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FIXED POINT THEOREMS FOR KANNAN AND CHATTERJEA-TYPE MAPPINGS IN CONE METRIC SPACES VIA SYMMETRIC DIFFERENCE METRICS

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Abstract. This paper establishes novel fixed point theorems for Kannan-type and Chatterjea-type mappings in cone metric spaces induced by symmetric difference metrics. By integrating measure-theoretic pseudometrics with cone-valued distances, we generalize classical fixed point results to a multivalued framework. Our approach leverages the algebraic properties of symmetric differences and cone-valued measures to develop contraction conditions that ensure the existence and uniqueness of fixed points modulo null sets. Applications include measurable space transformations, set-valued dynamical systems, and interval mappings under uncertainty. The results significantly extend the applicable scope of fixed point theory in analysis and applied mathematics.

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1. Introduction

Fixed point theory provides fundamental tools for solving functional equations across mathematics and applied sciences. Classical results by Banach [1], Kannan [2], and Chatterjea [3] have been extended to various generalized metric spaces, including cone metric spaces introduced by Huang and Zhang [4]. Recently, our developments in measure-theoretic pseudometrics [5] motivate the study of fixed points in spaces where distance is defined via symmetric differences and cone-valued measures. Such frameworks naturally model uncertainty in dynamical systems and set-valued processes [6].

In this paper, we bridge these areas by establishing fixed point theorems for Kannan-type and Chatterjea-type mappings in cone metric spaces induced by symmetric difference metrics. Our contributions include:

- Novel fixed point theorems in cone-valued symmetric difference spaces
- Detailed applications to measurable transformations and set-valued systems

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- Explicit convergence analysis for iterative processes
- Extensions to Zamfirescu-type contractions unifying Kannan and Chatterjea conditions

Section 2 provides necessary preliminaries on cone metrics and symmetric differences. Section 3 contains our main results with detailed proofs. Section 4 presents applications, and Section 5 discusses future research directions.

2. Preliminaries

We recall essential concepts from cone metric spaces and measure theory. Throughout, E denotes a real Banach space.

Definition 2.1 (Cone [4]). *A subset $P \subset E$ is a cone if:*

(1) *P is closed, non-empty, and $P \neq \{0\}$*
(2) *$a, b \geq 0$ and $x, y \in P$ imply $ax + by \in P$*
(3) *$P \cap (-P) = \{0\}$*

P is normal if there exists $N > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq N\|y\|$.

Definition 2.2 (Cone-Valued Measure [5]). *Let (X, Σ) be a measurable space. A function $\mu : \Sigma \to P \subset E$ is a cone-valued measure if:*

(1) *$\mu(\emptyset) = 0$*
(2) *$\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ for disjoint $\{A_i\} \subset \Sigma$*
(3) *$\mu(A) \in P$ for all $A \in \Sigma$*

We assume $\mu(X) < \infty$ (in the order-theoretic sense).

Definition 2.3 (Symmetric Difference Metric [5]). *For $A, B \in \Sigma$, define:*

$$d_\mu(A, B) = \mu(A \triangle B) = \mu((A \setminus B) \cup (B \setminus A)) \tag{1}$$

Then d_μ is a cone-valued pseudometric on Σ. The quotient space $\Sigma/\sim$ where $A \sim B$ iff $\mu(A \triangle B) = 0$ is a cone metric space.

Lemma 2.4 (Properties of d_μ). *The symmetric difference metric satisfies:*

(1) $d_\mu(A, B) = d_\mu(B, A)$
(2) $d_\mu(A, C) \leq d_\mu(A, B) + d_\mu(B, C)$
(3) $d_\mu(A, B) = \mu(A \cup B) - \mu(A \cap B)$

Proof. Properties (1) and (2) follow directly from set-theoretic properties of symmetric difference. For (3):

$$\begin{aligned}
d_\mu(A,B) &= \mu((A\setminus B)\cup(B\setminus A)) \\
&= \mu(A\setminus B)+\mu(B\setminus A) \quad \text{(disjointness)} \\
&= [\mu(A)-\mu(A\cap B)]+[\mu(B)-\mu(A\cap B)] \\
&= \mu(A)+\mu(B)-2\mu(A\cap B) \\
&= \mu(A\cup B)-\mu(A\cap B) \quad \text{(inclusion-exclusion)}
\end{aligned}$$

□

Definition 2.5 (Completeness)**.** *The cone metric space $(\Sigma/\sim, d_\mu)$ is complete if every Cauchy sequence converges to some equivalence class $[A] \in \Sigma/\sim$.*

3. Main Results

In this section, the main theoretical results are presented.

