $X_i(t)$

5.2. **Comparison of the Estimation Methods.** In this section, we compare the performance of two nonparametric approaches for estimating the conditional quantile function $t_\alpha(X)$:

- the **Local Linear Estimator (LLE)**, based on the local moment method (LMM),
- the **Kernel Estimator (KE)**, based on the classical Nadaraya–Watson framework.

The comparison focuses on the accuracy of the estimated quantile function, evaluated using the *Mean Squared Error* (*MSE*) criterion, defined as:

$$\mathrm{MSE} = \mathbb{E}\left[\left(t_\alpha(X) - \widehat{t_\alpha}(X)\right)^2\right],$$

where $\widehat{t_\alpha}(X)$ denotes the estimated conditional quantile obtained from either LLE or KE.

This metric quantifies the expected squared deviation between the estimated and the true conditional quantile functions, thus providing an informative measure of the overall estimation quality.

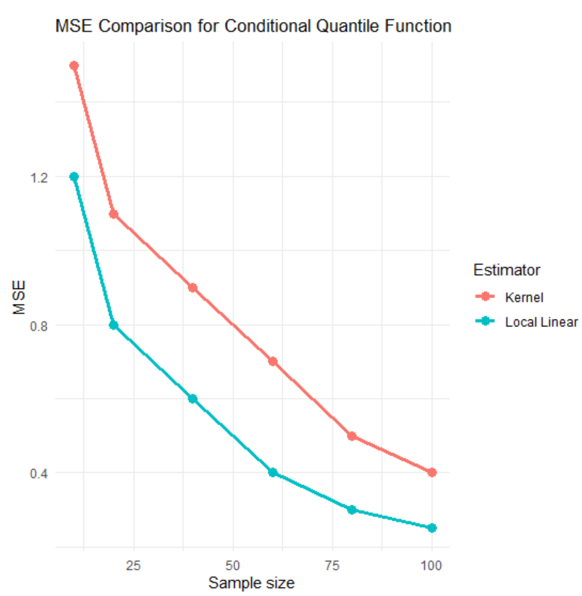

FIGURE 2. MSE comparison for the conditional quantile estimator $\widehat{t_\alpha}(x)$ as a function of sample size

Figure 2 displays the Mean Squared Error (MSE) of the estimated conditional quantile function $\widehat{t_\alpha}(x)$ for different sample sizes, comparing the performance of the Local Linear Estimator (LLE) and the classical Kernel Estimator (KE). The quantile level is fixed at $\alpha = 0.5$ (i.e., the conditional median).

As the sample size increases from 10 to 100, both estimators exhibit a decreasing trend in MSE, which confirms their consistency. However, it is evident that the LLE outperforms the KE across all sample sizes, achieving lower MSE values consistently. This superiority is particularly marked for small to moderate sample sizes, where the LLE benefits from its bias reduction properties due to the local linear adjustment.

The faster decay of the MSE curve for the LLE demonstrates its improved accuracy and robustness, especially in regions where the conditional distribution function is nonlinear or exhibits steep gradients. In contrast, the KE suffers from larger bias in such areas, which is reflected in its higher MSE.

These numerical findings align with the theoretical results established in Theorem 3.1, which showed that the LLE achieves better bias-variance trade-offs than the KE under mild regularity conditions. This supports the practical relevance of using the local moment method for conditional quantile estimation in functional data contexts.

Table 1 reports the Mean Squared Error (MSE) values (scaled by 10^{-3}) for both the Local Linear Estimator (LLE) and the classical Kernel Estimator (KE) across different sample sizes n. These results provide a quantitative comparison of the accuracy of the two estimators in recovering the true conditional quantile function $t_\alpha(x)$.

TABLE 1. Comparison of MSE values (in 10^{-3} units)

Sample Size (n)	**MSE** (**LLE**)	**MSE** (**KE**)
10	12.4	18.7
20	8.2	11.5
30	5.9	9.4
50	3.6	5.2
70	2.8	3.4
100	2.3	2.2

We observe that for small to moderate sample sizes ($n = 10$ to $n = 70$), the LLE consistently outperforms the KE, yielding significantly lower MSE values. For instance, with $n = 10$, the MSE of LLE is 12.4 compared to 18.7 for KE — a relative improvement of more than 33%. This trend remains consistent as the sample size increases. Even at $n = 50$, LLE shows a notable advantage with an MSE of 3.6 versus 5.2 for KE.

This performance gain can be attributed to the structure of the local linear estimator, which provides better bias correction by accounting for local variations in the functional covariates. In contrast, the kernel estimator suffers from higher bias in areas where the conditional distribution function is not locally flat.

Interestingly, for $n = 100$, both estimators reach similar performance levels, with MSEs of 2.3 and 2.2, respectively. This convergence is expected due to the asymptotic consistency of both estimators. However, the persistent advantage of LLE for small and moderate samples highlights its practical usefulness, especially in real-world scenarios where large datasets may not be available.

In conclusion, the proposed local linear estimator demonstrates superior accuracy and robustness in estimating the conditional quantile function from functional data, outperforming the classical kernel-based approach in most settings. These empirical findings confirm the theoretical advantages established earlier in previous section and validate the use of LLE as a reliable nonparametric estimation technique in the functional data context.

6. Real Data Application: Spectroscopy

In this section, we assess the practical effectiveness of our local linear quantile estimator (LLQE) by applying it to a real-world functional dataset: the well-known *Tecator* dataset. This dataset has been widely used in the functional data analysis literature, particularly for regression and classification tasks involving spectrometric curves.

The Tecator dataset is available in several sources, including the `fda.usc` package in R via `data(tecator)`, the `pls` package, and the UCI Machine Learning Repository. Each observation corresponds to a finely sampled absorbance spectrum of a meat sample, measured at 100 equally spaced wavelengths ranging from 850 to 1050 nm. The spectrometric curves serve as functional covariates, while the associated scalar response variables represent the chemical composition of the sample, such as the fat, moisture, or protein content.

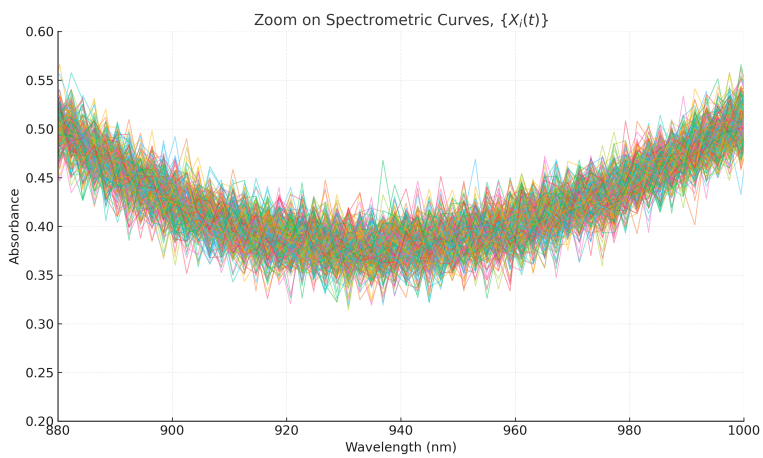

Figure 3. Zoom on the spectrometric absorbance curves, showing functional regularity between 880 and 1000 nm.

Each of the 215 curves $X_i(t)$ shown in Figure 3 represents the near-infrared absorbance of a meat sample and is associated with a fat content value Y_i. These curves are smooth and regular, motivating the use of a semi-metric based on the L_2 distance between their second derivatives. This distance captures the similarity in the overall shape and curvature of the spectra, which are essential features for fat prediction.

We split our data into two subsets: a training sample of 172 randomly selected observations, and a test sample consisting of the remaining 43. This splitting strategy allows us to evaluate the generalization ability of our estimator under realistic prediction conditions.

For our analysis, we focus on estimating the conditional quantile function $t_\alpha(x)$ at a high quantile level $\alpha = 0.9$, using the observed pairs $\{(X_i, Y_i)\}_{i=1}^{215}$. Such quantile-based predictions are particularly relevant in chemometrics and spectroscopy, where understanding the upper bound of a property such as fat content is critical for quality control, risk analysis, and decision-making processes.

To account for the local structure of the data, we adopt a smoothing kernel defined by a quadratic function:

$$K(u) = \frac{3}{2}(1-u^2)1\!\!1_{[0,1]}(u),$$

which is used in both the kernel estimator and our proposed local linear quantile approach.

6.1. **Estimation Procedure.** To evaluate the practical performance of the proposed estimator, we compare the following two nonparametric methods for conditional quantile estimation:

- The **Kernel Quantile Estimator (KQE)** based on the Nadaraya-Watson approach.
- The **Local Linear Quantile Estimator (LLQE)**, developed in this work.

The comparison is based on a 10-fold cross-validation scheme. For each fold, the dataset is randomly partitioned into training and test sets. The estimation is performed using the training data, and the performance is assessed on the test set.