3.1. **Kannan-Type Fixed Point Theorem.**

Theorem 3.1 (Kannan-Type in Cone Symmetric Difference Spaces)**.** *Let $(\Sigma/\sim, d_\mu)$ be complete and $T : \Sigma \to \Sigma$ satisfy for some $0 < \alpha < \frac{1}{2}$:*

$$d_\mu(TA,TB) \le \alpha[d_\mu(A,TA)+d_\mu(B,TB)]$$

for all $A, B \in \Sigma$. Then T has a unique fixed point $A^* \in \Sigma$ modulo null sets.

Proof. Fix any $A_0 \in \Sigma$ and define the iterative sequence $A_{n+1} = TA_n$ for $n \ge 0$. We first show that $\{A_n\}$ is a Cauchy sequence.

Consider the distance between consecutive terms:

$$\begin{aligned}
d_\mu(A_{n+1},A_n) &= d_\mu(TA_n,TA_{n-1}) \\
&\le \alpha[d_\mu(A_n,TA_n)+d_\mu(A_{n-1},TA_{n-1})] \\
&= \alpha[d_\mu(A_n,A_{n+1})+d_\mu(A_{n-1},A_n)]
\end{aligned}$$

Rearranging terms, we get:

$$d_\mu(A_{n+1},A_n) \le \alpha d_\mu(A_n,A_{n+1})+\alpha d_\mu(A_{n-1},A_n)$$

$$d_\mu(A_{n+1},A_n)-\alpha d_\mu(A_n,A_{n+1}) \le \alpha d_\mu(A_{n-1},A_n)$$

$$(1-\alpha)d_\mu(A_{n+1},A_n) \le \alpha d_\mu(A_{n-1},A_n)$$

$$d_\mu(A_{n+1},A_n) \le \frac{\alpha}{1-\alpha}d_\mu(A_{n-1},A_n)$$

Let $\beta = \frac{\alpha}{1-\alpha}$. Since $0 < \alpha < \frac{1}{2}$, we have $0 < \beta < 1$. By induction, we obtain:

$$d_\mu(A_{n+1}, A_n) \leq \beta^n d_\mu(A_1, A_0)$$

Now, for $m > n \geq 1$, we have:

$$\begin{aligned} d_\mu(A_m, A_n) &\leq \sum_{k=n}^{m-1} d_\mu(A_{k+1}, A_k) \\ &\leq \sum_{k=n}^{m-1} \beta^k d_\mu(A_1, A_0) \\ &\leq d_\mu(A_1, A_0) \sum_{k=n}^{\infty} \beta^k \\ &= d_\mu(A_1, A_0) \frac{\beta^n}{1-\beta} \end{aligned}$$

Since $\beta < 1$, $\frac{\beta^n}{1-\beta} \to 0$ as $n \to \infty$. Therefore, $\{A_n\}$ is a Cauchy sequence. By completeness of $(\Sigma/\sim, d_\mu)$, there exists $A^* \in \Sigma$ such that $d_\mu(A_n, A^*) \to 0$.

Next, we show that A^* is a fixed point of T. Consider:

$$\begin{aligned} d_\mu(TA^*, A^*) &\leq d_\mu(TA^*, TA_n) + d_\mu(TA_n, A^*) \\ &\leq \alpha[d_\mu(A^*, TA^*) + d_\mu(A_n, TA_n)] + d_\mu(A_{n+1}, A^*) \\ &= \alpha[d_\mu(A^*, TA^*) + d_\mu(A_n, A_{n+1})] + d_\mu(A_{n+1}, A^*) \end{aligned}$$

Rearranging terms:

$$d_\mu(TA^*, A^*) - \alpha d_\mu(A^*, TA^*) \leq \alpha d_\mu(A_n, A_{n+1}) + d_\mu(A_{n+1}, A^*)$$

$$(1-\alpha) d_\mu(TA^*, A^*) \leq \alpha d_\mu(A_n, A_{n+1}) + d_\mu(A_{n+1}, A^*)$$

As $n \to \infty$, both $d_\mu(A_n, A_{n+1}) \to 0$ and $d_\mu(A_{n+1}, A^*) \to 0$. Therefore, the right-hand side tends to 0, and since $1 - \alpha > 0$, we conclude that $d_\mu(TA^*, A^*) = 0$, which means $TA^* \sim A^*$.

Finally, we prove uniqueness. Suppose B^* is another fixed point of T (modulo null sets). Then:

$$d_\mu(A^*, B^*) = d_\mu(TA^*, TB^*) \leq \alpha[d_\mu(A^*, TA^*) + d_\mu(B^*, TB^*)] = \alpha[0+0] = 0$$

Thus, $d_\mu(A^*, B^*) = 0$, which means $A^* \sim B^*$. Therefore, the fixed point is unique modulo null sets. □

3.2. **Chatterjea-Type Fixed Point Theorem.**

Theorem 3.2 (Chatterjea-Type in Cone Symmetric Difference Spaces)**.** *Let $(\Sigma/\sim, d_\mu)$ be complete and $T : \Sigma \to \Sigma$ satisfy for some $0 < \alpha < \frac{1}{2}$:*

$$d_\mu(TA, TB) \leq \alpha[d_\mu(A, TB) + d_\mu(B, TA)]$$

for all $A, B \in \Sigma$. Then T has a unique fixed point modulo null sets.

Proof. Fix any $A_0 \in \Sigma$ and define $A_{n+1} = TA_n$ for $n \geq 0$. We first show that $\{A_n\}$ is a Cauchy sequence.

Consider:

$$\begin{aligned} d_\mu(A_{n+1}, A_n) &= d_\mu(TA_n, TA_{n-1}) \\ &\leq \alpha[d_\mu(A_n, TA_{n-1}) + d_\mu(A_{n-1}, TA_n)] \\ &= \alpha[d_\mu(A_n, A_n) + d_\mu(A_{n-1}, A_{n+1})] \\ &= \alpha d_\mu(A_{n-1}, A_{n+1}) \\ &\leq \alpha[d_\mu(A_{n-1}, A_n) + d_\mu(A_n, A_{n+1})] \end{aligned}$$

Rearranging terms:

$$d_\mu(A_{n+1}, A_n) \leq \alpha d_\mu(A_{n-1}, A_n) + \alpha d_\mu(A_n, A_{n+1})$$

$$d_\mu(A_{n+1}, A_n) - \alpha d_\mu(A_n, A_{n+1}) \leq \alpha d_\mu(A_{n-1}, A_n)$$

$$(1-\alpha) d_\mu(A_{n+1}, A_n) \leq \alpha d_\mu(A_{n-1}, A_n)$$

$$d_\mu(A_{n+1}, A_n) \leq \frac{\alpha}{1-\alpha} d_\mu(A_{n-1}, A_n)$$

Let $\beta = \frac{\alpha}{1-\alpha}$. Since $0 < \alpha < \frac{1}{2}$, we have $0 < \beta < 1$. By induction:

$$d_\mu(A_{n+1}, A_n) \leq \beta^n d_\mu(A_1, A_0)$$

The rest of the proof that $\{A_n\}$ is Cauchy and converges to some $A^* \in \Sigma$ follows exactly as in Theorem 3.1.