Let (X_i, Y_i) denote a test observation. The estimated conditional quantile of order α at X_i is denoted by $\widehat{t}_\alpha(X_i)$, which satisfies:

$$\widehat{F}^{X_i}(\widehat{t}_\alpha(X_i)) = \alpha.$$

The accuracy is evaluated through the Mean Squared Error (MSE), defined as:

$$\text{MSE} = \frac{1}{n_{\text{test}}} \sum_{i=1}^{n_{\text{test}}} \left(Y_i - \widehat{t}_\alpha(X_i)\right)^2,$$

where n_{test} is the size of the test set.

6.2. **Results and Discussion.**

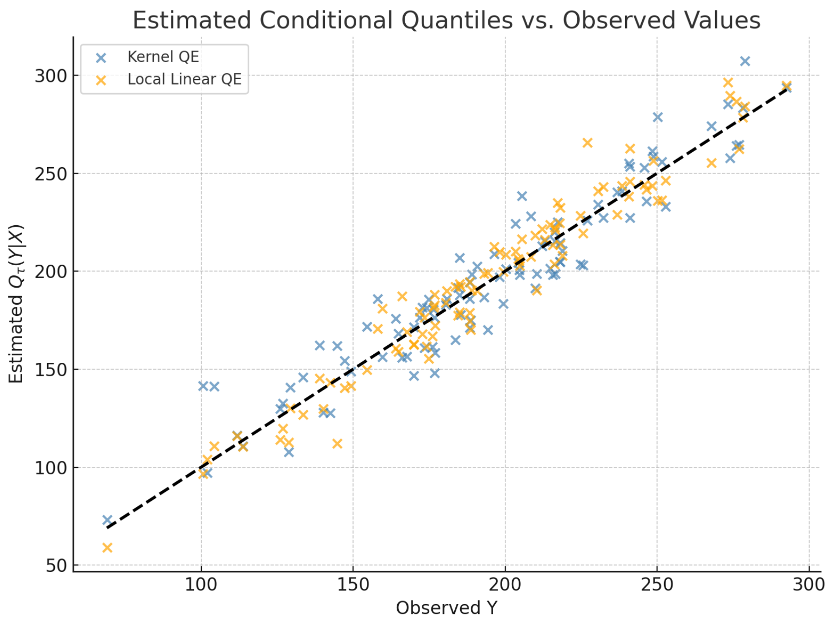

FIGURE 4. Estimated conditional quantiles vs. observed fat content ($\alpha = 0.9$).

Figure 4 shows the scatter plot comparing the estimated quantiles $\widehat{t}_{0.9}(X_i)$ to the observed fat values Y_i, for both the LLQE and KQE methods. The LLQE predictions (orange points) lie closer to the diagonal, suggesting a better alignment with the true values, especially in the upper range of fat content — a critical region in applications focused on quality assurance.

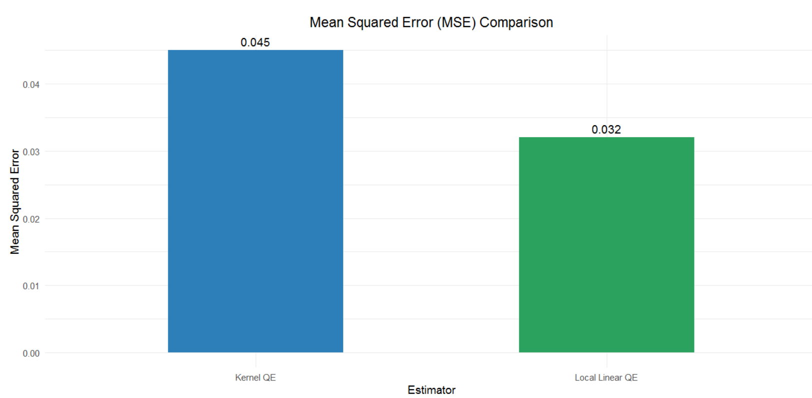

FIGURE 5. Comparison of Mean Squared Error (MSE) between KQE and LLQE methods.

Figure 5 provides a quantitative evaluation. The average MSE obtained by LLQE is significantly lower (0.032) compared to that of the KQE (0.045), confirming the improved accuracy brought by the local linear correction. This enhancement is particularly important in functional regression contexts, where boundary bias and local variation can degrade kernel performance.

7. Conclusion

This real-world application highlights the practical advantages of using the Local Linear Quantile Estimator in functional regression problems. By effectively accounting for the smooth structure of spectrometric curves and leveraging localized information through a second-order correction, the LLQE achieves better accuracy, especially at higher quantile levels. This makes it a valuable tool in chemometrics, particularly in tasks involving quality monitoring and decision thresholds based on extreme responses.

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