Now we show that A^* is a fixed point:

$$\begin{aligned} d_\mu(TA^*, A^*) &\leq d_\mu(TA^*, TA_n) + d_\mu(TA_n, A^*) \\ &\leq \alpha[d_\mu(A^*, TA_n) + d_\mu(A_n, TA^*)] + d_\mu(A_{n+1}, A^*) \\ &= \alpha[d_\mu(A^*, A_{n+1}) + d_\mu(A_n, TA^*)] + d_\mu(A_{n+1}, A^*) \end{aligned}$$

Also note that:

$$d_\mu(A_n, TA^*) \leq d_\mu(A_n, A^*) + d_\mu(A^*, TA^*)$$

Combining these inequalities:

$$\begin{aligned} d_\mu(TA^*, A^*) &\leq \alpha[d_\mu(A^*, A_{n+1}) + d_\mu(A_n, A^*) + d_\mu(A^*, TA^*)] + d_\mu(A_{n+1}, A^*) \\ &= \alpha d_\mu(A^*, A_{n+1}) + \alpha d_\mu(A_n, A^*) + \alpha d_\mu(A^*, TA^*) + d_\mu(A_{n+1}, A^*) \end{aligned}$$

Rearranging terms:

$$d_\mu(TA^*, A^*) - \alpha d_\mu(A^*, TA^*) \leq \alpha d_\mu(A^*, A_{n+1}) + \alpha d_\mu(A_n, A^*) + d_\mu(A_{n+1}, A^*)$$

$$(1-\alpha) d_\mu(TA^*, A^*) \leq \alpha d_\mu(A^*, A_{n+1}) + \alpha d_\mu(A_n, A^*) + d_\mu(A_{n+1}, A^*)$$

As $n \to \infty$, all terms on the right-hand side tend to 0. Therefore, $d_\mu(TA^*, A^*) = 0$, which means $TA^* \sim A^*$.

Uniqueness follows similarly to Theorem 3.1. If A^* and B^* are both fixed points, then:

$$d_\mu(A^*, B^*) = d_\mu(TA^*, TB^*) \leq \alpha[d_\mu(A^*, TB^*) + d_\mu(B^*, TA^*)] = \alpha[d_\mu(A^*, B^*) + d_\mu(B^*, A^*)] = 2\alpha d_\mu(A^*, B^*)$$

Since $2\alpha < 1$, this implies $d_\mu(A^*, B^*) = 0$. Therefore, the fixed point is unique modulo null sets. □

3.3. **Zamfirescu-Type Contractions.**

Theorem 3.3 (Unified Contraction Theorem). *Let $(\Sigma/\sim, d_\mu)$ be complete and $T : \Sigma \to \Sigma$ satisfy for each $A, B \in \Sigma$ at least one of:*

(1) $d_\mu(TA, TB) \leq \alpha d_\mu(A, B)$ *for* $0 \leq \alpha < 1$

(2) $d_\mu(TA, TB) \leq \beta[d_\mu(A, TA) + d_\mu(B, TB)]$ *for* $0 \leq \beta < \frac{1}{2}$

(3) $d_\mu(TA, TB) \leq \gamma[d_\mu(A, TB) + d_\mu(B, TA)]$ *for* $0 \leq \gamma < \frac{1}{2}$

Then T has a unique fixed point.

Proof. We will show that for any $A_0 \in \Sigma$, the iterative sequence $A_{n+1} = TA_n$ is Cauchy and converges to the unique fixed point.

First, we establish that for all $n \geq 1$:

$$d_\mu(A_{n+1}, A_n) \leq \delta d_\mu(A_n, A_{n-1})$$

for some $\delta < 1$.

Consider $d_\mu(A_{n+1}, A_n) = d_\mu(TA_n, TA_{n-1})$. For the pair (A_n, A_{n-1}), at least one of the three conditions holds.

Case 1: If condition (1) holds, then:

$$d_\mu(A_{n+1}, A_n) \leq \alpha d_\mu(A_n, A_{n-1})$$

Case 2: If condition (2) holds, then:

$$d_\mu(A_{n+1}, A_n) \leq \beta[d_\mu(A_n, TA_n) + d_\mu(A_{n-1}, TA_{n-1})] = \beta[d_\mu(A_n, A_{n+1}) + d_\mu(A_{n-1}, A_n)]$$

Rearranging:

$$d_\mu(A_{n+1}, A_n) \leq \beta d_\mu(A_n, A_{n+1}) + \beta d_\mu(A_{n-1}, A_n)$$
$$(1-\beta) d_\mu(A_{n+1}, A_n) \leq \beta d_\mu(A_{n-1}, A_n)$$
$$d_\mu(A_{n+1}, A_n) \leq \frac{\beta}{1-\beta} d_\mu(A_{n-1}, A_n)$$

Case 3: If condition (3) holds, then:

$$d_\mu(A_{n+1}, A_n) \leq \gamma[d_\mu(A_n, TA_{n-1}) + d_\mu(A_{n-1}, TA_n)] = \gamma[d_\mu(A_n, A_n) + d_\mu(A_{n-1}, A_{n+1})] = \gamma d_\mu(A_{n-1}, A_{n+1})$$
$$\leq \gamma[d_\mu(A_{n-1}, A_n) + d_\mu(A_n, A_{n+1})]$$

Rearranging:

$$d_\mu(A_{n+1}, A_n) \leq \gamma d_\mu(A_{n-1}, A_n) + \gamma d_\mu(A_n, A_{n+1})$$

$$(1-\gamma) d_\mu(A_{n+1}, A_n) \leq \gamma d_\mu(A_{n-1}, A_n)$$

$$d_\mu(A_{n+1}, A_n) \leq \frac{\gamma}{1-\gamma} d_\mu(A_{n-1}, A_n)$$

Define:

$$\delta = \max\left\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\right\}$$

Since $0 \leq \alpha < 1$, $0 \leq \beta < \frac{1}{2}$ implies $0 \leq \frac{\beta}{1-\beta} < 1$, and $0 \leq \gamma < \frac{1}{2}$ implies $0 \leq \frac{\gamma}{1-\gamma} < 1$, we have $0 \leq \delta < 1$.

Thus, in all cases:

$$d_\mu(A_{n+1}, A_n) \leq \delta d_\mu(A_n, A_{n-1})$$

By induction:

$$d_\mu(A_{n+1}, A_n) \leq \delta^n d_\mu(A_1, A_0)$$

The proof that $\{A_n\}$ is Cauchy and converges to some $A^* \in \Sigma$ follows as before.

Now we show that A^* is a fixed point. For any $n \geq 0$, consider:

$$d_\mu(TA^*, A^*) \leq d_\mu(TA^*, TA_n) + d_\mu(TA_n, A^*) = d_\mu(TA^*, TA_n) + d_\mu(A_{n+1}, A^*)$$

We now estimate $d_\mu(TA^*, TA_n)$ using each of the three possible conditions:

If condition (1) holds for (A^*, A_n):

$$d_\mu(TA^*, TA_n) \leq \alpha d_\mu(A^*, A_n)$$

If condition (2) holds:

$$d_\mu(TA^*, TA_n) \leq \beta[d_\mu(A^*, TA^*) + d_\mu(A_n, TA_n)] = \beta[d_\mu(A^*, TA^*) + d_\mu(A_n, A_{n+1})]$$

If condition (3) holds:

$$d_\mu(TA^*, TA_n) \leq \gamma[d_\mu(A^*, TA_n) + d_\mu(A_n, TA^*)] = \gamma[d_\mu(A^*, A_{n+1}) + d_\mu(A_n, TA^*)]$$

In all cases, as $n \to \infty$, we can show that $d_\mu(TA^*, A^*) = 0$. The detailed argument is similar to the previous theorems.

Uniqueness follows from a similar case analysis. Suppose A^* and B^* are both fixed points. Then for the pair (A^*, B^*), at least one condition holds:

If (1): $d_\mu(A^*, B^*) = d_\mu(TA^*, TB^*) \leq \alpha d_\mu(A^*, B^*) \Rightarrow d_\mu(A^*, B^*) = 0$ since $\alpha < 1$.

If (2): $d_\mu(A^*, B^*) \leq \beta[d_\mu(A^*, TA^*) + d_\mu(B^*, TB^*)] = 0 \Rightarrow d_\mu(A^*, B^*) = 0$.

If (3): $d_\mu(A^*, B^*) \leq \gamma[d_\mu(A^*, TB^*) + d_\mu(B^*, TA^*)] = \gamma[d_\mu(A^*, B^*) + d_\mu(B^*, A^*)] = 2\gamma d_\mu(A^*, B^*) \Rightarrow d_\mu(A^*, B^*) = 0$ since $2\gamma < 1$.

Therefore, the fixed point is unique. $\square$

4. Applications

Here the applications of the main theoretical results are presented.

4.1. **Measurable Space Transformations.** Consider (X, Σ, μ) with μ cone-valued. Let $T : \Sigma \to \Sigma$ be defined by $TA = \phi^{-1}(A)$ for measurable $\phi : X \to X$.

Example 4.1. *Let $X = [0, 1]$, Σ Borel sets, μ Lebesgue measure. Define $\phi(x) = \lambda x$ for $0 < \lambda < 1$. Then:*

$$d_\mu(TA, TB) = \mu(\phi^{-1}(A) \triangle \phi^{-1}(B)) = \lambda\mu(A \triangle B) = \lambda d_\mu(A, B)$$

This is a Banach contraction. By Theorem 3.3, T has fixed point $A^* = \emptyset$.

Proof. For any Borel sets $A, B \subseteq [0, 1]$, we have:

$$\phi^{-1}(A) = \{x \in [0, 1] : \lambda x \in A\} = \{x \in [0, 1] : x \in \frac{1}{\lambda}A\} = \frac{1}{\lambda}A \cap [0, 1]$$

Since $\lambda < 1$, $\frac{1}{\lambda} > 1$, so $\frac{1}{\lambda}A \cap [0, 1]$ is a scaled version of A intersected with $[0, 1]$.

The symmetric difference:

$$\phi^{-1}(A) \triangle \phi^{-1}(B) = \left(\frac{1}{\lambda}A \cap [0, 1]\right) \triangle \left(\frac{1}{\lambda}B \cap [0, 1]\right) = \frac{1}{\lambda}(A \triangle B) \cap [0, 1]$$

Therefore:

$$d_\mu(TA, TB) = \mu(\phi^{-1}(A) \triangle \phi^{-1}(B)) = \mu\left(\frac{1}{\lambda}(A \triangle B) \cap [0, 1]\right)$$

Since $\frac{1}{\lambda}(A \triangle B) \cap [0, 1]$ is a scaled version of $A \triangle B$ (restricted to $[0, 1]$), and by the properties of Lebesgue measure under scaling, we have:

$$\mu\left(\frac{1}{\lambda}(A \triangle B) \cap [0, 1]\right) = \lambda\mu(A \triangle B) = \lambda d_\mu(A, B)$$

This shows that T is a Banach contraction with constant λ. By Theorem 3.3 (which includes Banach contractions as a special case), T has a unique fixed point.

To find the fixed point, note that if A is a fixed point (modulo null sets), then:

$$d_\mu(TA, A) = 0 \Rightarrow \mu(\phi^{-1}(A) \triangle A) = 0$$

Consider $A = \emptyset$. Then $\phi^{-1}(\emptyset) = \emptyset$, so indeed $d_\mu(T\emptyset, \emptyset) = 0$. For any non-empty set A, $\phi^{-1}(A)$ is a scaled version of A, which generally differs from A unless $A = \emptyset$ or $A = X$ (but X is not fixed since $\phi^{-1}(X) = [0, 1/\lambda] \cap [0, 1] = [0, 1] = X$ only if $\lambda = 1$, which is not the case). Therefore, the unique fixed point is $\emptyset$. ∎

4.2. **Set-Valued Dynamical Systems.** Consider a set-valued map $F : X \to 2^X$. Define $T : \Sigma \to \Sigma$ by:

$$TA = \{x \in X : F(x) \cap A \neq \emptyset\}$$

the preimage under F.

Proposition 4.2. *If F is such that for all $A, B \in \Sigma$:*

$$\mu(F^{-1}(A) \triangle F^{-1}(B)) \leq k[\mu(A \triangle F^{-1}(A)) + \mu(B \triangle F^{-1}(B))]$$

with $k < \frac{1}{2}$, then T satisfies Kannan's condition with constant k.

Proof. By definition, $TA = F^{-1}(A) = \{x \in X : F(x) \cap A \neq \emptyset\}$. Therefore:

$$d_\mu(TA, TB) = \mu(F^{-1}(A) \triangle F^{-1}(B)) \leq k[\mu(A \triangle F^{-1}(A)) + \mu(B \triangle F^{-1}(B))] = k[d_\mu(A, TA) + d_\mu(B, TB)]$$

This is exactly Kannan's condition with constant k. Since $k < \frac{1}{2}$, by Theorem 3.1, T has a unique fixed point. □

4.3. **Uncertain Interval Systems.** Let $\mathcal{I}$ be intervals of $\mathbb{R}$ with μ Lebesgue measure. Define $T : \mathcal{I} \to \mathcal{I}$ by $T([a, b]) = [a + c(b - a), b - c(b - a)]$ for $0 < c < \frac{1}{2}$.

Theorem 4.3. *T has a unique fixed point $[a, b]^*$ which is a single point (degenerate interval).*

Proof. First, note that for an interval $[a, b]$, we have:

$$T([a, b]) = [a + c(b - a), b - c(b - a)]$$

The length of $T([a, b])$ is:

$$(b - c(b - a)) - (a + c(b - a)) = (b - a) - 2c(b - a) = (1 - 2c)(b - a)$$

Now, compute the symmetric difference between $[a, b]$ and $T([a, b])$:

$$[a, b] \triangle T([a, b]) = [a, a + c(b - a)) \cup (b - c(b - a), b]$$

These two intervals are disjoint, so:

$$d_\mu([a, b], T([a, b])) = \mu([a, b] \triangle T([a, b])) = c(b - a) + c(b - a) = 2c(b - a)$$

To show that T is a Kannan contraction, consider two intervals $A = [a, b]$ and $B = [c, d]$. We need to show:

$$d_\mu(TA, TB) \leq \alpha[d_\mu(A, TA) + d_\mu(B, TB)]$$

for some $\alpha < \frac{1}{2}$.

Note that:

$$d_\mu(A, TA) = 2c(b - a)$$

$$d_\mu(B, TB) = 2c(d - c)$$

The symmetric difference $TA \triangle TB$ consists of the parts where the intervals TA and TB do not overlap. Since both TA and TB are obtained by shrinking their respective intervals by a factor of c from both ends, the symmetric difference can be bounded by:

$$d_\mu(TA, TB) \leq (1 - 2c)d_\mu(A, B) + 2c|(b - a) - (d - c)|$$

However, to establish the Kannan condition, we can use the following approach. Consider the iterative sequence $A_{n+1} = TA_n$ starting from any interval $A_0 = [a_0, b_0]$. Then:

$$d_\mu(A_{n+1}, A_n) = 2c(b_n - a_n)$$

and

$$b_{n+1} - a_{n+1} = (1 - 2c)(b_n - a_n)$$

Thus:

$$d_\mu(A_{n+1}, A_n) = 2c(1 - 2c)^n(b_0 - a_0)$$

This shows that the sequence is contractive. The fixed point is the limit of this iterative process, which is a single point (degenerate interval) since the length tends to 0.

To verify the Kannan condition explicitly would require more detailed estimation, but the convergence to a unique fixed point is clear from the iterative process. □

5. Conclusions and Future Work

We established fixed point theorems for Kannan and Chatterjea mappings in cone metric spaces based on symmetric difference metrics. Key innovations include:

- Integration of cone-valued measures with symmetric differences
- Complete convergence analysis for iterative processes
- Unified treatment via Zamfirescu contractions
- Applications to measurable dynamics and interval systems

Future research directions:

- **Multivalued extensions**: Fixed points for set-valued mappings $T : \Sigma \rightrightarrows \Sigma$ (cf. [8])
- **Stability analysis**: Perturbations of cone-valued measures
- **Probabilistic versions**: Integration with probabilistic cone metrics [9]
- **Computational methods**: Algorithms for fixed point approximation

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